Digital Signal Processing

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DIGITAL SIGNAL PROCESSING
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To my Gurus

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and

To my well wishers

Er. SRI KONERU SATYANARAYANA

Honorable Chancellor, K.L. University, Vijayawada, A.P.

SRI B. VENU GOPALA KRISHNA

Chairman, Sasi Educational Society, Velivennu, W.G. District, A.P.

Contents

Preface

Reflecting over 38 years of experience in the classroom, this comprehensive textbook on Digital Signal Processing is developed to provide a solid grounding in the foundation of this subject. Using a student-friendly writing style, the text introduces the reader to the concepts of Digital Signal Processing in a simple and lucid manner. The text is suitable for use as one-semester course material by undergraduate students of Electronics and Communication Engineering, Telecommunication Engineering, Electronics and Instrumentation Engineering and Electrical and Electronics Engineering. It will also be useful to AMIE and grad IETE courses. This book is organised in 11 chapters. The outline of the book is as follows:

Signals constitute an important part of our daily life. Standard discrete-time signals, basic operations on signals, classification of signals and classification of the systems are discussed in Chapter 1. Also operations on signals and determination of the type of a given signal and determination of the type of a given system are illustrated with numerous examples.

Convolution and correlation of signals are very important in communication. Convolution is a mathematical way of combining two signals to form a third signal. Correlation, which is similar to convolution, compares two signals to determine the degree of similarity between them. The determination of linear convolution of two signals by various methods, determination of periodic convolution of signals using various methods, cross correlation and autocorrelation of signals, power spectral density and energy spectral density are covered in Chapter 2.

Z-transform is a very powerful mathematical technique for analysis of discrete-time systems. Unilateral and bilateral *Z*-transform, inverse *Z*-transform, ROC and its properties, properties and theorems of *Z*-transform and solution of difference equations using *Z*-transform are introduced in Chapter 3.

Realization of a discrete-time system means obtaining a network corresponding to the difference equation or transfer function of the system. Various methods of realization of discrete-time systems are described in Chapter 4.

Discrete-time Fourier transform (DTFT) is a method of representing a discrete-time signal in frequency domain. It is popular for digital signal processing because using this the complicated convolution operation of two sequences in time domain can be converted into a much simpler operation of multiplication in frequency domain. The DTFT, its properties and its use in the analysis of signals are explained in Chapter 5.

The Fourier series representation of a periodic discrete-time sequence is called discrete Fourier series (DFS). The discrete Fourier transform (DFT) is a sampled version of DTFT. The discrete Fourier series and its properties, the DFT and its properties, performing linear and circular convolutions using DFT, inverse discrete Fourier transform (IDFT), sectioned convolution using overlap-add method and overlap-save method are discussed in Chapter 6.

Fast Fourier Transform (FFT), a method developed by Cooley and Turkey is an algorithm for computing the DFT efficiently. The efficiency is achieved by adopting a divide and conquer approach which is based on decomposition of an N-point DFT into successively smaller DFTs and then combining them to give total transform. The computation of DFT by decimation-in-time (DIT) FFT algorithm and decimation-in-frequency (DIF) FFT algorithm, and computation of IDFT also by DIT FFT algorithm and DIF FFT algorithm and computation of DFT when N is a complex number by DIT FFT algorithm and DIF FFT algorithm are given in Chapter 7.

A filter is a frequency selective network. Filters are of two types—Finite impulse response (FIR) filters and Infinite impulse response (IIR) filters. IIR filters are designed by considering all the infinite samples of the impulse response. They are of recursive type. Design of IIR filters by approximation of derivatives, by impulse invariant transformation method and by bilinear transformation method are introduced in Chapter 8. Also design of low-pass Butterworth filters, low-pass Chebyshev filters and inverse Chebyshev filters are discussed and illustrated with examples. Analog and digital frequency transformations are also covered in this chapter.

FIR filters are designed by considering only a finite number of samples of the impulse response. They are usually implemented by non-recursive structures. Various design techniques for FIR filters—The Fourier series method, the window method and the frequency sampling method—are discussed and illustrated with examples in Chapter 9. Various windows like rectangular, triangular, Hamming, Hanning, Blackman and Kaiser windows are also explained in this chapter.

Discrete-time systems that process data at more than one sampling rate are known as multi-rate systems. Different sampling rates can be obtained using an up sampler and a down sampler. The basic operations in multi-rate processing to achieve this are decimation and interpolation. Up sampling, down sampling, decimation, interpolation, sampling rate conversion by a non-integer factor, polyphase decomposition, filter design for FIR decimators and interpolators and few applications of multi-rate signal processing are described in Chapter 10.

A programmable digital signal processor (P-DSP) is a specialized microprocessor designed specifically for digital signal processing, generally in real time computing. P-DSPs have many advantages over advanced microprocessors. Various architectures for P-DSPs, special address modes for P-DSPs, on-chip peripherals, architecture of TMS320C50 processor, CPU of that processor are discussed in Chapter 11.

A large number of typical examples have been worked out, so that the reader can understand the related concepts clearly. Extensive short questions with answers are given at the end of each chapter to enable the students to prepare for the examinations thoroughly. Review questions, fill in the blank type questions, objective type multiple choice questions and numerical problems are included at the end of each chapter to enable the students to build a clear understanding of the subject matter discussed in the text and also to assess their learning. The answers to all these are also given at the end of the book. Almost all the solved and unsolved problems presented in this book have been classroom tested. MATLAB programs and the results are given at the end of each chapter (Chapters 1 to 10) for typical examples. At the end of Chapter 11 few C-Programs and the results are given for the benefit of the students.

I express my profound gratitude to all those without whose assistance and cooperation, this book would not have been successfully completed. I wish to thank Dr. P. Srihari, Professor and Head of ECE department DIET, Anakapalli, Mr. G.N. Satapathi, and Sri TJV Subrahmanyeswara Rao, Associate Professor of KLUCE, Vijayawada for their help.

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A. Anand Kumar

Symbols, Notations and Abbreviations

SYMBOLS AND NOTATIONS

ω	Digital frequency
Ω	Analog frequency
$*$	Linear convolution
$\oplus$	Circular convolution
T	Transformation
F	Fourier transform
F^{-1}	Inverse Fourier transform
Z	Z-transform
Z^{-1}	Inverse Z-transform
$\longleftrightarrow$	Used for indicating a transform pair
$x(n)$	Time signal
$x(-n)$	Time reversed sequence
$X(s)$, $X(\omega)$ or $X(z)$	Transformed signal
$\lvert a \rvert$	Magnitude of the complex quantity a. Absolute value of a, if a is real valued
$\delta(n)$	Unit-sample sequence
$u(n)$	Unit-step sequence
$r(n)$	Unit-ramp sequence
$p(n)$	Unit parabolic sequence
Hz	Hertz
$X(k)$	DFT sequence
P	Average power
E	Total energy
$h(n)$	Impulse response
$H(s)$, $H(\omega)$, $H(z)$	Transfer function
W_N	Twiddle factor
$R_{xx}(n)$	Autocorrelation of $x(n)$
$R_{xy}(n)$	Cross correlation of $x(n)$ and $y(n)$

D	Decimation factor
I	Interpolation factor
$w(n)$	Window sequence
ω_c	Digital cutoff frequency
Ω_c	Analog cutoff frequency
ω_1	Pass band edge frequency
ω_2	Stop band edge frequency
$H_d(\omega)$	Desired frequency response
$h_d(n)$	Desired impulse response

ABBREVIATIONS

DSP	Digital Signal Processing
LTI	Linear Time Invariant
LTV	Linear Time Varying
LSI	Linear Shift Invariant
BIBO	Bounded-Input, Bounded-Output
FIR	Finite Impulse Response
IIR	Infinite Impulse Response
DFT	Discrete Fourier Transform
IDFT	Inverse Discrete Fourier Transform
DTFT	Discrete-Time Fourier Transform
IDTFT	Inverse Discrete-Time Fourier Transform
ROC	Region of Convergence
CTFT	Continuous-Time Fourier Transform
DFS	Discrete Fourier Series
FFT	Fast Fourier Transform
IFFT	Inverse Fast Fourier Transform
DIT	Decimation-In-Time
DIF	Decimation-In-Frequency
P-DSP	Programmable Digital Signal Processor
MAC	Multiplier Accumulator
ALU	Arithmetic Logic Unit
I/O	Input/Output
VLIW	Very Long Instruction Word
TDM	Time Division Multiplexing
RISC	Restricted Instruction Set Computer
CISC	Complicated Instruction Set Computer
CPU	Central Processing Unit
PLU	Parallel Logic Unit
CALU	Central Arithmetic Logic Unit
ARAU	Auxiliary Register Arithmetic Unit
PC	Program Counter
DMA	Direct Memory Access
PLL	Phase Locked Loop
HLL	High Level Language
A/D	Analog-To-Digital
D/A	Digital-To-Analog

1

Discrete-Time Signals and Systems

1.1 INTRODUCTION

Signals constitute an important part of our daily life. Anything that carries some information is called a signal. A signal is defined as a single-valued function of one or more independent variables which contain some information. A signal is also defined as a physical quantity that varies with time, space or any other independent variable. A signal may be represented in time domain or frequency domain. Human speech is a familiar example of a signal. Electric current and voltage are also examples of signals. A signal can be a function of one or more independent variables. A signal may be a function of time, temperature, position, pressure, distance etc. If a signal depends on only one independent variable, it is called a one-dimensional signal, and if a signal depends on two independent variables, it is called a two-dimensional signal.

A system is defined as an entity that acts on an input signal and transforms it into an output signal. A system is also defined as a set of elements or fundamental blocks which are connected together and produces an output in response to an input signal. It is a cause-and-effect relation between two or more signals. The actual physical structure of the system determines the exact relation between the input $x(n)$ and the output $y(n)$, and specifies the output for every input. Systems may be single-input and single-output systems or multi-input and multi-output systems.

Signal processing is a method of extracting information from the signal which in turn depends on the type of signal and the nature of information it carries. Thus signal processing is concerned with representing signals in the mathematical terms and extracting information by carrying out algorithmic operations on the signal. Digital signal processing has many advantages over analog signal processing. Some of these are as follows:

Digital circuits do not depend on precise values of digital signals for their operation. Digital circuits are less sensitive to changes in component values. They are also less sensitive to variations in temperature, ageing and other external parameters.

In a digital processor, the signals and system coefficients are represented as binary words. This enables one to choose any accuracy by increasing or decreasing the number of bits in the binary word.

Digital processing of a signal facilitates the sharing of a single processor among a number of signals by time sharing. This reduces the processing cost per signal.

Digital implementation of a system allows easy adjustment of the processor characteristics during processing.

Linear phase characteristics can be achieved only with digital filters. Also multirate processing is possible only in the digital domain. Digital circuits can be connected in cascade without any loading problems, whereas this cannot be easily done with analog circuits.

Storage of digital data is very easy. Signals can be stored on various storage media such as magnetic tapes, disks and optical disks without any loss. On the other hand, stored analog signals deteriorate rapidly as time progresses and cannot be recovered in their original form.

Digital processing is more suited for processing very low frequency signals such as seismic signals.

Though the advantages are many, there are some drawbacks associated with processing a signal in digital domain. Digital processing needs 'pre' and 'post' processing devices like analog-to-digital and digital-to-analog converters and associated reconstruction filters. This increases the complexity of the digital system. Also, digital techniques suffer from frequency limitations. Digital systems are constructed using active devices which consume power whereas analog processing algorithms can be implemented using passive devices which do not consume power. Moreover, active devices are less reliable than passive components. But the advantages of digital processing techniques outweigh the disadvantages in many applications. Also the cost of DSP hardware is decreasing continuously. Consequently, the applications of digital signal processing are increasing rapidly.

The digital signal processor may be a large programmable digital computer or a small microprocessor programmed to perform the desired operations on the input signal. It may also be a hardwired digital processor configured to perform a specified set of operations on the input signal.

DSP has many applications. Some of them are: Speech processing, Communication, Biomedical, Consumer electronics, Seismology and Image processing.

The block diagram of a DSP system is shown in Figure 1.1.

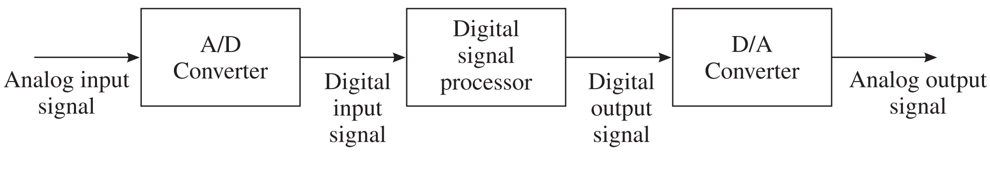


Figure 1.1 Block diagram of a digital signal processing system.

In this book we discuss only about discrete one-dimensional signals and consider only single-input and single-output discrete-time systems. In this chapter, we discuss about various basic discrete-time signals available, various operations on discrete-time signals and classification of discrete-time signals and discrete-time systems.

1.2 REPRESENTATION OF DISCRETE-TIME SIGNALS

Discrete-time signals are signals which are defined only at discrete instants of time. For those signals, the amplitude between the two time instants is just not defined. For discrete-time signal the independent variable is time n, and it is represented by $x(n)$.

There are following four ways of representing discrete-time signals:

1. Graphical representation
2. Functional representation
3. Tabular representation
4. Sequence representation

1.2.1 Graphical Representation

Consider a signal $x(n)$ with values

$$x(-2) = -3, \quad x(-1) = 2, \quad x(0) = 0, \quad x(1) = 3, \quad x(2) = 1 \quad \text{and} \quad x(3) = 2$$

This discrete-time signal can be represented graphically as shown in Figure 1.2.

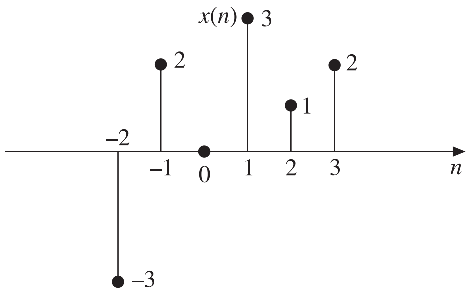


Figure 1.2 Graphical representation of discrete-time signal.

1.2.2 Functional Representation

In this, the amplitude of the signal is written against the values of n. The signal given in section 1.2.1 can be represented using the functional representation as follows:

$$x(n) = \begin{cases} -3 & \text{for } n = -2 \\ 2 & \text{for } n = -1 \\ 0 & \text{for } n = 0 \\ 3 & \text{for } n = 1 \\ 1 & \text{for } n = 2 \\ 2 & \text{for } n = 3 \end{cases}$$

Another example is:

$$x(n) = 2^n u(n)$$

or

$$x(n) = \begin{cases} 2^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

1.2.3 Tabular Representation

In this, the sampling instant n and the magnitude of the signal at the sampling instant are represented in the tabular form. The signal given in section 1.2.1 can be represented in tabular form as follows:

n	-2	-1	0	1	2	3
$x(n)$	-3	2	0	3	1	2

1.2.4 Sequence Representation

A finite duration sequence given in section 1.2.1 can be represented as follows:

$$x(n) = \left\{ \begin{matrix} -3, 2, 0, 3, 1, 2 \\ \uparrow \end{matrix} \right\}$$

Another example is:

$$x(n) = \left\{ \begin{matrix} \ldots 2, 3, 0, 1, -2, \ldots \\ \uparrow \end{matrix} \right\}$$

The arrow mark $\uparrow$ denotes the $n = 0$ term. When no arrow is indicated, the first term corresponds to $n = 0$.

So a finite duration sequence, that satisfies the condition $x(n) = 0$ for $n < 0$ can be represented as:

$$x(n) = \{3, 5, 2, 1, 4, 7\}$$

Sum and product of discrete-time sequences

The sum of two discrete-time sequences is obtained by adding the corresponding elements of sequences

$$\{C_n\} = \{a_n\} + \{b_n\} \quad \rightarrow \quad C_n = a_n + b_n$$

The product of two discrete-time sequences is obtained by multiplying the corresponding elements of the sequences.

$$\{C_n\} = \{a_n\}\{b_n\} \quad \rightarrow \quad C_n = a_n b_n$$

The multiplication of a sequence by a constant k is obtained by multiplying each element of the sequence by that constant.

$$\{C_n\} = k\{a_n\} \quad \rightarrow \quad C_n = k a_n$$

1.3 ELEMENTARY DISCRETE-TIME SIGNALS

There are several elementary signals which play vital role in the study of signals and systems. These elementary signals serve as basic building blocks for the construction of more complex signals. Infact, these elementary signals may be used to model a large number of physical signals, which occur in nature. These elementary signals are also called standard signals.

The standard discrete-time signals are as follows:

1. Unit step sequence
2. Unit ramp sequence
3. Unit parabolic sequence
4. Unit impulse sequence
5. Sinusoidal sequence
6. Real exponential sequence
7. Complex exponential sequence

1.3.1 Unit Step Sequence

The step sequence is an important signal used for analysis of many discrete-time systems. It exists only for positive time and is zero for negative time. It is equivalent to applying a signal whose amplitude suddenly changes and remains constant at the sampling instants forever after application. In between the discrete instants it is zero. If a step function has unity magnitude, then it is called unit step function.

The usefulness of the unit-step function lies in the fact that if we want a sequence to start at $n = 0$, so that it may have a value of zero for $n < 0$, we only need to multiply the given sequence with unit step function $u(n)$.

The discrete-time unit step sequence $u(n)$ is defined as:

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

The shifted version of the discrete-time unit step sequence $u(n - k)$ is defined as:

$$u(n-k) = \begin{cases} 1 & \text{for } n \geq k \\ 0 & \text{for } n < k \end{cases}$$

It is zero if the arugment $(n - k) < 0$ and equal to 1 if the arugment $(n - k) \geq 0$.

The graphical representation of $u(n)$ and $u(n - k)$ is shown in Figure 1.3[(a) and (b)].

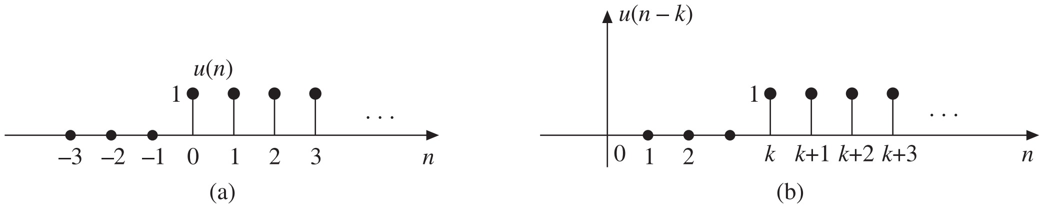


Figure 1.3 Discrete-time (a) Unit step function (b) Shifted unit step function.

1.3.2 Unit Ramp Sequence

The discrete-time unit ramp sequence $r(n)$ is that sequence which starts at $n = 0$ and increases linearly with time and is defined as:

$$r(n) = \begin{cases} n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

or $$r(n) = nu(n)$$

It starts at $n = 0$ and increases linearly with n.

The shifted version of the discrete-time unit ramp sequence $r(n - k)$ is defined as:

$$r(n-k) = \begin{cases} n-k & \text{for } n \geq k \\ 0 & \text{for } n < k \end{cases}$$

or $$r(n - k) = (n - k)\ u(n - k)$$

The graphical representation of $r(n)$ and $r(n - 2)$ is shown in Figure 1.4[(a) and (b)].

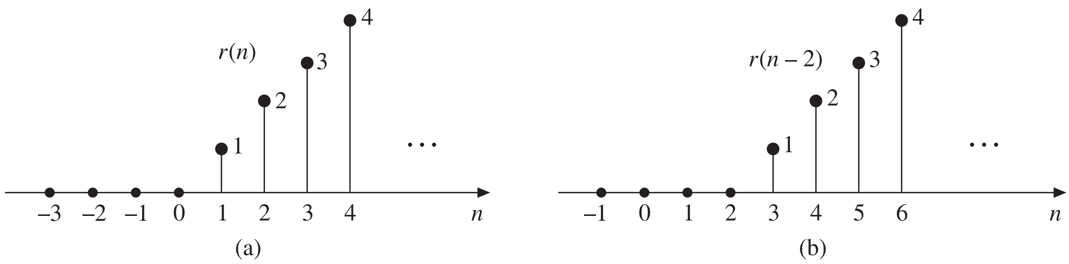


Figure 1.4 Discrete-time (a) Unit ramp sequence (b) Shifted ramp sequence.

1.3.3 Unit Parabolic Sequence

The discrete-time unit parabolic sequence $p(n)$ is defined as:

$$p(n) = \begin{cases} \dfrac{n^2}{2} & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

or $$p(n) = \frac{n^2}{2} u(n)$$

The shifted version of the discrete-time unit parabolic sequence $p(n - k)$ is defined as:

$$p(n-k) = \begin{cases} \dfrac{(n-k)^2}{2} & \text{for } n \ge k \\ 0 & \text{for } n < k \end{cases}$$

or $$p(n-k) = \frac{(n-k)^2}{2} u(n-k)$$

The graphical representation of $p(n)$ and $p(n - 3)$ is shown in Figure 1.5[(a) and (b)].

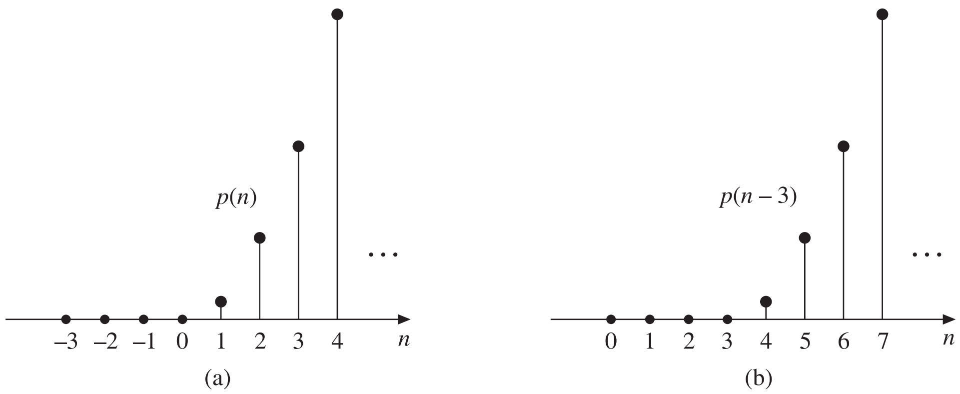


Figure 1.5 Discrete-time (a) Parabolic sequence (b) Shifted parabolic sequence.

1.3.4 Unit Impulse Function or Unit Sample Sequence

The discrete-time unit impulse function $\delta(n)$, also called unit sample sequence, is defined as:

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \ne 0 \end{cases}$$

This means that the unit sample sequence is a signal that is zero everywhere, except at $n = 0$, where its value is unity. It is the most widely used elementary signal used for the analysis of signals and systems.

The shifted unit impulse function $\delta(n - k)$ is defined as:

$$\delta(n-k) = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{for } n \ne k \end{cases}$$

The graphical representation of $\delta(n)$ and $\delta(n - k)$ is shown in Figure 1.6[(a) and (b)].

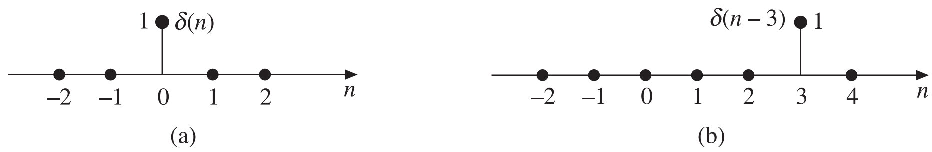


Figure 1.6 Discrete-time (a) Unit sample sequence (b) Delayed unit sample sequence.

Properties of discrete-time unit sample sequence

1. $\delta(n) = u(n) - u(n-1)$

2. $\delta(n-k) = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{for } n \neq k \end{cases}$

3. $x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$

4. $\sum_{n=-\infty}^{\infty} x(n)\,\delta(n-n_0) = x(n_0)$

Relation between the unit sample sequence and the unit step sequence

The unit sample sequence $\delta(n)$ and the unit step sequence $u(n)$ are related as:

$$u(n) = \sum_{m=0}^{n} \delta(m), \quad \delta(n) = u(n) - u(n-1)$$

1.3.5 Sinusoidal Sequence

The discrete-time sinusoidal sequence is given by

$$x(n) = A\,\sin(\omega n + \phi)$$

where A is the amplitude, ω is angular frequency, ϕ is phase angle in radians and n is an integer.

The period of the discrete-time sinusoidal sequence is:

$$N = \frac{2\pi}{\omega}\,m$$

where N and m are integers.

All continuous-time sinusoidal signals are periodic, but discrete-time sinusoidal sequences may or may not be periodic depending on the value of ω.

For a discrete-time signal to be periodic, the angular frequency ω must be a rational multiple of 2π. The graphical representation of a discrete-time sinusoidal signal is shown in Figure 1.7.

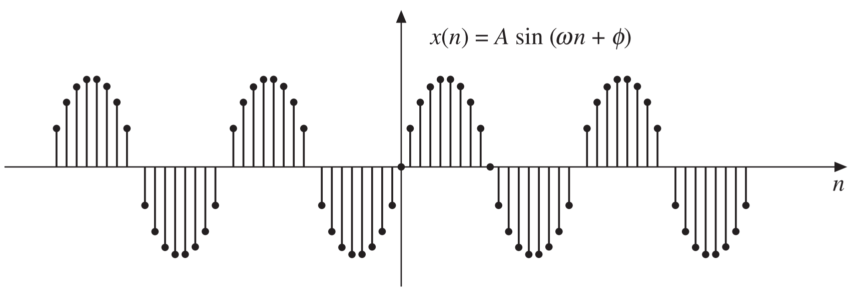


Figure 1.7 Discrete-time sinusoidal signal.

1.3.6 Real Exponential Sequence

The discrete-time real exponential sequence a^n is defined as:

$$x(n) = a^n \quad \text{for all } n$$

Figure 1.8 illustrates different types of discrete-time exponential signals.

When $a > 1$, the sequence grows exponentially as shown in Figure 1.8(a).

When $0 < a < 1$, the sequence decays exponentially as shown in Figure 1.8(b).

When $a < 0$, the sequence takes alternating signs as shown in Figure 1.8[(c) and (d)].

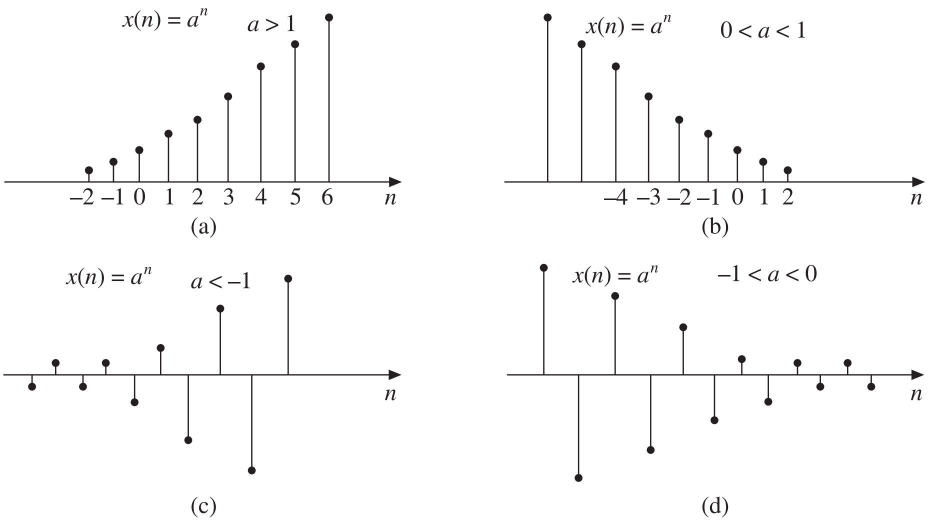


Figure 1.8 Discrete-time exponential signal a^n for (a) $a > 1$ (b) $0 < a < 1$ (c) $a < -1$ (d) $-1 < a < 0$.

1.3.7 Complex Exponential Sequence

The discrete-time complex exponential sequence is defined as:

$$\begin{aligned} x(n) &= a^n e^{j(\omega_0 n + \phi)} \\ &= a^n \cos(\omega_0 n + \phi) + ja^n \sin(\omega_0 n + \phi) \end{aligned}$$

For $|a| = 1$, the real and imaginary parts of complex exponential sequence are sinusoidal.

For $|a| > 1$, the amplitude of the sinusoidal sequence exponentially grows as shown in Figure 1.9(a).

For $|a| < 1$, the amplitude of the sinusoidal sequence exponentially decays as shown in Figure 1.9(b).

EXAMPLE 1.1 Find the following summations:

(a) $\sum_{n=-\infty}^{\infty} e^{3n}\,\delta(n-3)$ (b) $\sum_{n=-\infty}^{\infty} \delta(n-2)\cos 3n$

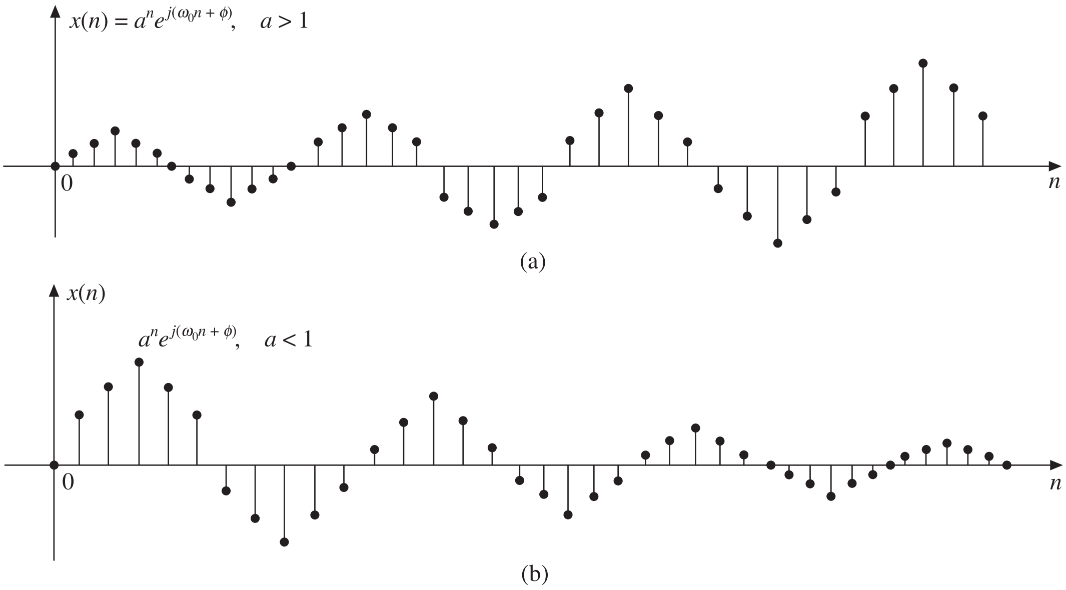


Figure 1.9 Complex exponential sequence $x(n) = a^n e^{j(\omega_0 n + \phi)}$ for (a) $a > 1$ (b) $a < 1$.

(c) $\sum_{n=-\infty}^{\infty} n^2 \delta(n+4)$

(d) $\sum_{n=-\infty}^{\infty} \delta(n-2)\, e^{n^2}$

(e) $\sum_{n=0}^{\infty} \delta(n+1)\, 4^n$

Solution:

(a) Given $$\sum_{n=-\infty}^{\infty} e^{3n}\, \delta(n-3)$$

We know that $$\delta(n-3) = \begin{cases} 1 & \text{for } n = 3 \\ 0 & \text{elsewhere} \end{cases}$$

$\therefore$ $$\sum_{n=-\infty}^{\infty} e^{3n} \delta(n-3) = [e^{3n}]_{n=3} = e^9$$

(b) Given $$\sum_{n=-\infty}^{\infty} \delta(n-2) \cos 3n$$

We know that $$\delta(n-2) = \begin{cases} 1 & \text{for } n = 2 \\ 0 & \text{elsewhere} \end{cases}$$

$\therefore$ $$\sum_{n=-\infty}^{\infty} \delta(n-2) \cos 3n = [\cos 3n]_{n=2} = \cos 6$$

(c) Given $$\sum_{n=-\infty}^{\infty} n^2\delta(n+4)$$

We know that $$\delta(n+4) = \begin{cases} 1 & \text{for } n=-4 \\ 0 & \text{elsewhere} \end{cases}$$

$\therefore$ $$\sum_{n=-\infty}^{\infty} n^2\delta(n+4) = [n^2]_{n=-4} = 16$$

(d) Given $$\sum_{n=-\infty}^{\infty} \delta(n-2)\, e^{n^2}$$

We know that $$\delta(n-2) = \begin{cases} 1 & \text{for } n=2 \\ 0 & \text{elsewhere} \end{cases}$$

$\therefore$ $$\sum_{n=-\infty}^{\infty} \delta(n-2)\, e^{n^2} = [e^{n^2}]_{n=2} = e^{2^2} = e^4$$

(e) Given $$\sum_{n=0}^{\infty} \delta(n+1)\, 4^n$$

We know that $$\delta(n+1) = \begin{cases} 1 & \text{for } n=-1 \\ 0 & \text{for } n\neq -1 \end{cases}$$

$\therefore$ $$\sum_{n=0}^{\infty} \delta(n+1)\, 4^n = 0$$

1.4 BASIC OPERATIONS ON SEQUENCES

When we process a sequence, this sequence may undergo several manipulations involving the independent variable or the amplitude of the signal.

The basic operations on sequences are as follows:

1. Time shifting
2. Time reversal
3. Time scaling
4. Amplitude scaling
5. Signal addition
6. Signal multiplication

The first three operations correspond to transformation in independent variable n of a signal. The last three operations correspond to transformation on amplitude of a signal.

1.4.1 Time Shifting

The time shifting of a signal may result in time delay or time advance. The time shifting operation of a discrete-time signal $x(n)$ can be represented by

$$y(n) = x(n - k)$$

This shows that the signal $y(n)$ can be obtained by time shifting the signal $x(n)$ by k units. If k is positive, it is delay and the shift is to the right, and if k is negative, it is advance and the shift is to the left.

An arbitrary signal $x(n)$ is shown in Figure 1.10(a). $x(n - 3)$ which is obtained by shifting $x(n)$ to the right by 3 units (i.e. delay $x(n)$ by 3 units) is shown in Figure 1.10(b). $x(n + 2)$ which is obtained by shifting $x(n)$ to the left by 2 units (i.e. advancing $x(n)$ by 2 units) is shown in Figure 1.10(c).

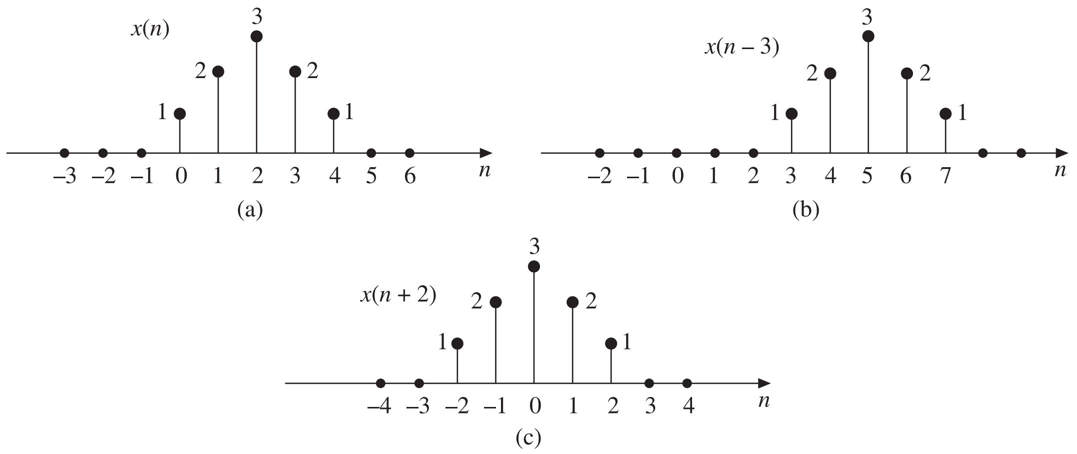


Figure 1.10 (a) Sequence $x(n)$ (b) $x(n - 3)$ (c) $x(n + 2)$.

1.4.2 Time Reversal

The time reversal also called time folding of a discrete-time signal $x(n)$ can be obtained by folding the sequence about $n = 0$. The time reversed signal is the reflection of the original signal. It is obtained by replacing the independent variable n by $-n$. Figure 1.11(a) shows an arbitrary discrete-time signal $x(n)$, and its time reversed version $x(-n)$ is shown in Figure 1.11(b). Figure 1.11[(c) and (d)] shows the delayed and advanced versions of reversed signal $x(-n)$.

The signal $x(-n + 3)$ is obtained by delaying (shifting to the right) the time reversed signal $x(-n)$ by 3 units of time. The signal $x(-n - 3)$ is obtained by advancing (shifting to the left) the time reversed signal $x(-n)$ by 3 units of time.

Figure 1.12 shows other examples for time reversal of signals.

EXAMPLE 1.2 Sketch the following signals:

(a) $u(n+2)\,u(-n+3)$ (b) $x(n) = u(n+4) - u(n-2)$

Solution:

(a) Given $x(n) = u(n+2)\,u(-n+3)$

The signal $u(n + 2)\,u(-n + 3)$ can be obtained by first drawing the signal $u(n + 2)$ as shown in Figure 1.13(a), then drawing $u(-n + 3)$ as shown in Figure 1.13(b),

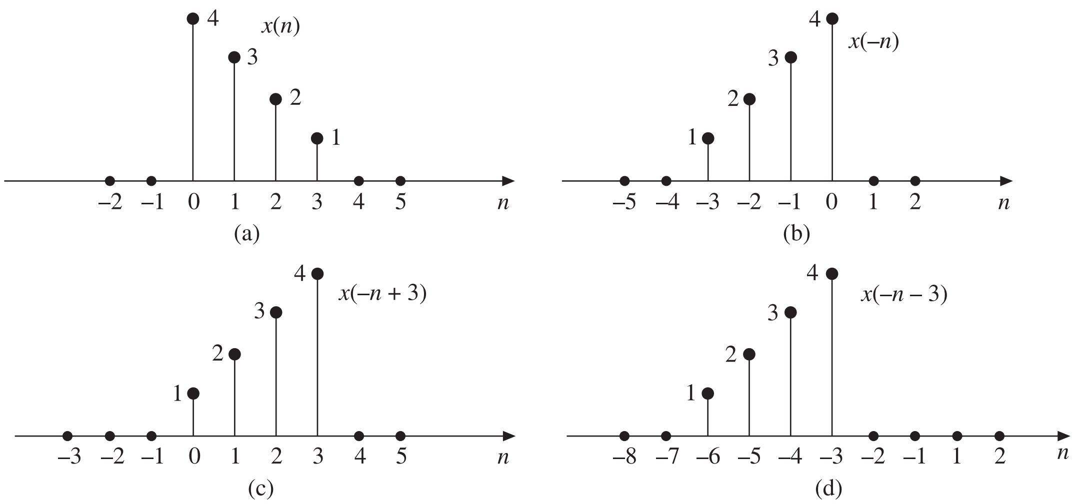


Figure 1.11 (a) Original signal $x(n)$ (b) Time reversed signal $x(-n)$ (c) Time reversed and delayed signal $x(-n+3)$ (d) Time reversed and advanced signal $x(-n-3)$.

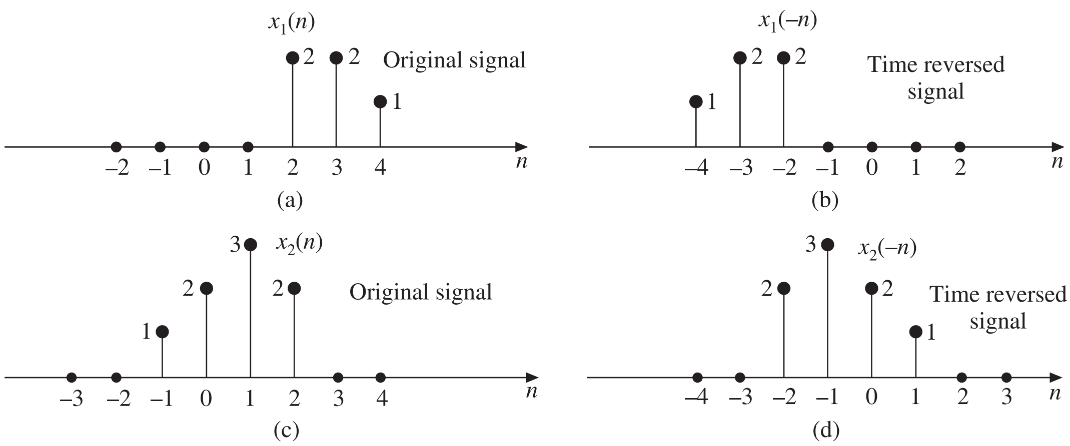


Figure 1.12 Time reversal operations.

and then multiplying these sequences element by element to obtain $u(n + 2)\, u(-n + 3)$ as shown in Figure 1.13(c).

$$x(n) = 0 \quad \text{for} \quad n < -2 \quad \text{and} \quad n > 3; \; x(n) = 1 \quad \text{for} \quad -2 < n < 3$$

(b) Given $\qquad x(n) = u(n + 4) - u(n - 2)$

The signal $u(n + 4) - u(n - 2)$ can be obtained by first plotting $u(n + 4)$ as shown in Figure 1.14(a), then plotting $u(n - 2)$ as shown in Figure 1.14(b), and then subtracting each element of $u(n - 2)$ from the corresponding element of $u(n + 4)$ to obtain the result shown in Figure 1.14(c).

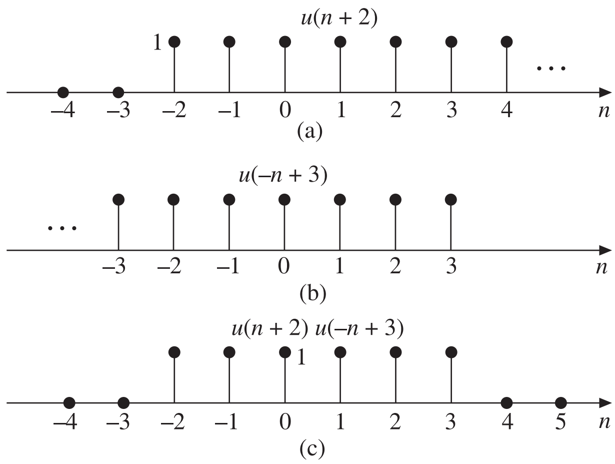


Figure 1.13 Plots of (a) $u(n + 2)$ (b) $u(-n + 3)$ (c) $u(n + 2)\,u(-n + 3)$.

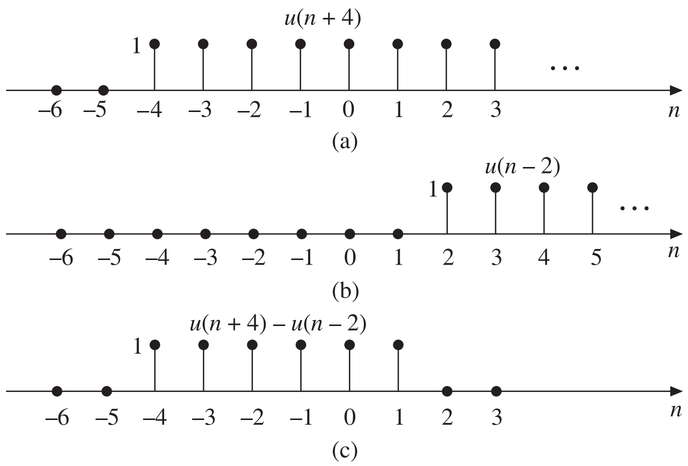


Figure 1.14 Plots of (a) $u(n + 4)$ (b) $u(n - 2)$ (c) $u(n + 4) - u(n - 2)$.

1.4.3 Amplitude Scaling

The amplitude scaling of a discrete-time signal can be represented by

$$y(n) = ax(n)$$

where a is a constant.

The amplitude of $y(n)$ at any instant is equal to a times the amplitude of $x(n)$ at that instant. If $a > 1$, it is amplification and if $a < 1$, it is attenuation. Hence the amplitude is rescaled. Hence the name amplitude scaling.

Figure 1.15(a) shows a signal $x(n)$ and Figure 1.15(b) shows a scaled signal $y(n) = 2x(n)$.

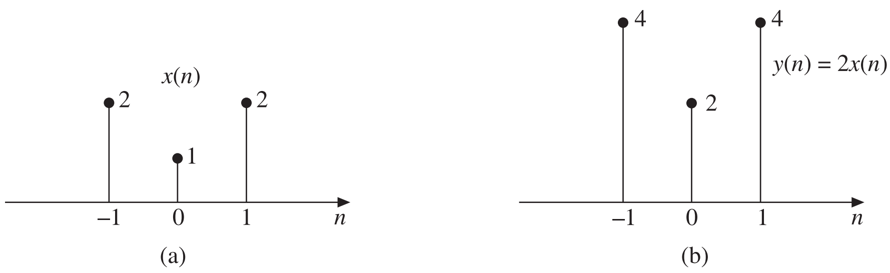


Figure 1.15 Plots of (a) Signal $x(n)$ (b) $y(n) = 2x(n)$.

1.4.4 Time Scaling

Time scaling may be time expansion or time compression. The time scaling of a discrete-time signal $x(n)$ can be accomplished by replacing n by an in it. Mathematically, it can be expressed as:

$$y(n) = x(an)$$

When $a > 1$, it is time compression and when $a < 1$, it is time expansion.

Let $x(n)$ be a sequence as shown in Figure 1.16(a). If $a = 2$, $y(n) = x(2n)$. Then

$$\begin{aligned} y(0) &= x(0) = 1 \\ y(-1) &= x(-2) = 3 \\ y(-2) &= x(-4) = 0 \\ y(1) &= x(2) = 3 \\ y(2) &= x(4) = 0 \end{aligned}$$

and so on.

So to plot $x(2n)$ we have to skip odd numbered samples in $x(n)$.

We can plot the time scaled signal $y(n) = x(2n)$ as shown in Figure 1.16(b). Here the signal is compressed by 2.

If $a = (1/2)$, $y(n) = x(n/2)$, then

$$\begin{aligned} y(0) &= x(0) = 1 \\ y(2) &= x(1) = 2 \\ y(4) &= x(2) = 3 \\ y(6) &= x(3) = 4 \\ y(8) &= x(4) = 0 \\ y(-2) &= x(-1) = 2 \\ y(-4) &= x(-2) = 3 \\ y(-6) &= x(-3) = 4 \\ y(-8) &= x(-4) = 0 \end{aligned}$$

We can plot $y(n) = x(n/2)$ as shown in Figure 1.16(c). Here the signal is expanded by 2. All odd components in $x(n/2)$ are zero because $x(n)$ does not have any value in between the sampling instants.

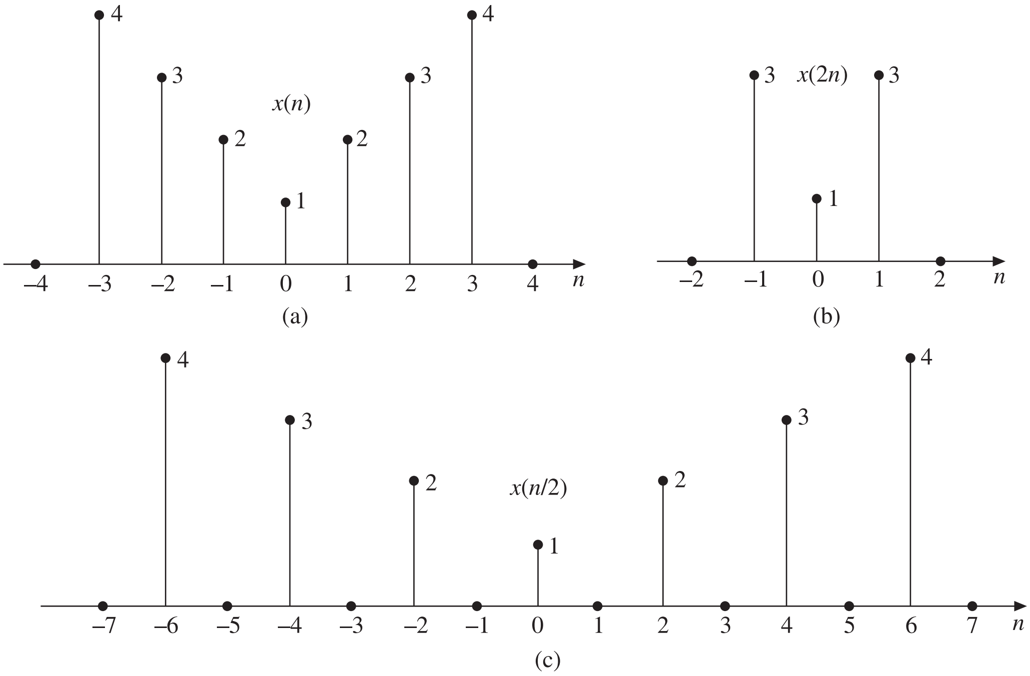


Figure 1.16 Discrete-time scaling (a) Plot of $x(n)$ (b) Plot of $x(2n)$ (c) Plot of $x(n/2)$.

Time scaling is very useful when data is to be fed at some rate and is to be taken out at a different rate.

1.4.5 Signal Addition

In discrete-time domain, the sum of two signals $x_1(n)$ and $x_2(n)$ can be obtained by adding the corresponding sample values and the subtraction of $x_2(n)$ from $x_1(n)$ can be obtained by subtracting each sample of $x_2(n)$ from the corresponding sample of $x_1(n)$ as illustrated below.

If $\quad x_1(n) = \{1, 2, 3, 1, 5\}$ and $x_2(n) = \{2, 3, 4, 1, -2\}$

Then $\quad x_1(n) + x_2(n) = \{1 + 2, 2 + 3, 3 + 4, 1 + 1, 5 - 2\} = \{3, 5, 7, 2, 3\}$

and $\quad x_1(n) - x_2(n) = \{1 - 2, 2 - 3, 3 - 4, 1 - 1, 5 + 2\} = \{-1, -1, -1, 0, 7\}$

1.4.6 Signal Multiplication

The multiplication of two discrete-time sequences can be performed by multiplying their values at the sampling instants as shown below.

If $\quad x_1(n) = \{1, -3, 2, 4, 1.5\}$ and $x_2(n) = \{2, -1, 3, 1.5, 2\}$

Then
$$x_1(n)\, x_2(n) = \{1 \times 2, -3 \times -1, 2 \times 3, 4 \times 1.5, 1.5 \times 2\}$$
$$= \{2, 3, 6, 6, 3\}$$

EXAMPLE 1.3 Express the signals shown in Figure 1.17 as the sum of singular functions.

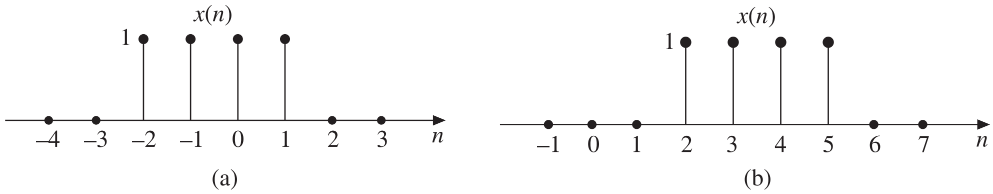


Figure 1.17 Waveforms for Example 1.3.

Solution:

(a) The given signal shown in Figure 1.17(a) is:

$$x(n) = \delta(n+2) + \delta(n+1) + \delta(n) + \delta(n-1)$$

$$x(n) = \begin{cases} 0 & \text{for } n \le -3 \\ 1 & \text{for } -2 \le n \le 1 \\ 0 & \text{for } n \ge 2 \end{cases}$$

$$\therefore \qquad x(n) = u(n+2) - u(n-2)$$

(b) The signal shown in Figure 1.17(b) is:

$$x(n) = \delta(n-2) + \delta(n-3) + \delta(n-4) + \delta(n-5)$$

$$x(n) = \begin{cases} 0 & \text{for } n \le 1 \\ 1 & \text{for } 2 \le n \le 5 \\ 0 & \text{for } n \ge 6 \end{cases}$$

$$\therefore \qquad x(n) = u(n-2) - u(n-6)$$

1.5 CLASSIFICATION OF DISCRETE-TIME SIGNALS

The signals can be classified based on their nature and characteristics in the time domain. They are broadly classified as: (i) continuous-time signals and (ii) discrete-time signals.

The signals that are defined for every instant of time are known as continuous-time signals. The continuous-time signals are also called analog signals. They are denoted by $x(t)$. They are continuous in amplitude as well as in time. Most of the signals available are continuous-time signals.

The signals that are defined only at discrete instants of time are known as discrete-time signals. The discrete-time signals are continuous in amplitude, but discrete in time. For discrete-time signals, the amplitude between two time instants is just not defined. For discrete-time signals, the independent variable is time n. Since they are defined only at discrete instants of time, they are denoted by a sequence $x(nT)$ or simply by $x(n)$ where n is an integer.

Figure 1.18 shows the graphical representation of discrete-time signals. The discrete-time signals may be inherently discrete or may be discrete versions of the continuous-time signals.

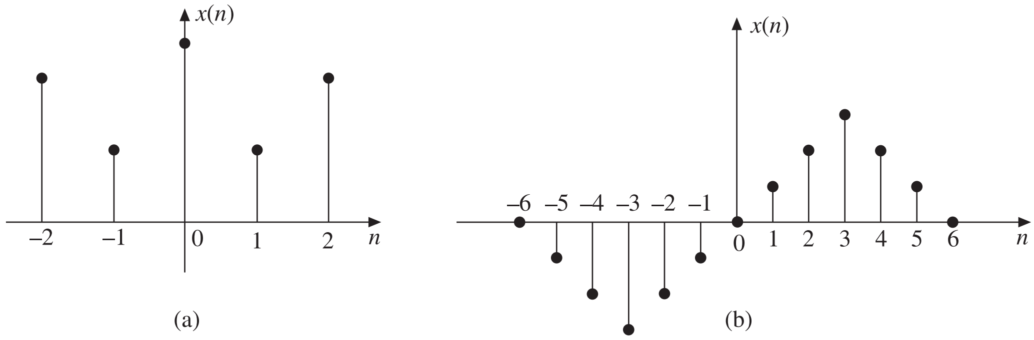


Figure 1.18 Discrete-time signals.

Both continuous-time and discrete-time signals are further classified as follows:

1. Deterministic and random signals
2. Periodic and non-periodic signals
3. Energy and power signals
4. Causal and non-causal signals
5. Even and odd signals

1.5.1 Deterministic and Random Signals

A signal exhibiting no uncertainty of its magnitude and phase at any given instant of time is called deterministic signal. A deterministic signal can be completely represented by mathematical equation at any time and its nature and amplitude at any time can be predicted.

Examples: Sinusoidal sequence $x(n) = \cos \omega n$, Exponential sequence $x(n) = e^{j\omega n}$, ramp sequence $x(n) = \alpha n$.

A signal characterized by uncertainty about its occurrence is called a non-deterministic or random signal. A random signal cannot be represented by any mathematical equation. The behaviour of such a signal is probabilistic in nature and can be analyzed only stochastically. The pattern of such a signal is quite irregular. Its amplitude and phase at any time instant cannot be predicted in advance. A typical example of a non-deterministic signal is thermal noise.

1.5.2 Periodic and Non-periodic Sequences

A signal which has a definite pattern and repeats itself at regular intervals of time is called a periodic signal, and a signal which does not repeat at regular intervals of time is called a non-periodic or aperiodic signal.

A discrete-time signal $x(n)$ is said to be periodic if it satisfies the condition $x(n) = x(n + N)$ for all integers n.

The smallest value of N which satisfies the above condition is known as fundamental period.

If the above condition is not satisfied even for one value of n, then the discrete-time signal is aperiodic. Sometimes aperiodic signals are said to have a period equal to infinity.

The angular frequency is given by

$$\omega = \frac{2\pi}{N}$$

$\therefore$ Fundamental period
$$N = \frac{2\pi}{\omega}$$

The sum of two discrete-time periodic sequences is always periodic.

Some examples of discrete-time periodic/non-periodic signals are shown in Figure 1.19.

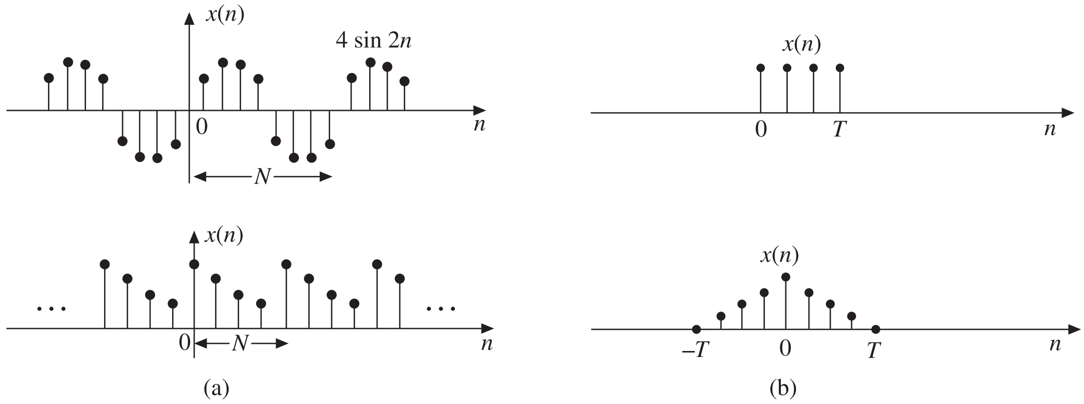


Figure 1.19 Examples of discrete-time: (a) Periodic and (b) Non-periodic signals.

EXAMPLE 1.4 Show that the complex exponential sequence $x(n) = e^{j\omega_0 n}$ is periodic only if $\omega_0/2\pi$ is a rational number.

Solution: Given
$$x(n) = e^{j\omega_0 n}$$

$x(n)$ will be periodic if
$$x(n + N) = x(n)$$

i.e.
$$e^{j[\omega_0(n+N)]} = e^{j\omega_0 n}$$

i.e.
$$e^{j\omega_0 N}\, e^{j\omega_0 n} = e^{j\omega_0 n}$$

This is possible only if
$$e^{j\omega_0 N} = 1$$

This is true only if
$$\omega_0 N = 2\pi k$$

where k is an integer.

$$\therefore \qquad \frac{\omega_0}{2\pi} = \frac{k}{N} \quad \text{Rational number}$$

This shows that the complex exponential sequence $x(n) = e^{j\omega_0 n}$ is periodic if $\omega_0/2\pi$ is a rational number.

EXAMPLE 1.5 Let $x(t)$ be the complex exponential signal, $x(t) = e^{j\omega_0 t}$ with radian frequency ω_0 and fundamental period $T = 2\pi/\omega_0$. Consider the discrete-time sequence $x(n)$ obtained by the uniform sampling of $x(t)$ with sampling interval T_s, i.e.,

$$x(n) = x(nT_s) = e^{jn\omega_0 T_s}$$

Show that $x(n)$ is periodic if the ratio of the sampling interval T_s to the fundamental period T of $x(t)$, i.e., T_s/T is a rational number.

Solution: Given

$$x(t) = e^{j\omega_0 t}$$

$$x(n) = x(nT_s) = e^{jn\omega_0 T_s}$$

where T_s is the sampling interval. Then,

$$\text{Fundamental period } T = \frac{2\pi}{\omega_0}$$

$$\omega_0 = \frac{2\pi}{T}$$

If $x(n)$ is periodic with the fundamental period N, then

$$x(n + N) = x(n)$$

i.e.

$$e^{j(n+N)\omega_0 T_s} = e^{jn\omega_0 T_s}$$

i.e.

$$e^{jn\omega_0 T_s} \quad e^{jN\omega_0 T_s} = e^{jn\omega_0 T_s}$$

This is true only if

$$e^{jN\omega_0 T_s} = 1$$

i.e.

$$N\omega_0 T_s = 2\pi m$$

where m is a positive integer.

i.e.

$$N\frac{2\pi}{T}T_s = 2\pi m$$

$\therefore$

$$\frac{T_s}{T} = \frac{m}{N} = \text{Rational number}$$

This shows that $x(n)$ is periodic if the ratio of the sampling interval to the fundamental period of $x(t)$, T_s/T is a rational number.

EXAMPLE 1.6 Obtain the condition for discrete-time sinusoidal signal to be periodic.

Solution: In case of continuous-time signals, all sinusoidal signals are periodic. But in discrete-time case, not all sinusoidal sequences are periodic.

Consider a discrete-time signal given by

$$x(n) = A \sin(\omega_0 n + \theta)$$

where A is amplitude, ω_0 is frequency and θ is phase shift.

A discrete-time signal is periodic if and only if

$$x(n) = x(n + N) \text{ for all } n$$

Now, $\quad x(n + N) = A \sin[\omega_0(n + N) + \theta] = A \sin(\omega_0 n + \theta + \omega_0 N)$

Therefore, $x(n)$ and $x(n + N)$ are equal if $\omega_0 N = 2\pi m$. That is, there must be an integer m such that

$$\omega_0 = \frac{2\pi m}{N} = 2\pi \left[\frac{m}{N}\right]$$

or

$$N = 2\pi \left[\frac{m}{\omega_0}\right]$$

From the above equation we find that, for the discrete-time signal to be periodic, the fundamental frequency ω_0 must be a rational multiple of 2π. Otherwise the discrete-time signal is aperiodic. The smallest value of positive integer N, for some integer m, is the fundamental period.

EXAMPLE 1.7 Determine whether the following discrete-time signals are periodic or not. If periodic, determine the fundamental period.

(a) $\sin(0.02\pi n)$ (b) $\sin(5\pi n)$

(c) $\cos 4n$ (d) $\sin \frac{2\pi n}{3} + \cos \frac{2\pi n}{5}$

(e) $\cos\left(\frac{n}{6}\right)\cos\left(\frac{n\pi}{6}\right)$ (f) $\cos\left(\frac{\pi}{2} + 0.3n\right)$

(g) $e^{j(\pi/2)n}$ (h) $1 + e^{j2\pi n/3} - e^{j4\pi n/7}$

Solution:

(a) Given $\quad x(n) = \sin(0.02\pi n)$

Comparing it with $\quad x(n) = \sin(2\pi f n)$

we have $\quad 0.02\pi = 2\pi f \quad$ or $\quad f = \frac{0.02\pi}{2\pi} = 0.01 = \frac{1}{100} = \frac{k}{N}$

Here f is expressed as a ratio of two integers with $k = 1$ and $N = 100$. So it is rational. Hence the given signal is periodic with fundamental period $N = 100$.

(b) Given $\quad x(n) = \sin(5\pi n)$

Comparing it with $\quad x(n) = \sin(2\pi f n)$

we have $\quad 2\pi f = 5\pi \quad$ or $\quad f = \frac{5}{2} = \frac{k}{N}$

Here f is a ratio of two integers with $k = 5$ and $N = 2$. Hence it is rational. Hence the given signal is periodic with fundamental period $N = 2$.

(c) Given $\quad x(n) = \cos 4n$

Comparing it with $\quad x(n) = \cos 2\pi fn$

we have $\quad 2\pi f = 4 \quad \text{or} \quad f = \dfrac{2}{\pi}$

Since $f = (2/\pi)$ is not a rational number, $x(n)$ is not periodic.

(d) Given $$x(n) = \sin \frac{2\pi n}{3} + \cos \frac{2\pi n}{5}$$

Comparing it with $\quad x(n) = \sin 2\pi f_1 n + \cos 2\pi f_2 n$

we have $\quad 2\pi f_1 = \dfrac{2\pi}{3} \quad \text{or} \quad f_1 = \dfrac{1}{3} = \dfrac{k_1}{N_1}$

$\therefore \quad N_1 = 3$

and $\quad 2\pi f_2 n = \dfrac{2\pi}{5} \quad \text{or} \quad f_2 = \dfrac{1}{5}$

$\therefore \quad N_2 = 5$

Since $\dfrac{N_1}{N_2} = \dfrac{3}{5}$ is a ratio of two integers, the sequence $x(n)$ is periodic. The period of $x(n)$ is the LCM of N_1 and N_2. Here LCM of $N_1 = 3$ and $N_2 = 5$ is 15. Therefore, the given sequence is periodic with fundamental period $N = 15$.

(e) Given $$x(n) = \cos\left(\frac{n}{6}\right) \cos\left(\frac{n\pi}{6}\right)$$

Comparing it with $\quad x(n) = \cos(2\pi f_1 n) \cos(2\pi f_2 n)$

we have $\quad 2\pi f_1 n = \dfrac{n}{6} \quad \text{or} \quad f_1 = \dfrac{1}{12\pi}$

which is not rational.

And $\quad 2\pi f_2 n = \dfrac{n\pi}{6} \quad \text{or} \quad f_2 = \dfrac{1}{12}$

which is rational.

Thus, $\cos(n/6)$ is non-periodic and $\cos(n\pi/6)$ is periodic. $x(n)$ is non-periodic because it is the product of periodic and non-periodic signals.

(f) Given $$x(n) = \cos\left(\frac{\pi}{2} + 0.3n\right)$$

Comparing it with $\quad x(n) = \cos(2\pi fn + \theta)$

we have $\quad 2\pi fn = 0.3n$ and phase shift $\theta = \dfrac{\pi}{2}$

$\therefore \qquad f = \dfrac{0.3}{2\pi} = \dfrac{3}{20\pi}$

which is not rational.

Hence, the signal $x(n)$ is non-periodic.

(g) Given $\qquad x(n) = e^{j(\pi/2)n}$

Comparing it with $\qquad x(n) = e^{j2\pi fn}$

we have $\qquad 2\pi f = \dfrac{\pi}{2} \quad \text{or} \quad f = \dfrac{1}{4} = \dfrac{k}{N}$

which is rational.

Hence, the given signal $x(n)$ is periodic with fundamental period $N = 4$.

(h) Given $\qquad x(n) = 1 + e^{j2\pi n/3} - e^{j4\pi n/7}$

Let $\qquad x(n) = 1 + e^{j2\pi n/3} - e^{j4\pi n/7} = x_1(n) + x_2(n) + x_3(n)$

where $\qquad x_1(n) = 1,\ x_2(n) = e^{j2\pi n/3} \quad \text{and} \quad x_3(n) = e^{j4\pi n/7}$

$x_1(n) = 1$ is a d.c. signal with an arbitrary period $N_1 = 1$

$$x_2(n) = e^{j2\pi n/3} = e^{j2\pi f_2 n}$$

$\therefore \qquad \dfrac{2\pi n}{3} = 2\pi f_2 n \quad \text{or} \quad f_2 = \dfrac{1}{3} = \dfrac{k_2}{N_2} \quad$ where $N_2 = 3$

Hence $x_2(n)$ is periodic with period $N_2 = 3$.

$$x_3(n) = e^{j4\pi n/7} = e^{j2\pi f_3 n}$$

$\therefore \qquad \dfrac{4\pi n}{7} = 2\pi f_3 n \quad \text{or} \quad f_3 = \dfrac{2}{7} = \dfrac{k_3}{N_3} \quad$ where $N_3 = \dfrac{7}{2}$

Now, $\qquad \dfrac{N_1}{N_2} = \dfrac{1}{3} = \text{Rational number}$

$$\frac{N_1}{N_3} = \frac{1}{7/2} = \frac{2}{7} = \text{Rational number}$$

The LCM of $\qquad N_1, N_2, N_3 = \dfrac{7}{2} \times 3 = \dfrac{21}{2}$

$\therefore$ The given signal $x(n)$ is periodic with fundamental period $N = 10.5$.

1.5.3 Energy and Power Signals

Signals may also be classified as energy signals and power signals. However there are some signals which can neither be classified as energy signals nor power signals.

The total energy E of a discrete-time signal $x(n)$ is defined as:

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

and the average power P of a discrete-time signal $x(n)$ is defined as:

$$P = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

or $P = \dfrac{1}{N}\displaystyle\sum_{n=0}^{n-1} |x(n)|^2$ for a digital signal with $x(n) = 0$ for $n < 0$.

A signal is said to be an energy signal if and only if its total energy E over the interval $(-\infty, \infty)$ is finite (i.e., $0 < E < \infty$). For an energy signal, average power $P = 0$. Non-periodic signals which are defined over a finite time (also called time limited signals) are the examples of energy signals. Since the energy of a periodic signal is always either zero or infinite, any periodic signal cannot be an energy signal.

A signal is said to be a power signal, if its average power P is finite (i.e., $0 < P < \infty$). For a power signal, total energy $E = \infty$. Periodic signals are the examples of power signals. Every bounded and periodic signal is a power signal. But it is true that a power signal is not necessarily a bounded and periodic signal.

Both energy and power signals are mutually exclusive, i.e. no signal can be both energy signal and power signal.

The signals that do not satisfy the above properties are neither energy signals nor power signals. For example, $x(n) = u(n)$, $x(n) = nu(n)$, $x(n) = n^2u(n)$.

These are signals for which neither P nor E are finite. If the signals contain infinite energy and zero power or infinite energy and infinite power, they are neither energy nor power signals.

If the signal amplitude becomes zero as $|n| \to \infty$, it is an energy signal, and if the signal amplitude does not become zero as $|n| \to \infty$, it is a power signal.

EXAMPLE 1.8 Find which of the following signals are energy signals, power signals, neither energy nor power signals:

(a) $\left(\dfrac{1}{2}\right)^n u(n)$ (b) $e^{j[(\pi/3)n+(\pi/2)]}$

(c) $\sin\left(\dfrac{\pi}{3}n\right)$ (d) $u(n) - u(n-6)$

(e) $nu(n)$ (f) $r(n) - r(n-4)$

Solution:

(a) Given $x(n) = \left(\dfrac{1}{2}\right)^n u(n)$

Energy of the signal $E = \underset{N\to\infty}{\text{Lt}} \sum_{n=-N}^{N} |x(n)|^2$

$$= \underset{N\to\infty}{\text{Lt}} \sum_{n=-N}^{N} \left[\left(\frac{1}{2}\right)^n\right]^2 u(n)$$

$$= \underset{N\to\infty}{\text{Lt}} \sum_{n=0}^{N} \left(\frac{1}{4}\right)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{1-(1/4)} = \frac{4}{3} \text{ joule}$$

Power of the signal $P = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=0}^{N} \left(\frac{1}{4}\right)^n$$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \left[\frac{1-(1/4)^{N+1}}{1-(1/4)}\right]$$

$$= 0$$

The energy is finite and power is zero. Therefore, $x(n)$ is an energy signal.

(b) Given $x(n) = e^{j[(\pi/3)n + (\pi/2)]}$

Energy of the signal $E = \underset{N\to\infty}{\text{Lt}} \sum_{n=-N}^{N} \left|e^{j[(\pi/3)n + (\pi/2)]}\right|^2$

$$= \underset{N\to\infty}{\text{Lt}} \sum_{n=-N}^{N} 1$$

$$= \underset{N\to\infty}{\text{Lt}} [2N+1] = \infty$$

Power of the signal $P = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} \left|e^{j[(\pi/3)n + (\pi/2)]}\right|^2$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} 1$$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} [2N+1] = 1 \text{ watt}$$

The energy is infinite and the power is finite. Therefore, it is a power signal.

(c) Given $$x(n) = \sin\left(\frac{\pi}{3}n\right)$$

Energy of the signal $$E = \underset{N\to\infty}{\text{Lt}} \sum_{n=-N}^{N} \sin^2\left(\frac{\pi}{3}n\right)$$

$$= \underset{N\to\infty}{\text{Lt}} \sum_{n=-N}^{N} \frac{1-\cos[(2\pi/3)n]}{2}$$

$$= \frac{1}{2} \underset{N\to\infty}{\text{Lt}} \sum_{n=-N}^{N} \left(1-\cos\frac{2\pi}{3}n\right)$$

$$= \infty$$

Power of the signal $$P = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} \sin^2\left(\frac{\pi}{3}n\right)$$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1-\cos[(2\pi/3)n]}{2}$$

$$= \frac{1}{2} \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1}[2N+1] = \frac{1}{2} \text{ watt}$$

The energy is infinite and power is finite. Therefore, it is a power signal.

(d) Given $$x(n) = u(n) - u(n-6)$$

Energy of the signal $$E = \underset{N\to\infty}{\text{Lt}} \sum_{n=-N}^{N} [u(n)-u(n-6)]^2$$

$$= \underset{N\to\infty}{\text{Lt}} \sum_{n=0}^{5} 1 = 6 \text{ joule}$$

Power of the signal $$P = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} [u(n)-u(n-6)]^2$$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=0}^{5} 1 = 0$$

Energy is finite and power is zero. Therefore, it is an energy signal.

(e) Given $$x(n) = nu(n)$$

Energy of the signal $$E = \underset{N\to\infty}{\text{Lt}} \sum_{n=-N}^{N} [n]^2\, u(n)$$

$$= \underset{N\to\infty}{\text{Lt}} \sum_{n=0}^{N} [n^2]$$

$$= \infty$$

$$\text{Power of the signal } P = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} [n]^2 u(n)$$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=0}^{N} n^2$$

$$= \infty$$

Energy is infinite and power is also infinite. Therefore, it is neither energy signal nor power signal.

(f) Given $x(n) = r(n) - r(n-4)$

$$\text{Energy of the signal } E = \underset{N\to\infty}{\text{Lt}} \sum_{n=-N}^{N} [r(n) - r(n-4)]^2$$

$$= \underset{N\to\infty}{\text{Lt}} \left[\sum_{n=0}^{4} n^2 + \sum_{n=5}^{N} (4)^2 \right]$$

$$= \infty$$

$$\text{Power of the signal } P = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} [r(n) - r(n-4)]^2$$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \left[\sum_{n=0}^{4} n^2 + \sum_{n=5}^{N} (4)^2 \right]$$

$$= 8 \text{ watt}$$

Energy is infinite and power is finite. Therefore, it is a power signal.

EXAMPLE 1.9 Find whether the signal

$$x(n) = \begin{cases} n^2 & 0 \le n \le 3 \\ 10 - n & 4 \le n \le 6 \\ n & 7 \le n \le 9 \\ 0 & \text{otherwise} \end{cases}$$

is a power signal or an energy signal. Also find the energy and power of the signal.

Solution: The given signal is a non-periodic finite duration signal. So it has finite energy and zero average power. So it is an energy signal.

$$\text{Energy of the signal } E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=0}^{3} (n^2)^2 + \sum_{n=4}^{6} (10-n)^2 + \sum_{n=7}^{9} (n)^2$$

$$= \sum_{n=0}^{3} n^4 + \sum_{n=4}^{6} (100 + n^2 - 20n) + \sum_{n=7}^{9} n^2$$

$$= (0+1+16+81) + (36+25+16) + (49+64+81)$$

$$= 369 < \infty \text{ joule}$$

$$\text{Power of the signal } P = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \left[\sum_{n=0}^{3} (n^2)^2 + \sum_{n=4}^{6} (10-n)^2 + \sum_{n=7}^{9} (n)^2 \right]$$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} [369] = 0$$

Here energy is finite and power is zero. So it is an energy signal.

1.5.4 Causal and Non-causal Signals

A discrete-time signal $x(n)$ is said to be causal if $x(n) = 0$ for $n < 0$, otherwise the signal is non-causal. A discrete-time signal $x(n)$ is said to be anti-causal if $x(n) = 0$ for $n > 0$.

A causal signal does not exist for negative time and an anti-causal signal does not exist for positive time. A signal which exists in positive as well as negative time is called a non-casual signal.

$u(n)$ is a causal signal and $u(-n)$ an anti-causal signal, whereas $x(n) = 1$ for $-2 \le n \le 3$ is a non-causal signal.

EXAMPLE 1.10 Find which of the following signals are causal or non-causal.

(a) $x(n) = u(n+4) - u(n-2)$ (b) $x(n) = \left(\frac{1}{4}\right)^n u(n+2) - \left(\frac{1}{2}\right)^n u(n-4)$

(c) $x(n) = u(-n)$

Solution:

(a) Given $x(n) = u(n+4) - u(n-2)$
The given signal exists from $n = -4$ to $n = 1$. Since $x(n) \ne 0$ for $n < 0$, it is non-causal.

(b) Given $x(n) = \left(\frac{1}{4}\right)^n u(n+2) - \left(\frac{1}{2}\right)^n u(n-4)$

The given signal exists for $n < 0$ also. So it is non-causal.

(c) Given $x(n) = u(-n)$

The given signal exists only for $n < 0$. So it is anti-causal. It can be called non-causal also.

1.5.5 Even and Odd Signals

Any signal $x(n)$ can be expressed as sum of even and odd components. That is

$$x(n) = x_e(n) + x_o(n)$$

where $x_e(n)$ is even components and $x_o(n)$ is odd components of the signal.

Even (symmetric) signal

A discrete-time signal $x(n)$ is said to be an even (symmetric) signal if it satisfies the condition:

$$x(n) = x(-n) \quad \text{for all } n$$

Even signals are symmetrical about the vertical axis or time origin. Hence they are also called symmetric signals: cosine sequence is an example of an even signal. Some even signals are shown in Figure 1.20(a). An even signal is identical to its reflection about the origin. For an even signal $x_0(n) = 0$.

Odd (anti-symmetric) signal

A discrete-time signal $x(n)$ is said to be an odd (anti-symmetric) signal if it satisfies the condition:

$$x(-n) = -x(n) \quad \text{for all } n$$

Odd signals are anti-symmetrical about the vertical axis. Hence they are called anti-symmetric signals. Sinusoidal sequence is an example of an odd signal. For an odd signal $x_e(n) = 0$. Some odd signals are shown in Figure 1.20(b).

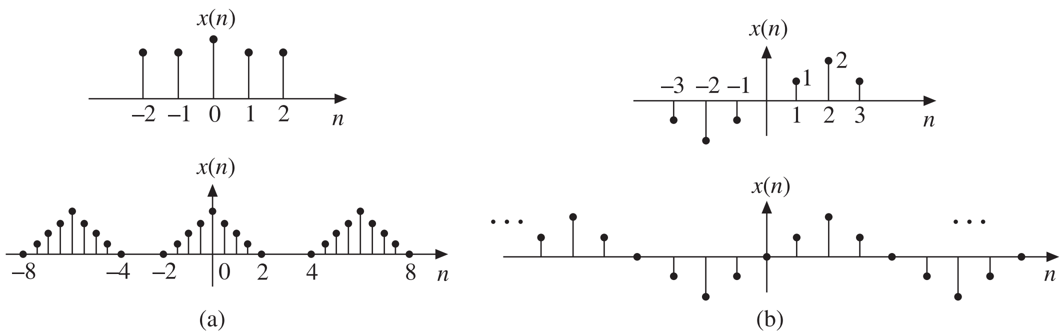


Figure 1.20 (a) Even sequences (b) Odd sequences.

Evaluation of even and odd parts of a signal

We have $$x(n) = x_e(n) + x_o(n)$$

$$\therefore \quad x(-n) = x_e(-n) + x_o(-n) = x_e(n) - x_o(n)$$

$$x(n) + x(-n) = x_e(n) + x_o(n) + x_e(n) - x_o(n) = 2x_e(n)$$

$$\therefore \quad x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

$$x(n) - x(-n) = [x_e(n) + x_o(n)] - [x_e(n) - x_o(n)] = 2x_o(n)$$

$$\therefore \quad x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

The product of two even or odd signals is an even signal and the product of even signal and odd signal is an odd signal.

We can prove this as follows:

Let $$x(n) = x_1(n)\, x_2(n)$$

(a) If $x_1(n)$ and $x_2(n)$ are both even, i.e.

$$x_1(-n) = x_1(n)$$

and $$x_2(-n) = x_2(n)$$

Then $$x(-n) = x_1(-n)x_2(-n) = x_1(n)x_2(n) = x(n)$$

Therefore, $x(n)$ is an even signal.

If $x_1(n)$ and $x_2(n)$ are both odd, i.e.

$$x_1(-n) = -x_1(n)$$

and $$x_2(-n) = -x_2(n)$$

Then $$x(-n) = x_1(-n)x_2(-n) = [-x_1(n)][-x_2(n)] = x_1(n)x_2(n) = x(n)$$

Therefore, $x(n)$ is an even signal.

(b) If $x_1(n)$ is even and $x_2(n)$ is odd, i.e.

$$x_1(-n) = x_1(n)$$

and $$x_2(-n) = -x_2(n)$$

Then $$x(-n) = x_1(-n)x_2(-n) = -x_1(n)x_2(n) = -x(n)$$

Therefore, $x(n)$ is an odd signal.

Thus, the product of two even signals or of two odd signals is an even signal, and the product of even and odd signals is an odd signal.

Every signal need not be either purely even signal or purely odd signal, but every signal can be decomposed into sum of even and odd parts.

EXAMPLE 1.11 Find the even and odd components of the following signals:

(a) $x(n) = \{-3, 1, \underset{\uparrow}{2}, -4, 2\}$ (b) $x(n) = \{-2, 5, \underset{\uparrow}{1}, -3\}$

(c) $x(n) = \{\underset{\uparrow}{5}, 4, 3, 2, 1\}$ (d) $x(n) = \{5, 4, 3, 2, \underset{\uparrow}{1}\}$

Solution:

(a) Given $x(n) = \{-3, 1, \underset{\uparrow}{2}, -4, 2\}$

$\therefore$ $x(-n) = \{2, -4, \underset{\uparrow}{2}, 1, -3\}$

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

$$= \frac{1}{2}[-3 + 2, 1 -4, 2 + 2, -4 + 1, 2 -3]$$

$$= \{-0.5, -1.5, \underset{\uparrow}{2}, -1.5, -0.5\}$$

$$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

$$= \frac{1}{2}[-3 - 2, 1 + 4, 2 - 2, -4 - 1, 2 + 3]$$

$$= \{-2.5, 2.5, \underset{\uparrow}{0}, -2.5, 2.5\}$$

(b) Given $x(n) = \{-2, 5, \underset{\uparrow}{1}, -3\}$

$$x(-n) = \{-3, \underset{\uparrow}{1}, 5, -2\}$$

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

$$= \frac{1}{2}[-2 + 0, 5 - 3, 1 + 1, -3 + 5, 0 - 2]$$

$$= \{-1, 1, \underset{\uparrow}{1}, 1, -1\}$$

$$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

$$= \frac{1}{2}[-2 - 0, 5 + 3, 1 - 1, -3 - 5, 0 + 2]$$

$$= \{-1, 4, \underset{\uparrow}{0}, - 4, 1\}$$

(c) Given $x(n) = \left\{\underset{\uparrow}{5}, 4, 3, 2, 1\right\}$

$n = 0, 1, 2, 3, 4$

$\therefore \quad x(n) = \underset{\uparrow}{5}, 4, 3, 2, 1$

$x(-n) = 1, 2, 3, 4, \underset{\uparrow}{5}$

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

$$= \frac{1}{2}[1, 2, 3, 4, 5 \underset{\uparrow}{+} 5, 4, 3, 2, 1]$$

$$= \left\{0.5, 1, 1.5, 2, \underset{\uparrow}{5}, 2, 1.5, 1, 0.5\right\}$$

$$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

$$= \frac{1}{2}[-1, -2, -3, -4, 5 \underset{\uparrow}{-} 5, 4, 3, 2, 1]$$

$$= \left\{-0.5, -1, -1.5, -2, \underset{\uparrow}{0}, 2, 1.5, 1, 0.5\right\}$$

(d) Given $x(n) = \left\{5, 4, 3, 2, \underset{\uparrow}{1}\right\}$

$n = -4, -3, -2, -1, 0$

$\therefore \quad x(n) = 5, 4, 3, 2, \underset{\uparrow}{1}$

$x(-n) = \underset{\uparrow}{1}, 2, 3, 4, 5$

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

$$= \frac{1}{2}[5, 4, 3, 2, 1 \underset{\uparrow}{+} 1, 2, 3, 4, 5]$$

$$= [2.5, 2, 1.5, 1, \underset{\uparrow}{1}, 1, 1.5, 2, 2.5]$$

$$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

$$= \frac{1}{2}[5, 4, 3, 2, 1 \underset{\uparrow}{-} 1, -2, -3, -4, -5]$$

$$= \frac{1}{2}[2.5, 2, 1.5, 1, \underset{\uparrow}{0}, -1, -1.5, -2, -2.5]$$

When the signal is given as a waveform

The even part of the signal can be found by folding the signal about the y-axis and adding the folded signal to the original signal and dividing the sum by two. The odd part of the signal can be found by folding the signal about y-axis and subtracting the folded signal from the original signal and dividing the difference by two as illustrated in Figure 1.21.

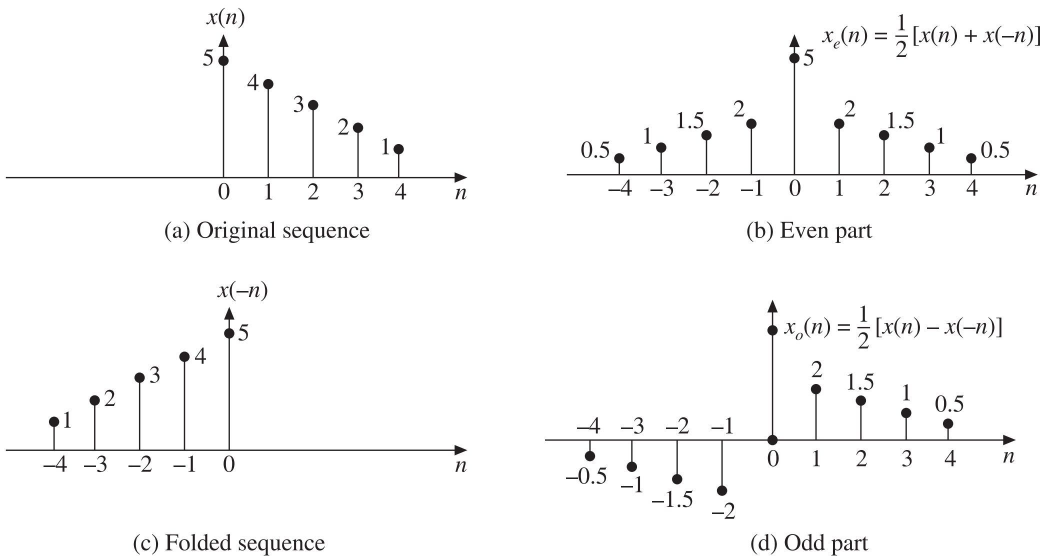


Figure 1.21 Graphical evaluation of even and odd parts.

1.6 CLASSIFICATION OF DISCRETE-TIME SYSTEMS

A system is defined as an entity that acts on an input signal and transforms it into an output signal. A system may also be defined as a set of elements or functional blocks which are connected together and produces an output in response to an input signal. The response or output of the system depends on the transfer function of the system. It is a cause and effect relation between two or more signals.

As signals, systems are also broadly classified into continuous-time and discrete-time systems. A continuous-time system is one which transforms continuous-time input signals into continuous-time output signals, whereas a discrete-time system is one which transforms discrete-time input signals into discrete-time output signals.

For example microprocessors, semiconductor memories, shift registers, etc. are discrete-time systems.

A discrete-time system is represented by a block diagram as shown in Figure 1.22. An arrow entering the box is the input signal (also called excitation, source or driving function) and an arrow leaving the box is an output signal (also called response). Generally, the input is denoted by $x(n)$ and the output is denoted by $y(n)$.

The relation between the input $x(n)$ and the output $y(n)$ of a system has the form:

$$y(n) = \text{Operation on } x(n)$$

Mathematically,

$$y(n) = T[x(n)]$$

which represents that $x(n)$ is transformed to $y(n)$. In other words, $y(n)$ is the transformed version of $x(n)$.

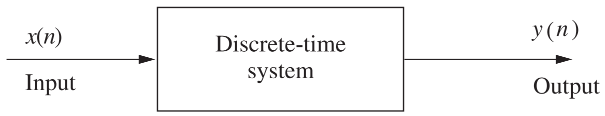


Figure 1.22 Block diagram of discrete-time system.

Both continuous-time and discrete-time systems are further classified as follows:

1. Static (memoryless) and dynamic (memory) systems
2. Causal and non-causal systems
3. Linear and non-linear systems
4. Time-invariant and time varying systems
5. Stable and unstable systems.
6. Invertible and non-invertible systems
7. FIR and IIR systems

1.6.1 Static and Dynamic Systems

A system is said to be static or memoryless if the response is due to present input alone, i.e., for a static or memoryless system, the output at any instant n depends only on the input applied at that instant n but not on the past or future values of input or past values of output.

For example, the systems defined below are static or memoryless systems.

$$y(n) = x(n)$$

$$y(n) = 2x^2(n)$$

In contrast, a system is said to be dynamic or memory system if the response depends upon past or future inputs or past outputs. A summer or accumulator, a delay element is a discrete-time system with memory.

For example, the systems defined below are dynamic or memory systems.

$$y(n) = x(2n)$$

$$y(n) = x(n) + x(n - 2)$$

$$y(n) + 4y(n - 1) + 4y(n - 2) = x(n)$$

Any discrete-time system described by a difference equation is a dynamic system.

A purely resistive electrical circuit is a static system, whereas an electric circuit having inductors and/or capacitors is a dynamic system.

A discrete-time LTI system is memoryless (static) if its impulse response $h(n)$ is zero for $n \neq 0$. If the impulse response is not identically zero for $n \neq 0$, then the system is called dynamic system or system with memory.

EXAMPLE 1.12 Find whether the following systems are dynamic or not:

(a) $y(n) = x(n + 2)$ (b) $y(n) = x^2(n)$

(c) $y(n) = x(n - 2) + x(n)$

Solution:

(a) Given $$y(n) = x(n + 2)$$

The output depends on the future value of input. Therefore, the system is dynamic.

(b) Given $$y(n) = x^2(n)$$

The output depends on the present value of input alone. Therefore, the system is static.

(c) Given $$y(n) = x(n - 2) + x(n)$$

The system is described by a difference equation. Therefore, the system is dynamic.

1.6.2 Causal and Non-causal Systems

A system is said to be causal (or non-anticipative) if the output of the system at any instant n depends only on the present and past values of the input but not on future inputs, i.e., for a causal system, the impulse response or output does not begin before the input function is applied, i.e., a causal system is non anticipatory.

Causal systems are real time systems. They are physically realizable.

The impulse response of a causal system is zero for $n < 0$, since $\delta(n)$ exists only at $n = 0$,

i.e. $$h(n) = 0 \quad \text{for} \quad n < 0$$

The examples for causal systems are:

$$y(n) = nx(n)$$

$$y(n) = x(n - 2) + x(n - 1) + x(n)$$

A system is said to be non-causal (anticipative) if the output of the system at any instant n depends on future inputs. They are anticipatory systems. They produce an output even before the input is given. They do not exist in real time. They are not physically realizable.

A delay element is a causal system, whereas an image processing system is a non-causal system.

The examples for non-causal systems are:

$$y(n) = x(n) + x(2n)$$

$$y(n) = x^2(n) + 2x(n + 2)$$

EXAMPLE 1.13 Check whether the following systems are causal or not:

(a) $y(n) = x(n) + x(n-2)$ (b) $y(n) = x(2n)$

(c) $y(n) = \sin[x(n)]$ (d) $y(n) = x(-n)$

Solution:

(a) Given $y(n) = x(n) + x(n-2)$

For $n = -2$ $y(-2) = x(-2) + x(-4)$

For $n = 0$ $y(0) = x(0) + x(-2)$

For $n = 2$ $y(2) = x(2) + x(0)$

For all values of n, the output depends only on the present and past inputs. Therefore, the system is causal.

(b) Given $y(n) = x(2n)$

For $n = -2$ $y(-2) = x(-4)$

For $n = 0$ $y(0) = x(0)$

For $n = 2$ $y(2) = x(4)$

For positive values of n, the output depends on the future values of input. Therefore, the system is non-causal.

(c) Given $y(n) = \sin[x(n)]$

For $n = -2$ $y(-2) = \sin[x(-2)]$

For $n = 0$ $y(0) = \sin[x(0)]$

For $n = 2$ $y(2) = \sin[x(2)]$

For all values of n, the output depends only on the present value of input. Therefore, the system is causal.

(d) Given $y(n) = x(-n)$

For $n = -2$ $y(-2) = x(2)$

For $n = 0$ $y(0) = x(0)$

For $n = 2$ $y(2) = x(-2)$

For negative values of n, the output depends on the future values of input. Therefore, the system is non-causal.

1.6.3 Linear and Non-linear Systems

A system which obeys the principle of superposition and principle of homogeneity is called a linear system and a system which does not obey the principle of superposition and homogeneity is called a non-linear system.

Homogeneity property means a system which produces an output $y(n)$ for an input $x(n)$ must produce an output $ay(n)$ for an input $ax(n)$.

Superposition property means a system which produces an output $y_1(n)$ for an input $x_1(n)$ and an output $y_2(n)$ for an input $x_2(n)$ must produce an output $y_1(n) + y_2(n)$ for an input $x_1(n) + x_2(n)$.

Combining them we can say that a system is linear if an arbitrary input $x_1(n)$ produces an output $y_1(n)$ and an arbitrary input $x_2(n)$ produces an output $y_2(n)$, then the weighted sum of inputs $ax_1(n) + bx_2(n)$ where a and b are constants produces an output $ay_1(n) + by_2(n)$ which is the sum of weighted outputs.

$$T(ax_1(n) + bx_2(n)] = aT[x_1(n)] + bT[x_2(n)]$$

Simply we can say that a system is linear if the output due to weighted sum of inputs is equal to the weighted sum of outputs.

In general, if the describing equation contains square or higher order terms of input and/or output and/or product of input/output and its difference or a constant, the system will definitely be non-linear.

Few examples of linear systems are filters, communication channels etc.

EXAMPLE 1.14 Check whether the following systems are linear or not:

(a) $y(n) = n^2 x(n)$ (b) $y(n) = x(n) + \dfrac{1}{2x(n-2)}$

(c) $y(n) = 2x(n) + 4$ (d) $y(n) = x(n) \cos \omega n$

(e) $y(n) = |x(n)|$ (f) $y(n) = \dfrac{1}{N} \sum_{k=0}^{N-1} x(n-k)$

Solution:

(a) Given $y(n) = n^2 x(n)$

$$y(n) = T[x(n)] = n^2 x(n)$$

Let an input $x_1(n)$ produce an output $y_1(n)$.

$$\therefore \qquad y_1(n) = T[x_1(n)] = n^2 x_1(n)$$

Let an input $x_2(n)$ produce an output $y_2(n)$.

$$\therefore \qquad y_2(n) = T[x_2(n)] = n^2 x_2(n)$$

The weighted sum of outputs is:

$$ay_1(n) + by_2(n) = a[n^2 x_1(n)] + b[n^2 x_2(n)] = n^2 [ax_1(n) + bx_2(n)]$$

The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n) + bx_2(n)] = n^2 [ax_1(n) + bx_2(n)]$$

$$y_3(n) = ay_1(n) + by_2(n)$$

The weighted sum of outputs is equal to the output due to weighted sum of inputs. The superposition principle is satisfied. Therefore, the given system is linear.

(b) Given $$y(n) = x(n) + \frac{1}{2x(n-2)}$$

$$y(n) = T[x(n)] = x(n) + \frac{1}{2x(n-2)}$$

For an input $x_1(n)$,

$$y_1(n) = T[x_1(n)] = x_1(n) + \frac{1}{2x_1(n-2)}$$

For an input $x_2(n)$,

$$y_2(n) = T[x_2(n)] = x_2(n) + \frac{1}{2x_2(n-2)}$$

The weighted sum of outputs is:

$$ay_1(n) + by_2(n) = a\left[x_1(n) + \frac{1}{2x_1(n-2)}\right] + b\left[x_2(n) + \frac{1}{2x_2(n-2)}\right]$$

$$= [ax_1(n) + bx_2(n)] + \frac{a}{2x_1(n-2)} + \frac{b}{2x_2(n-2)}$$

The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n) + bx_2(n)] = [ax_1(n) + bx_2(n)] + \frac{1}{2[ax_1(n-2) + bx_2(n-2)]}$$

$$y_3(n) \neq ay_1(n) + by_2(n)$$

The weighted sum of outputs is not equal to the output due to weighted sum of inputs. The superposition principle is not satisfied. Therefore, the given system is non-linear.

(c) Given $$y(n) = 2x(n) + 4$$

$$y(n) = T[x(n)] = 2x(n) + 4$$

For an input $x_1(n)$,

$$y_1(n) = T[x_1(n)] = 2x_1(n) + 4$$

For an input $x_2(n)$,

$$y_2(n) = T[x_2(n)] = 2x_2(n) + 4$$

The weighted sum of outputs is:

$$ay_1(n) + by_2(n) = a[2x_1(n) + 4] + b[2x_2(n) + 4] = 2[ax_1(n) + bx_2(n)] + 4(a + b)$$

The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n) + bx_2(n)] = 2[ax_1(n) + bx_2(n)] + 4$$

$$y_3(n) \neq ay_1(n) + by_2(n)$$

The weighted sum of outputs is not equal to the output due to weighted sum of inputs. The superposition principle is not satisfied. Therefore, the given system is non-linear.

(d) Given $$y(n) = x(n) \cos \omega n$$

$$y(n) = T[x(n)] = x(n) \cos \omega n$$

For an input $x_1(n)$,

$$y_1(n) = T[x_1(n)] = x_1(n) \cos \omega n$$

For an input $x_2(n)$,

$$y_2(n) = T[x_2(n)] = x_2(n) \cos \omega n$$

The weighted sum of outputs is:

$$ay_1(n) + by_2(n) = ax_1(n) \cos \omega n + bx_2(n) \cos \omega n = [ax_1(n) + bx_2(n)] \cos \omega n$$

The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n) + bx_2(n)] = [ax_1(n) + bx_2(n)] \cos \omega n$$

$$y_3(n) = ay_1(n) + by_2(n)$$

The weighted sum of outputs is equal to the output due to weighted sum of inputs. The superposition principle is satisfied. Therefore, the given system is linear.

(e) Given $$y(n) = |x(n)|$$

$$y(n) = T[x(n)] = |x(n)|$$

For an input $x_1(n)$,

$$y_1(n) = T[x_1(n)] = |x_1(n)|$$

For an input $x_2(n)$,

$$y_2(n) = T[x_2(n)] = |x_2(n)|$$

The weighted sum of outputs is:

$$ay_1(n) + by_2(n) = a|x_1(n)| + b|x_2(n)|$$

The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n) + bx_2(n)] = |ax_1(n) + bx_2(n)|$$

$$y_3(n) \neq ay_1(n) + by_2(n)$$

The weighted sum of outputs is not equal to the output due to weighted sum of inputs. The superposition principle is not satisfied. Therefore, the given system is non-linear.

(f) Given $y(n) = \dfrac{1}{N} \sum_{k=0}^{N-1} x(n-k)$

$$y(n) = T[x(n)] = \frac{1}{N} \sum_{k=0}^{N-1} x(n-k)$$

For an input $x_1(n)$,

$$y_1(n) = T[x_1(n)] = \frac{1}{N}\sum_{k=0}^{N-1} x_1(n-k)$$

For an input $x_2(n)$,

$$y_2(n) = T[x_2(n)] = \frac{1}{N}\sum_{k=0}^{N-1} x_2(n-k)$$

The weighted sum of outputs is:

$$ay_1(n) + by_2(n) = a\frac{1}{N}\sum_{k=0}^{N-1} x_1(n-k) + b\frac{1}{N}\sum_{k=0}^{N-1} x_2(n-k)$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} [ax_1(n-k) + bx_2(n-k)]$$

The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n) + bx_2(n)] = \frac{1}{N}\sum_{k=0}^{N-1} [ax_1(n-k) + bx_2(n-k)]$$

$$y_3(n) = ay_1(n) + by_2(n)$$

The weighted sum of outputs is equal to the output due to weighted sum of inputs. The superposition principle is satisfied. Therefore, the given system is linear.

1.6.4 Shift-invariant and Shift-varying Systems

Time-invariance is the property of a system which makes the behaviour of the system independent of time. This means that the behaviour of the system does not depend on the time at which the input is applied. For discrete-time systems, the time invariance property is called shift invariance.

A system is said to be shift-invariant if its input/output characteristics do not change with time, i.e., if a time shift in the input results in a corresponding time shift in the output as shown in Figure 1.23, i.e.

If $\quad T[x(n)] = y(n)$

Then $\quad T[x(n-k)] = y(n-k)$

A system not satisfying the above requirements is called a time-varying system (or shift-varying system). A time-invariant system is also called a fixed system.

The time-invariance property of the given discrete-time system can be tested as follows:

Let $x(n)$ be the input and let $x(n-k)$ be the input delayed by k units.

$y(n) = T[x(n)]$ be the output for the input $x(n)$.

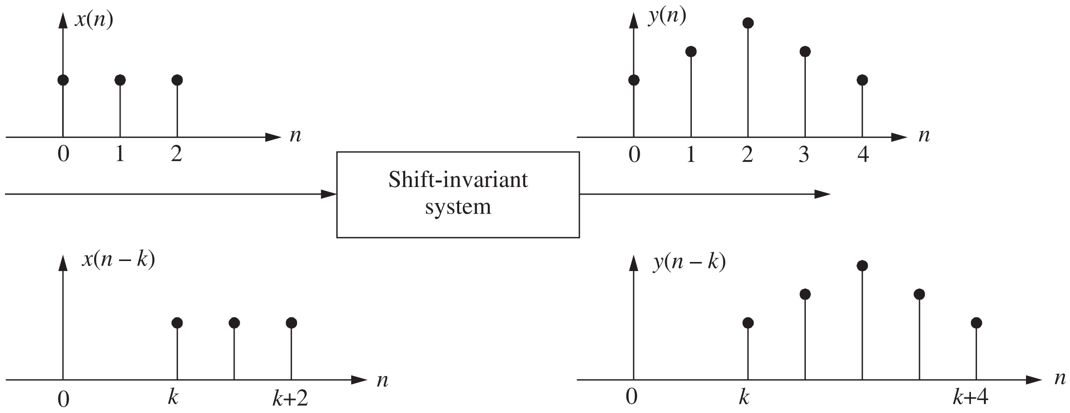


Figure 1.23 Time-invariant system.

$y(n, k) = T[x(n-k)] = y(n)\Big|_{x(n)=x(n-k)}$ be the output for the delayed input $x(n-k)$.

$y(n-k) = y(n)\Big|_{n=n-k}$ be the output delayed by k units.

If $$y(n, k) = y(n-k)$$

i.e. if delayed output is equal to the output due to delayed input for all possible values of k, then the system is time-invariant.

On the other hand, if

$$y(n, k) \neq y(n-k)$$

i.e. if the delayed output is not equal to the output due to delayed input, then the system is time-variant.

If the discrete-time system is described by difference equation, the time invariance can be found by observing the coefficients of the difference equation.

If all the coefficients of the difference equation are constants, then the system is time-invariant. If even one of the coefficient is function of time, then the system is time-variant.

The system described by

$$y(n) + 3y(n-1) + 5y(n-2) = 2x(n)$$

is time-invariant system because all the coefficients are constants.

The system described by

$$y(n) - 2ny(n-1) + 3n^2y(n-2) = x(n) + x(n-1)$$

is time-varying system because all the coefficients are not constant (Two are functions of time).

The systems satisfying both linearity and time-invariant conditions are called **linear, time-invariant** systems, or simply **LTI** systems.

EXAMPLE 1.15 Determine whether the following systems are time-invariant or not:

(a) $y(n) = x(n/2)$ (b) $y(n) = x(n)$

(c) $y(n) = x^2(n-2)$ (d) $y(n) = x(n) + nx(n-2)$

Solution:

(a) Given $$y(n) = x\left(\frac{n}{2}\right)$$

$$y(n) = T[x(n)] = x\left(\frac{n}{2}\right)$$

The output due to input delayed by k units is:

$$y(n, k) = T[x(n-k)] = y(n)\big|_{x(n)=x(n-k)} = x\left(\frac{n}{2} - k\right)$$

The output delayed by k units is:

$$y(n-k) = y(n)\big|_{n=n-k} = x\left(\frac{n-k}{2}\right)$$

$$y(n, k) \neq y(n-k)$$

i.e. the delayed output is not equal to the output due to delayed input. Therefore, the system is time-variant.

(b) Given $$y(n) = x(n)$$

$$y(n) = T[x(n)] = x(n)$$

The output due to input delayed by k units is:

$$y(n, k) = T[x(n-k)] = y(n)\big|_{x(n)=x(n-k)} = x(n-k)$$

The output delayed by k units is:

$$y(n-k) = y(n)\big|_{n=n-k} = x(n-k)$$

$$y(n, k) = y(n-k)$$

i.e. the delayed output is equal to the output due to delayed input. Therefore, the system is time-invariant.

(c) Given $$y(n) = x^2(n-2)$$

$$y(n) = T[x(n)] = x^2(n-2)$$

The output due to input delayed by k units is:

$$y(n, k) = T[x(n-k)] = y(n)\big|_{x(n)=x(n-k)} = x^2(n-2-k)$$

The output delayed by k units is:

$$y(n-k) = y(n)\big|_{n=n-k} = x^2(n-2-k)$$

$$y(n, k) = y(n-k)$$

i.e. the delayed output is equal to the output due to delayed input. Therefore, the system is time-invariant.

(d) Given $$y(n) = x(n) + nx(n-2)$$

$$y(n) = T[x(n)] = x(n) + nx(n-2)$$

The output due to input delayed by k units is:

$$y(n, k) = T[x(n-k)] = y(n)\big|_{x(n)=x(n-k)} = x(n-k) + nx(n-2-k)$$

The output delayed by k units is:

$$y(n-k) = y(n)\big|_{n=n-k} = x(n-k) + (n-k)\,x(n-k-2)$$

$$y(n, k) \neq y(n-k)$$

i.e. the delayed output is not equal to the output due to delayed input. Therefore, the system is time-variant.

EXAMPLE 1.16 Show that the following systems are linear shift-invariant systems:

(a) $y(n) = x\left(\dfrac{n}{2}\right)$ (b) $y(n) = \begin{cases} x(n) + x(n-2) & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$

Solution: To show that a given system is a linear time-invariant system we have to show separately that it is linear and time-invariant.

(a) Given $$y(n) = x\left(\frac{n}{2}\right)$$

For inputs $x_1(n)$ and $x_2(n)$,

$$y_1(n) = x_1\left(\frac{n}{2}\right)$$

$$y_2(n) = x_2\left(\frac{n}{2}\right)$$

The weighted sum of outputs is:

$$ay_1(n) + by_2(n) = ax_1\left(\frac{n}{2}\right) + bx_2\left(\frac{n}{2}\right)$$

The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n) + bx_2(n)] = ax_1\left(\frac{n}{2}\right) + bx_2\left(\frac{n}{2}\right)$$

$$y_3(n) = ay_1(n) + by_2(n)$$

So the system is linear.

$$y(n, k) = y(n)\big|_{x(n)=x(n-k)} = x\left(\frac{n}{2} - k\right)$$

$$y(n-k) = y(n)\big|_{n=n-k} = x\left(\frac{n-k}{2}\right)$$

$$y(n, k) \neq y(n-k)$$

So the system is shift-varying.
Hence the given system is linear but shift-varying. It is not a linear shift-invariant system.

(b) Given $$y(n) = \begin{cases} x(n) + x(n-2) & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

For inputs $x_1(n)$ and $x_2(n)$,

$$y_1(n) = x_1(n) + x_1(n-2) \quad \text{for } n \geq 0$$

$$y_2(n) = x_2(n) + x_2(n-2) \quad \text{for } n \geq 0$$

The weighted sum of outputs is:

$$ay_1(n) + by_2(n) = a[x_1(n) + x_1(n-2)] + b[x_2(n) + x_2(n-2)]$$

The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n) + bx_2(n)] = [ax_1(n) + bx_2(n)] + ax_1(n-2) + bx_2(n-2)$$

$$y_3(n) = ay_1(n) + by_2(n)$$

So the system is linear.

$$y(n, k) = y(n)\big|_{x(n)=x(n-k)} = x(n-k) + x(n-2-k)$$

$$y(n-k) = y(n)\big|_{n=n-k} = x(n-k) + x(n-k-2)$$

$$y(n, k) = y(n-k)$$

So the system is time-invariant. Hence the given system is linear time-invariant.

EXAMPLE 1.17 Check whether the following systems are:

1. Static or dynamic
2. Linear or non-linear
3. Causal or non-causal, and
4. Shift-invariant or shift-variant

(a) $y(n) = \text{ev}\{x(n)\}$

(b) $y(n) = x(n)\, x(n-2)$

(c) $y(n) = \log_{10}|x(n)|$

(d) $y(n) = a^n u(n)$

(e) $y(n) = x^2(n) + \dfrac{1}{x^2(n-1)}$

Solution:

(a) Given $$y(n) = \text{ev}\{x(n)\}$$

$$y(n) = \text{ev}\{x(n)\} = \frac{1}{2}[x(n) + x(-n)]$$

1. For positive values of n, the output depends on past values of input and for negative values of n, the output depends on future values of input. Hence the system is dynamic.

2. $y(n) = T[x(n)] = \frac{1}{2}[x(n) + x(-n)]$

   For an input $x_1(n)$,

$$y_1(n) = \frac{1}{2}[x_1(n) + x_1(-n)]$$

   For an input $x_2(n)$,

$$y_2(n) = \frac{1}{2}[x_2(n) + x_2(-n)]$$

   The weighted sum of outputs is:

$$ay_1(n) + by_2(n) = a\frac{1}{2}[x_1(n) + x_1(-n)] + b\frac{1}{2}[x_2(n) + x_2(-n)]$$

$$= \frac{1}{2}\{[ax_1(n) + bx_2(n)] + [ax_1(-n) + bx_2(-n)]\}$$

   The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n) + bx_2(n)] = \frac{1}{2}\{[ax_1(n) + bx_2(n)] + [ax_1(-n) + bx_2(-n)]\}$$

$$y_3(n) = ay_1(n) + by_2(n)$$

   The weighted sum of outputs is equal to the output due to weighted sum of inputs. Hence superposition principle is valid and the system is linear.

3. $y(-2) = \frac{1}{2}[x(-2) + x(2)]$

   i.e. for negative values of n, the output depends on future values of input. Hence the system is non-causal.

4. Given $\qquad y(n) = \frac{1}{2}[x(n) + x(-n)]$

   The output due to input delayed by k units is:

$$y(n, k) = T[x(n-k)] = y(n)\big|_{x(n)=x(n-k)} = \frac{1}{2}[x(n-k) + x(-n-k)]$$

   The output delayed by k units is:

$$y(n-k) = y(n)\big|_{n=n-k} = \frac{1}{2}[x(n-k) + x(-n+k)]$$

$$y(n, k) \neq y(n-k)$$

   So the system is time-variant.
   So the given system is dynamic, linear, non-causal and time-variant.

(b) Given $$y(n) = x(n)\,x(n-2)$$

1. The output depends on past values of input. So it requires memory. Hence the system is dynamic.
2. The only term contains the product of input and delayed input. So the system is non-linear. This can be proved.

   Let an input $x_1(n)$ produce an output $y_1(n)$. Then
   $$y_1(n) = x_1(n)\,x_1(n-2)$$
   Let an input $x_2(n)$ produce an output $y_2(n)$. Then
   $$y_2(n) = x_2(n)\,x_2(n-2)$$
   The weighted sum of outputs is:
   $$ay_1(n) + by_2(n) = ax_1(n)\,x_1(n-2) + bx_2(n)\,x_2(n-2)$$
   The output due to weighted sum of inputs is:
   $$y_3(n) = T[ax_1(n) + bx_2(n)] = [ax_1(n) + bx_2(n)]\,[ax_1(n-2) + bx_2(n-2)]$$
   $$y_3(n) \neq ay_1(n) + by_2(n)$$
   Hence the system is non-linear.
3. The output depends only on the present and past values of input. It does not depend on future values of input. So the system is causal.
4. Given $$y(n) = x(n)x(n-2)$$
   The output due to input delayed by k units is:
   $$y(n,k) = y(n)\big|_{x(n)=x(n-k)} = x(n-k)\,x(n-2-k)$$
   The output delayed by k units is:
   $$y(n-k) = y(n)\big|_{n=n-k} = x(n-k)\;x(n-k-2)$$
   $$y(n,k) = y(n-k)$$
   Hence the system is shift-invariant.

   So the given system is dynamic, non-linear, causal and shift-invariant.

(c) Given $$y(n) = \log_{10}|x(n)|$$

1. The output depends on present value of input only. Hence the system is static.
2. Given $$y(n) = \log_{10}|x(n)|$$
   Let an input $x_1(n)$ produce an output $y_1(n)$. Then
   $$y_1(n) = \log_{10}|x_1(n)|$$
   Let an input $x_2(n)$ produce an output $y_2(n)$. Then
   $$y_2(n) = \log_{10}|x_2(n)|$$
   The weighted sum of outputs is:
   $$ay_1(n) + by_2(n) = a\log_{10}|x_1(n)| + b\log_{10}|x_2(n)|$$

The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n) + bx_2(n)] = \log_{10}|ax_1(n) + bx_2(n)|$$

$$y_3(n) \neq ay_1(n) + by_2(n)$$

Hence the system is non-linear.

3. The output does not depend upon future inputs. Hence the system is causal.

4. $y(n) = T[x(n)] = \log_{10}|x(n)|$

The output due to input delayed by k units is:

$$y(n, k) = T[x(n-k)] = y(n)\big|_{x(n)=x(n-k)} = \log_{10}|x(n-k)|$$

The output delayed by k units is:

$$y(n-k) = y(n)\big|_{n=n-k} = \log_{10}|x(n-k)|$$

$$y(n, k) = y(n-k)$$

Hence the system is shift-invariant.

So the given system is static, non-linear, causal and shift-invariant.

(d) Given $\qquad y(n) = a^n x(n)$

1. The output at any instant depends only on the present values of input. Hence the system is static.

2. Given $\qquad y(n) = a^n x(n)$

For an input $x_1(n)$,

$$y_1(n) = a^n x_1(n)$$

For an input $x_2(n)$,

$$y_2(n) = a^n x_2(n)$$

The weighted sum of outputs is:

$$py_1(n) + qy_2(n) = pa^n x_1(n) + qa^n x_2(n) = a^n[px_1(n) + qx_2(n)]$$

The output due to weighted sum of inputs is:

$$y_3(n) = T[px_1(n) + qx_2(n)] = a^n[px_1(n) + qx_2(n)]$$

$$y_3(n) = py_1(n) + qy_2(n)$$

Hence the system is linear.

3. The output depends only on the present input. It does not depend on future inputs. Hence the system is causal.

4. Given $\qquad y(n) = T[x(n)] = a^n x(n)$

The output due to input delayed by k units is:

$$y(n, k) = T[x(n-k)] = y(n)\big|_{x(n)=x(n-k)} = a^n x(n-k)$$

The output delayed by k units is:

$$y(n-k) = y(n)\big|_{n=n-k} = a^{n-k}x(n-k)$$

$$y(n,k) \neq y(n-k)$$

Hence the system is shift-variant.
So the given system is static, linear, causal and shift-variant.

(e) Given $$y(n) = x^2(n) + \frac{1}{x^2(n-1)}$$

1. The output at any instant depends upon past input. So memory is required. Hence the system is dynamic.

2. Given $$y(n) = x^2(n) + \frac{1}{x^2(n-1)}$$

   There are square terms of input. So the system is non-linear. This can be proved.
   For an input $x_1(n)$,

   $$y_1(n) = x_1^2(n) + \frac{1}{x_1^2(n-1)}$$

   For an input $x_2(n)$,

   $$y_2(n) = x_2^2(n) + \frac{1}{x_2^2(n-1)}$$

   The weighted sum of outputs is:

   $$ay_1(n) + by_2(n) = ax_1^2(n) + \frac{a}{x_1^2(n-1)} + bx_2^2(n) + \frac{b}{x_2^2(n-1)}$$

   The output due to weighted sum of inputs is:

   $$y_3(n) = T[ax_1(n) + bx_2(n)] = [ax_1(n) + bx_2(n)]^2 + \frac{1}{[ax_1(n-1) + bx_2(n-1)]^2}$$

   $$y_3(n) \neq ay_1(n) + by_2(n)$$

   Hence the system is non-linear.
3. The output does not depend on future values of input. Hence the system is causal.

4. Given $$y(n) = T[x(n)] = x^2(n) + \frac{1}{x^2(n-1)}$$

   The output due to input delayed by k units is:

   $$y(n,k) = T[x(n-k)] = y(n)\big|_{x(n)=x(n-k)} = x^2(n-k) + \frac{1}{x^2(n-1-k)}$$

The output delayed by k units is:

$$y(n-k) = y(n)\Big|_{n=n-k} = x^2(n-k) + \frac{1}{x^2(n-k-1)}$$

$$y(n, k) = y(n-k)$$

Hence the system is shift-invariant.
So the given system is dynamic, non-linear, causal and shift-invariant.

1.6.5 Stable and Unstable Systems

A bounded signal is a signal whose magnitude is always a finite value, i.e. $|x(n)| \leq M$, where M is a positive real finite number. For example a sinewave is a bounded signal. A system is said to be bounded-input, bounded-output (BIBO) stable, if and only if every bounded input produces a bounded output. The output of such a system does not diverge or does not grow unreasonably large.

Let the input signal $x(n)$ be bounded (finite), i.e.,

$$|x(n)| \leq M_x < \infty \quad \text{for all } n$$

where M_x is a positive real number. If

$$|y(n)| \leq M_y < \infty$$

i.e. if the output $y(n)$ is also bounded, then the system is BIBO stable. Otherwise, the system is unstable. That is, we say that a system is unstable even if one bounded input produces an unbounded output.

It is very important to know about the stability of the system. Stability indicates the usefulness of the system. The stability can be found from the impulse response of the system which is nothing but the output of the system for a unit impulse input. If the impulse response is absolutely summable for a discrete-time system, then the system is stable.

BIBO stability criterion

The necessary and sufficient condition for a discrete-time system to be BIBO stable is given by the expression:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

where $h(n)$ is the impulse response of the system. This is called BIBO stability criterion.

Proof: Consider a linear time-invariant system with $x(n)$ as input and $y(n)$ as output. The input and output of the system are related by the convolution integral.

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

Taking absolute values on both sides, we have

$$|y(n)| = \left|\sum_{k=-\infty}^{\infty} x(k)\,h(n-k)\right|$$

Using the fact that the absolute value of the sum of the product of two terms is always less than or equal to the sum of the product of their absolute values, we have

$$\left|\sum_{k=-\infty}^{\infty} x(k)h(n-k)\right| \le \sum_{k=-\infty}^{\infty} |x(k)||h(n-k)|$$

If the input $x(k)$ is bounded, i.e. there exists a finite number M_x such that,

$$|x(k)| \le M_x < \infty$$

$$|y(n)| \le M_x \sum_{k=-\infty}^{\infty} |h(n-k)|$$

Changing the variables by $m = n - k$, the output is bounded if

$$\sum_{n=-\infty}^{\infty} |h(m)| < \infty$$

Replacing m by n, we have

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

which is the necessary and sufficient condition for a system to be BIBO stable.

Figure 1.24 shows bounded and unbounded discrete-time signals. Figure 1.25 shows stable and unstable systems.

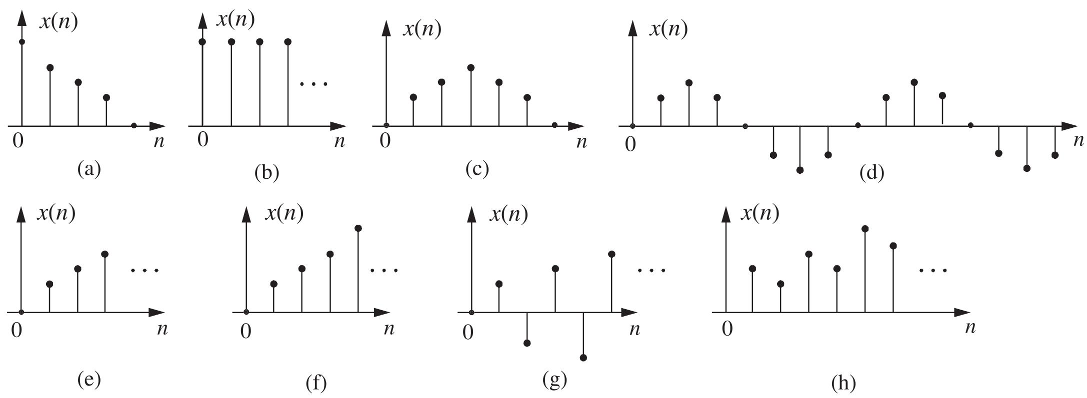


Figure 1.24 (a)–(d) Bounded signals (e)–(h) Unbounded signals.

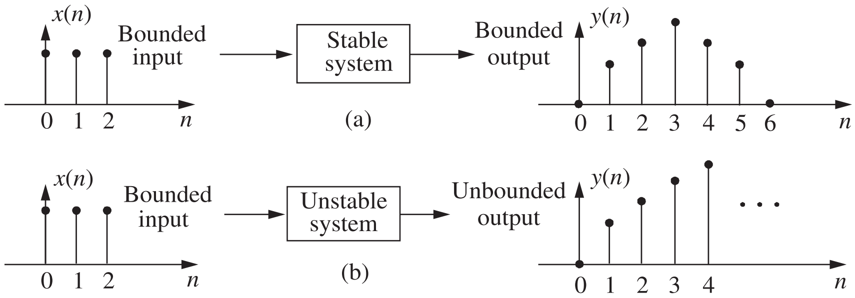


Figure 1.25 (a) Stable system (b) Unstable system.

The conditions for a BIBO stable system are given as follows:

1. If the system transfer function is a rational function, the degree of the numerator should not be larger than the degree of the denominator.
2. The poles of the system must lie inside the unit circle in the z-plane.
3. If a pole lies on the unit circle it must be a single order pole, i.e. no repeated pole lies on the unit circle.

EXAMPLE 1.18 Check the stability of the system defined by

(a) $y(n) = ax(n-7)$ (b) $y(n) = x(n) + \frac{1}{2}x(n-1) + \frac{1}{4}x(n-2)$

(c) $h(n) = a^n \quad \text{for } 0 < n < 11$ (d) $h(n) = 2^n u(n)$

(e) $h(n) = u(n)$

Solution:

(a) Given $y(n) = ax(n-7)$

Let $x(n) = \delta(n)$

Then $y(n) = h(n)$

$\therefore$ $h(n) = a\delta(n-7)$

$\therefore$ $h(n) = a \quad \text{for } n = 7$

$\quad = 0 \quad \text{for } n \neq 7$

A system is stable if its impulse response $h(n)$ is absolutely summable.

i.e. $$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

In this case,

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} a\delta(n-7) = a$$

Hence the given system is stable if the value of a is finite.

(b) Given $$y(n) = x(n) + \frac{1}{2}x(n-1) + \frac{1}{4}x(n-2)$$

Let $$x(n) = \delta(n)$$

Then $$y(n) = h(n)$$

$\therefore$ $$h(n) = \delta(n) + \frac{1}{2}\delta(n-1) + \frac{1}{4}\delta(n-2)$$

A discrete-time system is stable if

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

The given $h(n)$ has a value only at $n = 0$, $n = 1$ and $n = 2$. For all other values of n from $-\infty$ to ∞, $h(n) = 0$.

At $n = 0$, $h(0) = \delta(0) + \frac{1}{2}\delta(0-1) + \frac{1}{4}\delta(0-2) = \delta(0) + \frac{1}{2}\delta(-1) + \frac{1}{4}\delta(-2) = 1$

At $n = 1$, $h(1) = \delta(1) + \frac{1}{2}\delta(1-1) + \frac{1}{4}\delta(1-2) = \delta(1) + \frac{1}{2}\delta(0) + \frac{1}{4}\delta(-2) = \frac{1}{2}$

At $n = 2$, $h(2) = \delta(2) + \frac{1}{2}\delta(2-1) + \frac{1}{4}\delta(2-2) = \delta(2) + \frac{1}{2}\delta(1) + \frac{1}{4}\delta(0) = \frac{1}{4}$

$\therefore$ $$\sum_{n=-\infty}^{\infty} |h(n)| = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4} < \infty \text{ a finite value.}$$

Hence the system is stable.

(c) Given $$h(n) = a^n \quad \text{for } 0 < n < 11$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} |a^n| = \sum_{n=0}^{11} a^n = \frac{1-a^{12}}{1-a}$$

This value is finite for finite value of a. Hence the system is stable if a is finite.

(d) Given $$h(n) = 2^n u(n)$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} |2^n u(n)| = \sum_{n=0}^{\infty} 2^n = \infty$$

The impulse response is not absolutely summable. Hence this system is unstable.

(e) Given $$h(n) = u(n)$$

For stability,

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

In this case,

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=0}^{\infty} 1 = 1 + 1 + 1 + \cdots = \infty$$

So the output is not bounded and the system is unstable.

EXAMPLE 1.19 Check whether the following digital systems are BIBO stable or not:

(a) $y(n) = ax(n+1) + bx(n-1)$
(b) $y(n) =$ maximum of $[x(n), x(n-1), x(n-2)]$
(c) $y(n) = ax(n) + b$
(d) $y(n) = e^{-x(n)}$
(e) $y(n) = ax(n) + bx^2(n-1)$

Solution:

(a) Given $\quad y(n) = ax(n+1) + bx(n-1)$

If $\quad x(n) = \delta(n)$

then $\quad y(n) = h(n)$

Hence the impulse response is $h(n) = a\delta(n+1) + b\delta(n-1)$.

When $n = 0$, $\quad h(0) = a\delta(1) + b\delta(-1) = 0$

When $n = 1$, $\quad h(1) = a\delta(2) + b\delta(0) = b$

When $n = 2$, $\quad h(2) = a\delta(3) + b\delta(1) = 0$

In general,

$$h(n) = \begin{cases} b & \text{for } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

$\therefore$

$$\sum_{n=-\infty}^{\infty} |h(n)| = b$$

The necessary and sufficient condition for BIBO stability is:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

So the system is BIBO stable if $|b| < \infty$.

(b) Given $\quad y(n) =$ maximum of $[x(n), x(n-1), x(n-2)]$

If $\quad x(n) = \delta(n)$

Then $\quad y(n) = h(n)$

$\therefore \quad h(n) =$ maximum of $[\delta(n), \delta(n-1), \delta(n-2)]$

$h(0) =$ maximum of $[\delta(0), \delta(-1), \delta(-2)] = 1$

$h(1) =$ maximum of $[\delta(1), \delta(0), \delta(-1)] = 1$

$h(2) =$ maximum of $[\delta(2), \delta(1), \delta(0)] = 1$

$h(3) =$ maximum of $[\delta(3), \delta(2), \delta(1)] = 0$

Similarly, $h(4) = 0 = h(5) = h(6) \ldots$

$$\therefore \quad \sum_{n=-\infty}^{\infty} |h(n)| = |h(0)| + |h(1)| + |h(2)| + \cdots$$
$$= 1 + 1 + 1 + 0 + 0 + \cdots = 3$$

So the given system is BIBO stable.

(c) Given $y(n) = ax(n) + b$

If $x(n) = \delta(n)$

Then $y(n) = h(n)$

Hence the impulse response is $h(n) = a\delta(n) + b$

when $n = 0$, $h(0) = a\delta(0) + b = a + b$

when $n = 1$, $h(1) = a\delta(1) + b = b$

Here, $h(1) = h(2) = \ldots = h(n) = b$

Therefore,

$$h(n) = \begin{cases} a + b & \text{when } n = 0 \\ b & \text{when } n \neq 0 \end{cases}$$

The necessary and sufficient condition for BIBO stability is:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Therefore, $$\sum_{n=-\infty}^{\infty} |h(n)| = |h(0)| + |h(1)| + |h(2)| + \cdots + |h(n)| + \cdots + \cdots$$
$$= |a + b| + |b| + |b| + \cdots + |b| + \cdots$$

This series never converges since the ratio between the successive terms is one. Hence the given system is BIBO unstable.

(d) Given $y(n) = e^{-x(n)}$

If $x(n) = \delta(n)$

Then $y(n) = h(n)$

Hence the impulse response is $h(n) = e^{-\delta(n)}$.

When $n = 0$, $h(0) = e^{-\delta(0)} = e^{-1}$

When $n = 1$, $h(1) = e^{-\delta(1)} = e^{0} = 1$

In general,

$$h(n) = \begin{cases} e^{-1} & \text{when } n = 0 \\ 1 & \text{when } n \neq 0 \end{cases}$$

The necessary and sufficient condition for BIBO stability is:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Therefore, $$\sum_{n=-\infty}^{\infty} |h(n)| = |h(0)| + |h(1)| + |h(2)| + \cdots + |h(n)| + \cdots$$

$$= e^{-1} + 1 + 1 + 1 + \cdots + 1 + \cdots$$

Since the given sequence never converges, it is BIBO unstable.

(e) Given $\qquad y(n) = ax(n) + bx^2(n-1)$

If $\qquad x(n) = \delta(n)$

Then $\qquad y(n) = h(n)$

The above equation can be changed into $h(n) = a\delta(n) + b\delta^2(n-1)$.

When $n = 0$, $\quad h(0) = a\delta(0) + b\delta^2(-1) = a$

When $n = 1$, $\quad h(1) = a\delta(1) + b\delta^2(0) = b$

When $n = 2$, $\quad h(2) = a\delta(2) + b\delta^2(1) = 0$

Hence, $$\sum_{n=-\infty}^{\infty} |h(n)| = |h(0)| + |h(1)| + |h(2)| + \cdots + |h(n)| + \cdots$$

$$= |a| + |b| + 0 + 0 + \cdots$$

Hence, the given system is BIBO stable if $|a| + |b| < \infty$.

EXAMPLE 1.20 Determine whether each of the system with impulse response/output listed below is (i) causal, (ii) stable.

(a) $h(n) = 3^n u(-n)$ $\qquad$ (b) $h(n) = \cos\dfrac{n\pi}{2}$

(c) $h(n) = \delta(n) + \cos n\pi$ $\qquad$ (d) $h(n) = e^{3n}u(n-2)$

(e) $y(n) = \cos x(n)$ $\qquad$ (f) $y(n) = \sum_{k=-\infty}^{n+5} x(k)$

(g) $y(n) = \log|x(n)|$ $\qquad$ (h) $h(n) = [u(n) - u(n-15)]\, 2^n$

(i) $h(n) = 4^n u(3-n)$ $\qquad$ (j) $h(n) = e^{-5|n|}$

Solution:

(a) Given $\qquad h(n) = 3^n u(-n)$

$u(-n)$ exists for $-\infty < n \le 0$. Hence $h(n) \ne 0$ for $n < 0$. So the system is non-causal.

For stability, $$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

$$\sum_{n=-\infty}^{\infty} 3^n u(-n) = \sum_{n=-\infty}^{0} 3^n = \sum_{n=0}^{\infty} 3^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$$

$$= 1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \cdots$$

$$= \left(1 - \frac{1}{3}\right)^{-1} = \frac{1}{1-(1/3)} = \frac{3}{2} < \infty$$

So the system is stable.

(b) Given $$h(n) = \cos\frac{n\pi}{2}$$

$\cos(n\pi/2)$ exists for $-\infty < n < \infty$. So $h(n) \neq 0$ for $n < 0$. So, the system is non-causal.

For stability, $$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

$$\sum_{n=-\infty}^{\infty} \left|\cos\frac{n\pi}{2}\right| = \infty$$

because for odd values of n, $\left|\cos\frac{n\pi}{2}\right| = 0$ and for even values of n, $\left|\cos\frac{n\pi}{2}\right| = 1$.

So the system is unstable.

(c) Given $$h(n) = \delta(n) + \cos n\pi$$

$\delta(n) = 1$ for $n = 0$ and $\delta(n) = 0$ for $n \neq 0$
$|\cos n\pi| = 1$ for all values of n.

For stability, $$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = |h(-\infty)| + \cdots + |h(-1)| + |h(0)| + |h(1)| + \cdots + |h(\infty)|$$

$$= 1 + \cdots + 1 + 2 + 1 + \cdots + 1 = \infty$$

Therefore, the system is unstable.

(d) Given $$h(n) = e^{3n}u(n-2)$$

$u(n-2)$ exists only for $n \geq 2$. So $h(n) = 0$ for $n < 0$. Hence the system is causal.

For stability, $$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} \left|e^{3n}u(n-2)\right| = \sum_{n=2}^{\infty} e^{3n}$$

$$= e^6 + e^9 + e^{12} + \cdots$$

$$= \infty$$

Therefore, the system is unstable.

(e) Given $$y(n) = \cos x(n)$$

For the system to be stable, it has to satisfy the following condition:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

If $x(n) = \delta(n)$, then the impulse response is:

$$h(n) = \cos \delta(n)$$

For $n = 0$, $h(0) = \cos \delta(0) = \cos 1 = 0.54$

For $n = 1$, $h(1) = \cos \delta(1) = \cos 0 = 1$

For $n = 2$, $h(2) = \cos \delta(2) = \cos 0 = 1$

For $n = -1$, $h(-1) = \cos \delta(-1) = \cos 0 = 1$

For $n = -2$, $h(-2) = \cos \delta(-2) = \cos 0 = 1$

$$\sum_{n=-\infty}^{\infty} |h(n)| = |h(-\infty)| + \cdots + |h(-2)| + |h(-1)| + |h(0)| + |h(1)| + |h(2)| + \cdots + |h(\infty)|$$

$$= 1 + 1 + \cdots + 1 + 0.54 + 1 + 1 + \cdots + 1$$

$$= \infty$$

The system is unstable.

(f) Given $$y(n) = \sum_{k=-\infty}^{n+5} x(k)$$

For the system to be stable,

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Let $x(n) = \delta(n)$, then $y(n) = h(n)$. So for the given system

$$h(n) = \sum_{k=-\infty}^{n+5} \delta(k)$$

For $n = -6$, $$h(-6) = \sum_{k=-\infty}^{-1} \delta(k) = 0$$

For $n = -5$, $$h(-5) = \sum_{k=-\infty}^{0} \delta(k) = 1$$

For $n = 1$, $$h(1) = \sum_{k=-\infty}^{6} \delta(k) = 1$$

$\therefore$ For $n = -\infty$ to $n = -6$, $h(n) = 0$ and for $n = -5$ to $n = \infty$, $h(n) = 1$.

$\therefore$ $$\sum_{n=-\infty}^{\infty} |h(n)| = 0 + 0 + \cdots + 1 + 1 + 1 + \cdots + \infty = \infty$$

So the given system is unstable.

(g) Given $$y(n) = \log_{10}|x(n)|$$

The output depends only on the present input. Hence the system is causal. The impulse response is $h(n) = \log_{10}|\delta(n)|$.

$$h(0) = \log_{10}|\delta(0)| = \log_{10} 1 = 0$$

$$h(1) = \log_{10}|\delta(1)| = \log_{10} 0 = 0$$

$$h(2) = \log_{10}|\delta(2)| = \log_{10} 0 = 0$$

$$\therefore \qquad \sum_{n=-\infty}^{\infty} |h(n)| = 0 + 0 + \cdots = 0$$

(h) Given $$h(n) = [u(n) - u(n-15)]\, 2^n$$

$h(n) = 0$ for $n < 0$. So the system is causal.

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} [u(n) - u(n-15)]2^n$$

$$= \sum_{n=0}^{14} (1)\, 2^n = 1 + 2 + 2^2 + 2^3 + \cdots + 2^{14} < \infty$$

Therefore, the system is stable.

(i) Given $$h(n) = 4^n u(2-n)$$

$h(n) \neq 0$ for $n < 0$. So the system is non-causal.

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} 4^n u(2-n) = \sum_{n=-\infty}^{2} 4^n = \sum_{n=-\infty}^{0} 4^n + \sum_{n=1}^{2} 4^n$$

$$= \sum_{n=0}^{\infty} 4^{-n} + \sum_{n=1}^{2} 4^n = \left(1 - \frac{1}{4}\right)^{-1} + 4 + 4^2 = \left\{\frac{1}{[1-(1/4)]} + 20\right\} < \infty$$

Therefore, the system is stable.

(j) Given $$h(n) = e^{-5|n|}$$

The system is non-causal since $h(n) \neq 0$ for $n < 0$.

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} e^{-5n} = \sum_{n=-\infty}^{-1} e^{5n} + \sum_{n=0}^{\infty} e^{-5n} = \sum_{n=1}^{\infty} e^{-5n} + \sum_{n=0}^{\infty} e^{-5n}$$

$$= \frac{e^{-5}}{1-e^{-5}} + \frac{1}{1-e^{-5}} = \frac{1+e^{-5}}{1-e^{-5}} < \infty$$

Therefore, the system is stable.

EXAMPLE 1.21 Comment about the linearity, stability, time-invariance and causality for the following filter:

$$y(n) = 2x(n+1) + [x(n-1)]^2$$

***Solution*:** Given $y(n) = 2x(n+1) + [x(n-1)]^2$

1. There is a square term of delayed input [i.e. $x(n - 1)^2$] in the difference equation. So the system is non-linear.
2. The output depends on the future value of input [i.e. $2x(n + 1)$]. So the system is non-causal.
3. For $x(n) = \delta(n)$, $y(n) = h(n)$

$$\therefore \qquad h(n) = 2\delta(n+1) + \{\delta(n-1)\}^2$$

$$h(0) = 2\delta(1) + \{\delta(-1)\}^2 = 0 + 0 = 0$$

$$h(1) = 2\delta(2) + \{\delta(0)\}^2 = 0 + 1 = 1$$

$$h(-1) = 2\delta(0) + \{\delta(-2)\}^2 = 2 + 0 = 2$$

$$h(-2) = 2\delta(-1) + \{\delta(-3)\}^2 = 0 + 0 = 0$$

$$h(n) = 0, \text{ for any other } n$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = 0 + 1 + 2 + 0 + 0 + \cdots = 3 < \infty$$

Impulse response is absolutely summable. So the system is stable. Also we can say that since the output depends only on the delayed and advanced inputs, if the input is bounded the output is bounded. So the system is BIBO stable.

4. The output due to delayed input is:

$$y(n, k) = 2x(n+1-k) + \{x(n-1-k)\}^2$$

The delayed output is

$$y(n-k) = 2x(n-1+k) + \{x(n-k-1)\}^2$$

$$\therefore \qquad y(n, k) = y(n-k)$$

Therefore, the system is time-invariant. Also, we can say that since the system is described by constant coefficient difference equation, the system is time-invariant. So the given system is non-linear, stable, time-invariant and non-causal.

EXAMPLE 1.22 State whether the following system is linear, causal, time-invariant and stable:

$$y(n) + y(n-1) = x(n) + x(n-2)$$

Solution: Given $y(n) = -\,y(n-1) + x(n) + x(n-2)$

1. Let an input $x_1(n)$ produce an output $y_1(n)$ and an input $x_2(n)$ produce an output $y_2(n)$. Therefore, weighted sum of outputs is:

$$ay_1(n) + by_2(n) = -[ay_1(n-1) + by_2(n-1)] + [ax_1(n) + bx_2(n)] + [ax_1(n-2) + bx_2(n-2)]$$

The output due to weighted sum of inputs is:

$$y_3(n) = -\{ay_1(n-1) + by_2(n-1)\} + \{ax_1(n) + bx_2(n)\} + \{ax_1(n-2) + bx_2(n-2)\}$$

So the system is linear.

2. The output depends only on the present and past inputs and past outputs. So the system is causal.
3. All the coefficients of the differential equation are constants. So the system is time-invariant.
4. For $x(n) = \delta(n)$, $y(n) = h(n)$

$\therefore$
$$h(n) = -h(n-1) + \delta(n) + \delta(n-2)$$
$$h(0) = -h(-1) + \delta(0) + \delta(-2) = 1$$
$$h(1) = -h(0) + \delta(1) + \delta(-1) = -1$$
$$h(2) = -h(1) + \delta(2) + \delta(0) = 1 + 0 + 1 = 2$$
$$h(3) = -h(2) + \delta(3) + \delta(1) = -2 + 0 + 0 = -2$$
$$\sum_{n=-\infty}^{\infty} |h(n)| = 1 + 1 + 2 + 2 + \cdots = \infty$$

i.e. the impulse response is not absolutely summable. So the system is unstable. Therefore, the given system is non-linear, causal, time-variant and unstable.

EXAMPLE 1.23 Determine whether the following system is linear, stable, causal and time-invariant using appropriate tests:

$$y(n) = nx(n) + x(n+2) + y(n-2)$$

Solution: Given $y(n) = nx(n) + x(n+2) + y(n-2)$

1. Let an input $x_1(n)$ produce an output $y_1(n)$ and an input $x_2(n)$ produce an output $y_2(n)$. Then the weighted sum of outputs is:

$$ay_1(n) + by_2(n) = n[ax_1(n) + bx_2(n)] + [ax_1(n+2) + bx_2(n+2)] + [ay_1(n-2) + by_2(n-2)]$$

The output due to weighted sum of inputs is:

$$y_3(n) = n\{ax_1(n) + bx_2(n)\} + \{ax_1(n+2) + bx_2(n+2)\} + \{ay_1(n-2) + by_2(n-2)\}$$
$$y_3(n) = ay_1(n) + by_2(n)$$

So the system is linear.

2. For $x(n) = \delta(n)$, $y(n) = h(n)$

$\therefore$
$$h(n) = n\delta(n) + \delta(n+2) + h(n-2)$$
$$h(-2) = -2\delta(-2) + \delta(0) + h(-4) = 1$$
$$h(0) = 0\delta(0) + \delta(2) + h(-2) = 0 + 0 + 1 = 1$$
$$h(1) = 1\delta(1) + \delta(3) + h(-1) = 0$$
$$h(2) = 2\delta(2) + \delta(4) + h(0) = 1$$
$$h(3) = 3\delta(3) + \delta(5) + h(1) = 0$$
$$h(4) = 4\delta(4) + \delta(6) + h(2) = 1$$

$$\sum_{n=-\infty}^{\infty} |h(x)| = 1 + 0 + 1 + 0 + \cdots = \infty$$

So the system is unstable.

3. $y(2) = 2x(2) + x(4) + y(0)$

   The output depends on future inputs. So the system is non-causal.

4. The coefficient of $x(n)$ is a function of time. So it is a time varying system.

EXAMPLE 1.24 Find the linear, invariance, causality of the following systems:

(a) $y(n) = -ax(n-1) + x(n)$

(b) $y(n) = x(n^2) + x(-n)$

Solution: (a) Given $y(n) = -ax(n-1) + x(n)$

1. Let an input $x_1(n)$ produce an output $y_1(n)$ and an input $x_2(n)$ produce an output $y_2(n)$. Then the weighted sum of outputs is:

$$py_1(n) + qy_2(n) = -a[px_1(n-1) + qx_2(n-1)] + [px_1(n) + qx_2(n)]$$

The output due to weighted sum of inputs is:

$$y_3(n) = -a[px_1(n-1) + qx_2(n-1)] + [px_1(n) + qx_2(n)]$$

$$y_3(n) = py_1(n) + qy_2(n)$$

So the system is linear.

2. The output depends only on the present and past inputs. So the system is causal.
3. The output due to delayed input is:

$$y(n, k) = -ax(n-1-k) + x(n-k)$$

The delayed output is

$$y(n-k) = -ax(n-k-1) + x(n-k)$$

$$y(n, k) = y(n-k)$$

So the system is time-invariant. Therefore, the system is linear, causal and time-invariant.

(b) Given $y(n) = x(n^2) + x(-n)$

1. Let an input $x_1(n)$ produce an output $y_1(n)$ and an input $x_2(n)$ produce an output $y_2(n)$. Then the weighted sum of outputs is:

$$ay_1(n) + by_2(n) = [ax_1(n^2) + bx_2(n^2)] + [ax_1(-n) + bx_2(-n)]$$

The output due to weighted sum of inputs is:

$$y_3(n) = \{ax_1(n^2) + bx_2(n^2)\} + \{ax_1(-n) + bx_2(-n)\}$$

$$y_3(n) = ay_1(n) + by_2(n)\}$$

So the system is linear.

2. $y(-2) = x(4) + x(2)$

   $y(2) = x(4) + x(-2)$

   The output depends upon future inputs. So the system is non-causal.

3. The output due to delayed input is:

$$y(n, k) = x(n^2 - k) + x(-n - k)$$

The delayed output is:

$$y(n - k) = x\{(n - k)^2\} + x\{-(n - k)\}$$

$$y(n, k) \neq y(n - k)$$

So the system is time-variant. Therefore, the system is linear, non-causal and time-variant.

EXAMPLE 1.25 Test the causality and stability of the following system:

$$y(n) = x(n) - x(-n - 1) + x(n - 1)$$

Solution: Given $y(n) = x(n) - x(-n - 1) + x(n - 1)$

1. $y(-2) = x(-2) - x(1) + x(-3)$

   For negative values of n, the output depends on future values of input. So the system is non-causal.

2. For $x(n) = \delta(n)$, $y(n) = h(n)$

$$\therefore \quad h(n) = \delta(n) - \delta(-n - 1) + \delta(n - 1)$$

$$h(0) = \delta(0) - \delta(-1) + \delta(-1) = 1 - 0 + 0 = 1$$

$$h(1) = \delta(1) - \delta(-2) + \delta(0) = 0 - 0 + 1 = 1$$

$$h(-1) = \delta(-1) - \delta(0) + \delta(-2) = 0 - 1 + 0 = -1$$

$$h(n) = 0 \text{ for any other value of } n$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = 1 + 1 + 1 + 0 + 0 + \cdots = 3 < \infty$$

   i.e., the impulse response is absolutely summable. So the system is stable.

EXAMPLE 1.26 If a system is represented by the following difference equation:

$$y(n) = 3y^2(n - 1) - nx(n) + 4x(n - 1) - x(n + 1) \qquad n \geq 0$$

(a) Is the system linear? Explain.
(b) Is the system shift-invariant? Explain.
(c) Is the system causal? Why or why not?

Solution: Given $y(n) = 3y^2(n - 1) - nx(n) + 4x(n - 1) - x(n + 1) \qquad n \geq 0$

(a) No, the system is non-linear, because there is a square term of delayed output in the difference equation.
(b) No, the system is shift-variant because the coefficient of $x(n)$ is not a constant. It is a function of time.
(c) No, the system is non-causal because the output depends on future inputs.

EXAMPLE 1.27 Test the following systems for linearity, time-invariance, stability and causality.

(a) $y(n) = a^{\{x(n)\}}$ (b) $y(n) = \sin\left\{\dfrac{2\pi bfn}{F}\right\} x(n)$

Solution:

(a) Given $y(n) = a^{\{x(n)\}}$

1. Let an input $x_1(n)$ produce an output $y_1(n)$ and an input $x_2(n)$ produce an output $y_2(n)$. Then the weighted sum of outputs is:

$$py_1(n) + qy_2(n) = pa^{\{x_1(n)\}} + qa^{\{x_2(n)\}}$$

The output due to weighted sum of inputs is:

$$y_3(n) = a^{\{px_1(n) + qx_2(n)\}}$$

$$y_3(n) \neq py_1(n) + qy_2(n)$$

So the system is non-linear.

2. The output due to delayed input is:

$$y(n, k) = a^{\{x(n-k)\}}$$

The delayed output is:

$$y(n-k) = a^{\{x(n-k)\}}$$

$$y(n, k) = y(n-k)$$

So the system is shift-invariant.

3. When input $x(n) = \delta(n)$, $y(n) = h(n)$

$$h(n) = a^{\{\delta(n)\}}$$

$$h(0) = a^{\{\delta(0)\}} = a$$

$$h(n) = a^0 = 1 \quad \text{for any other } n$$

$$\sum_{n=-\infty}^{\infty} |h(x)| = 1 + 1 + \cdots + a + 1 + 1 + \cdots = \infty$$

The impulse response is not absolutely summable. So the system is unstable.

4. The output depends only on the present input. So the system is causal. Thus, the given system is non-linear, shift-invariant, unstable and causal.

(b) Given $y(n) = \sin\left\{\dfrac{2\pi bfn}{F}\right\} x(n) = \sin\dfrac{n\omega b}{F} x(n)$

1. Let an input $x_1(n)$ produce an output $y_1(n)$ and an input $x_2(n)$ produce an output $y_2(n)$. Then the weighted sum of outputs is:

$$py_1(n) + qy_2(n) = p\sin\left(\frac{n\omega b}{F}\right)x_1(n) + q\sin\left(\frac{n\omega b}{F}\right)x_2(n)$$

$$= \sin\left(\frac{n\omega b}{F}\right)[px_1(n) + qx_2(n)]$$

The output due to weighted sum of inputs is:

$$y_3(n) = \sin\left(\frac{n\omega b}{F}\right)[px_1(n) + qx_2(n)]$$

$$\therefore \qquad y_3(n) = py_1(n) + qy_2(n)$$

So the system is linear.

2. The output due to delayed input is:

$$y(n, k) = \sin\left(\frac{n\omega b}{F}\right) x(n-k)$$

The delayed output is:

$$y(n-k) = \sin\left[\frac{(n-k)\,\omega b}{F}\right] x(n-k)$$

$$y(n, k) \neq y(n-k)$$

So the system is shift-invariant.

3. When $x(n) = \delta(n)$, $y(n) = h(n)$

$$\therefore \qquad h(n) = \sin\left(\frac{n\omega b}{F}\right)\delta(n)$$

$$h(0) = \sin(0)\,\delta(0) = 0$$

$$h(1) = \sin\left(\frac{\omega b}{F}\right)\delta(1) = 0$$

$$h(-1) = \sin\left(\frac{-\omega b}{F}\right)\delta(-1) = 0$$

$$h(n) = 0 \qquad \text{for all other } n$$

$\sum_{n=-\infty}^{\infty} |h(n)| = 0.$ So the system is stable.

4. The output depends upon present input only. So the system is causal. Therefore, the given system is linear, shift-invariant, stable and causal.

1.6.6 FIR and IIR Systems

Linear time-invariant discrete-time systems can be classified according to the type of impulse response. If the impulse response sequence is of finite duration, the system is called a finite impulse response (FIR) system, and if the impulse response sequence is of infinite duration, the system is called an infinite impulse response (IIR) system.

An example of FIR system is described by

$$h(n) = \begin{cases} -2 & n = 2, 4 \\ 2 & n = 1, 3 \\ 0 & \text{otherwise} \end{cases}$$

An example of IIR system is described by

$$h(n) = 2^n u(n)$$

1.6.7 Invertible and Non-invertible Systems

A system is known as invertible only if an inverse system exists which when cascaded with the original system produces an output equal to the input of the first system. A system which does not satisfy this criterion is called a non-invertible system.

For $y(n) = 3x(n)$, the system is said to be invertible, whereas for $y(n) = 2x^2(n)$, the system is said to be non-invertible. Mathematically, a system is invertible if

$$x(n) = T^{-1}\{T[x(n)]\}$$

The block diagram representation of both an invertible and non-invertible system is shown in Figure 1.26.

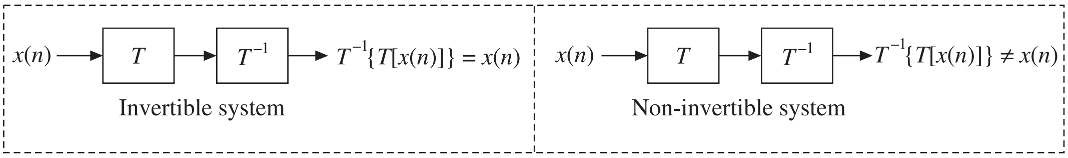


Figure 1.26 Invertibility property in discrete domain.

If the impulse response of the original system is $h(n)$, then the impulse response of the inverse system is $h^{-1}(n)$.

From Figure 1.27, we can find that

$$y(n) = x(n) * h(n)$$

and

$$\begin{aligned} p(n) &= y(n) * h^{-1}(n) \\ &= [x(n) * h(n)] * h^{-1}(n) \\ &= x(n) * [h(n) * h^{-1}(n)] \\ &= x(n) * \delta(n) = x(n) \qquad [\therefore\ h(n) * h^{-1}(n) = \delta(n)] \end{aligned}$$

As an example consider the cascading of a system and its inverse as shown in Figure 1.27.

For example if $\qquad y(n) = ax(n)$

Then

$$p(n) = \frac{1}{a}\, y(n) = x(n)$$

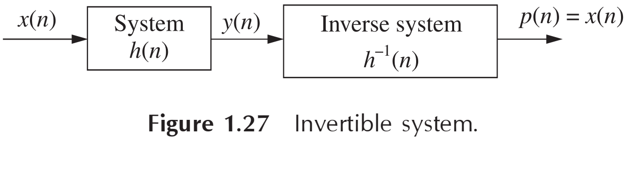


Figure 1.27 Invertible system.

1.7 REPRESENTATION OF AN ARBITRARY SEQUENCE

Any arbitrary sequence $x(n)$ can be represented in terms of delayed and weighted impulse sequence $\delta(n)$. Consider a finite 5 sample sequence shown in Figure 1.28(a). Figure 1.28(b) shows a unit impulse.

The sample $x(0)$ can be obtained by multiplying $x(0)$, the magnitude with unit impulse $\delta(n)$ as shown in Figure 1.28(c), i.e.

$$x(0)\,\delta(n) = \begin{cases} x(0) & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

Similarly the sample $x(-1)$ can be obtained by multiplying $x(-1)$, the magnitude with $\delta(n+1)$ which is one sample advanced unit impulse as shown in Figure 1.28(d). $x(-2)$, $x(1)$ and $x(2)$ can be obtained as shown in Figure 1.28[(e), (f) and (g)]. i.e.

$$x(-1)\,\delta(n+1) = \begin{cases} x(-1) & \text{for } n = -1 \\ 0 & \text{for } n \neq -1 \end{cases}$$

In the same way,

$$x(-2)\,\delta(n+2) = \begin{cases} x(-2) & \text{for } n = -2 \\ 0 & \text{for } n \neq -2 \end{cases}$$

$$x(1)\,\delta(n-1) = \begin{cases} x(1) & \text{for } n = 1 \\ 0 & \text{for } n \neq 1 \end{cases}$$

$$x(2)\,\delta(n-2) = \begin{cases} x(2) & \text{for } n = 2 \\ 0 & \text{for } n \neq 2 \end{cases}$$

In this case,

$$x(0)\,\delta(n) = \delta(n),\ x(1)\delta(n-1) = 3\delta(n-1),\ x_2\delta(n-2) = 2\delta(n-2)$$

$$x(-1)\delta(n+1) = 3\delta(n+1),\ x(-2)\delta(n+2) = 2\delta(n+2)$$

$$\therefore \qquad x(n) = 2\delta(n+2) + 3\delta(n+1) + \delta(n) + 3\delta(n-1) + 2\delta(n-2)$$

In general, we can write an infinite sequence $x(n)$ for $-\infty \le n \le \infty$ as:

$$x(n) = \cdots + x(-3)\delta(n+3) + x(-2)\delta(n+2) + x(-1)\delta(n+1) + x(0)\,\delta(n)$$
$$+\, x(1)\,\delta(n-1) + x(2)\delta(n-2) + x(3)\delta(n-3) + \cdots$$
$$= \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

where $\delta(n-k)$ is unity for $n = k$ and zero for all other terms.

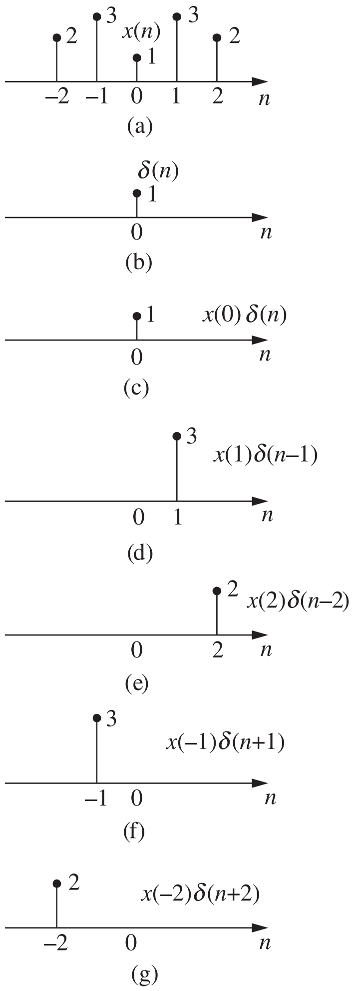


Figure 1.28 Representation of a sequence as a sum of delayed impulses.

EXAMPLE 1.28 Represent the sequence $x(n) = \left\{ \underset{\uparrow}{3, 1, -2, 1, 4, 2, 5, 1} \right\}$ as sum of shifted unit impulses.

Solution: Given
$$x(n) = \left\{ 3, 1, -2, \underset{\uparrow}{1}, 4, 2, 5, 1 \right\}$$
$$n = -3, -2, -1, 0, 1, 2, 3, 4$$
$$\begin{aligned} x(n) &= x(-3)\delta(n+3) + x(-2)\delta(n+2) + x(-1)\delta(n+1) + x(0)\delta(n) \\ &\quad + x(1)\delta(n-1) + x(2)\delta(n-2) + x(3)\delta(n-3) + x(4)\delta(n-4) \\ &= 3\delta(n+3) + \delta(n+2) - 2\delta(n+1) + \delta(n) + 4\delta(n-1) + 2\delta(n-2) + 5\delta(n-3) + \delta(n-4) \end{aligned}$$

SHORT QUESTIONS WITH ANSWERS

1. Define a signal.
Ans. A signal is defined as a single-valued function of one or more independent variables which contain some information.

2. What is one-dimensional signal?
Ans. A signal which depends on only one independent variable is called a one-dimensional signal.

3. What is signal modelling?
Ans. The representation of a signal by the mathematical expression is known as signal modelling.

4. What are the different types of representing discrete-time signals?
Ans. There are following four different types of representation of discrete-time signals:
(a) Graphical representation (b) Functional representation
(c) Tabular representation (d) Sequence representation

5. Define unit step sequence.
Ans. The discrete-time unit step sequence $u(n)$ is defined as:

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

6. Define unit ramp sequence.
Ans. The discrete-time unit ramp sequence $r(n)$ is defined as:

$$r(n) = \begin{cases} n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

or $$r(n) = nu(n)$$

7. Define unit parabolic sequence.
Ans. The discrete-time unit parabolic sequence $p(n)$ is defined as:

$$p(n) = \begin{cases} \dfrac{n^2}{2} & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

or $$p(n) = \frac{n^2}{2} u(n)$$

8. Define unit impulse sequence.
Ans. The discrete-time unit impulse sequence $\delta(n)$ is defined as:

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

9. Write the properties of unit impulse function.
Ans. The properties of discrete-time unit sample sequence are given as follows:

(a) $\delta(n) = u(n) - u(n-1)$ (b) $\delta(n-k) = \begin{cases} 1, & n = k \\ 0, & n \neq k \end{cases}$

(c) $x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$

10. Define a sinusoidal signal.
Ans. The discrete-time sinusoidal signal is given by

$$x(n) = A\cos(\omega_0 n + \phi)$$

where ω_0 is the frequency (in radians/sample), and ϕ is the phase (in radians).

11. Define a real exponential signal.
Ans. The discrete-time real exponential sequence is given by

$$x(n) = a^n u(n)$$

where a is a constant.

12. Define complex exponential signal.
Ans. The discrete-time complex exponential signal is given by

$$x(n) = a^n e^{j(\omega_0 n + \phi)}$$

where a is a constant.

13. What are the basic operations on discrete-time signals?
Ans. The basic set of operations on discrete-time signals are as follows:

(a) Time shifting (b) Time reversal
(c) Time scaling (d) Amplitude scaling
(e) Signal addition (f) Signal multiplication

14. How are discrete-time signals classified?
Ans. Discrete-time signals are classified according to their characteristics. Some of them are as follows:

(a) Deterministic and random signals
(b) Periodic and aperiodic signals
(c) Energy and power signals
(d) Even and odd signals
(e) Causal and non-causal signals

15. What are digital signals?
Ans. The signals that are discrete in time and quantized in amplitude are called digital signals.

16. Distinguish between deterministic and random signals.
Ans. A deterministic signal is a signal exhibiting no uncertainty of its magnitude and phase at any given instant of time. It can be represented by a mathematical equation, whereas a random signal is a signal characterized by uncertainty about its occurrence. It cannot be represented by a mathematical equation.

17. Distinguish between periodic and aperiodic signals.
Ans. A discrete-time sequence $x(n)$ is said to be periodic if it satisfies the condition:

$$x(n) = x(n + N) \quad \text{for all } n$$

whereas a discrete-time signal $x(n)$ is said to be aperiodic if the above condition is not satisfied even for one value of n.

18. What do you mean by fundamental period of a signal?
Ans. The smallest value of N that satisfies the condition $x(n + N) = x(n)$ for all values of n for discrete-time signals is called the fundamental period of the signal $x(n)$.

19. Are all sinusoidal sequences periodic?
Ans. In the case of discrete-time signals, not all sinusoidal sequences are periodic.

20. What is the condition to be satisfied for a discrete-time sinusoidal sequence to be periodic?
Ans. For the discrete-time sinusoidal sequence to be periodic, the condition to be satisfied is, the fundamental frequency ω_0 must be a rational multiple of 2π. Otherwise, the discrete-time signal is aperiodic.

21. What is the fundamental period of a discrete-time sinusoidal sequence?
Ans. The smallest value of positive integer N, for some integer m, which satisfies the equation $N = 2\pi(m/\omega_0)$ for a sinusoidal periodic signal is called the fundamental period of that signal.

22. Distinguish between energy and power signals.
Ans. An energy signal is one whose total energy E = finite value and whose average power $P = 0$, whereas a power signal is one whose average power P = finite value and total energy $E = \infty$.

23. Write the expressions for total energy E and average power P of a signal.
Ans. The expressions for total energy E and average power P of a signal are:

$$E = \underset{N\to\infty}{\mathrm{Lt}} \sum_{n=-N}^{N} |x(n)|^2$$

and $$P = \underset{N\to\infty}{\mathrm{Lt}} \frac{1}{2N} \sum_{n=-N}^{N} |x(n)|^2 \quad \text{for discrete-time signals.}$$

24. Do all the signals belong to either energy signal or power signal category?
Ans. No. Some signals may not correspond to either energy signal type or power signal type. Such signals are neither power signals nor energy signals.

25. Distinguish between even and odd signals.
Ans. A discrete-time signal $x(n)$ is said to be even (symmetric) signal if it satisfies the condition:

$$x(-n) = x(n) \quad \text{for all } n$$

whereas a discrete-time signal $x(n)$ is said to be odd (anti-symmetric) signal if it satisfies the condition:

$$x(-n) = -x(n) \quad \text{for all } n$$

26. Do all the signals correspond to either even or odd type?
Ans. No. All the signals need not necessarily belong to either even or odd type. There are signals which are neither even nor odd.

27. Can every signal be decomposed into even and odd parts?
Ans. Yes, every signal can be decomposed into even and odd parts.

28. Write the expressions for even and odd parts of a signal.
Ans. The even and odd parts of a discrete-time signal are given by

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

$$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

29. Distinguish between causal and non-causal signals.
Ans. A discrete-time signal $x(n)$ is said to be causal if $x(n) = 0$ for $n < 0$, otherwise the signal is non-causal.

30. Define anti-causal signal.
Ans. A discrete-time signal $x(n)$ is said to be anti-causal if $x(n) = 0$ for $n > 0$.

31. Define a system.
Ans. A system is defined as a physical device, that generates a response or output signal for a given input signal.

32. How are discrete-time systems classified?
Ans. The discrete-time systems are classified as follows:

(a) Static (memoryless) and dynamic (memory) systems
(b) Causal and non-causal systems
(c) Linear and non-linear systems
(d) Time-invariant and time varying systems
(e) Stable and unstable systems.
(f) Invertible and non-invertible systems

33. Define a discrete-time system.
Ans. A discrete-time system is a system which transforms discrete-time input signals into discrete-time output signals.

34. Define a static system.
Ans. A static or memoryless system is a system in which the response at any instant is due to present input alone, i.e. for a static or memoryless system, the output at any instant n depends only on the input applied at that instant n but not on the past or future values of input.

35. Define a dynamic system.
Ans. A dynamic or memory system is a system in which the response at any instant depends upon past or future inputs.

36. Define a causal system.
Ans. A causal (non-anticipative) system is a system whose output at any time n depends only on the present and past values of the input but not on future inputs.

37. Define a non-causal system.
Ans. A non-causal (anticipative) system is a system whose output at any time n depends on future inputs.

38. What is homogeneity property?
Ans. Homogeneity property means a system which produces an output $y(n)$ for an input $x(n)$ must produce an output $ay(n)$ for an input $ax(n)$.

39. What is superposition property?
Ans. Superposition property means a system which produces an output $y_1(n)$ for an input $x_1(n)$ and an output $y_2(n)$ for an input $x_2(n)$ must produce an output $y_1(n) + y_2(n)$ for an input $x_1(n) + x_2(n)$.

40. Define a linear system.
Ans. A linear system is a system which obeys the principle of superposition and principle of homogeneity.

41. Define a non-linear system.
Ans. A non-linear system is a system which does not obey the principle of superposition and principle of homogeneity.

42. Define a shift-invariant system.
Ans. A shift-invariant system is a system whose input/output characteristics do not change with time, i.e. a system for which a time shift in the input results in a corresponding time shift in the output.

43. Define a shift-variant system.
Ans. A shift-variant system is a system whose input/output characteristics change with time, i.e. a system for which a time shift in the input does not result in a corresponding time shift in the output.

44. Define a bounded input-bounded output stable system.
Ans. A bounded input-bounded output stable system is a system which produces a bounded output for every bounded input.

45. Define an unstable system.
Ans. An unstable system is a system which produces an unbounded output for a bounded input.

46. What is an invertible system?
Ans. An invertible system is a system which has a unique relation between its input and output.

47. What is a non-invertible system?
Ans. A non-invertible system is a system which does not have a unique relation between its input and output.

REVIEW QUESTIONS

1. Define various elementary discrete-time signals. Indicate them graphically.
2. What are the types of representation of discrete-time signals? Represent a sequence in all types.
3. What are the basic operations on discrete-time signals? Illustrate with an example.
4. How are discrete-time signals classified? Differentiate between them.
5. Write short notes on: (i) Complex exponential signals, (ii) Sinusoidal signals.
6. Write the properties of the unit impulse sequence.
7. Derive the relation between complex exponential and sinusoidal signals.
8. Write short notes on the following sequences:
   (a) Unit step (b) Unit impulse
   (c) Unit ramp (d) Sinusoidal sequence
9. Define a system. How are discrete-time systems classified? Define each one of them.
10. Distinguish between
    (a) Static (memoryless) and dynamic (memory) systems
    (b) Causal and non-causal systems
    (c) Linear and non-linear systems
    (d) Time-invariant and time varying systems
    (e) Stable and unstable systems
    (f) Invertible and non-invertible systems
    (g) FIR and IIR systems

FILL IN THE BLANKS

1. If a signal depends on only one independent variable, it is called a _______ signal.
2. The representation of a signal by mathematical expression is known as _______.
3. Discrete-time signals are _______ in time and _______ in amplitude.
4. The signals that are discrete in time and quantized in amplitude are called _______ signals.
5. The _______ of a signal can be obtained by folding the signal about $n = 0$.
6. A signal which can be described by a mathematical equation is called a _______ signal.
7. A signal which cannot be represented by a mathematical equation is called a _______ signal.
8. In the case of _______ -time signals, not all the sinusoidal signals are periodic.
9. For an even signal, $x(-n) =$ _______ for all n.
10. For an odd signal, $x(-n) =$ _______ for all n.

11. For an energy signal, E = ______ and P = ______.
12. For a power signal, P = ______ and E = ______.
13. For an anti-causal signal, $x(n) = 0$ for ______.
14. For a static system, the output does not depend on the ______ values of input.
15. For a dynamic system, the output depends on the ______ and/or ______ values of input.
16. Static systems are also called ______ systems.
17. Dynamic systems are also called ______ systems.
18. A causal system is one whose output depends on ______ values of input.
19. A non-causal system is one whose output depends on ______ values of input.
20. A causal system is also known as a ______ system.
21. A non-causal system is also known as an ______ system.
22. A ______ system is definitely a dynamic system.
23. A ______ system obeys the principle of superposition.
24. A ______ system does not obey the principle of superposition.
25. An LTI system is one which satisfies the properties of ______ and ______.
26. For a time-invariant system, its ______ do not change with time.
27. For a time-variant system, its ______ change with time.
28. A system is said to be stable if every ______ input produces a bounded output.
29. For a discrete-time system to be stable, its impulse response must be ______.
30. A system which has an unique relation between its input and output is called ______.
31. A system which does not have an unique relation between its input and output is called ______.

OBJECTIVE TYPE QUESTIONS

1. A signal can be represented in
   (a) time domain (b) frequency domain
   (c) both (a) and (b) (d) none of these
2. $\delta(n)$ =
   (a) $u(n) + u(n-1)$ (b) $u(n)\,u(n-1)$
   (c) $u(n) - u(n-1)$ (d) $u(n-1) - u(n)$
3. A deterministic signal has
   (a) no uncertainty (b) uncertainty
   (c) partial uncertainty (d) none of these

4. A random signal has
(a) no uncertainty (b) uncertainty
(c) partial uncertainty (d) none of these

5. The fundamental period of a discrete-time complex exponential sequence is $N =$
(a) $\frac{2\pi}{\omega_0}$ (b) $\frac{2\pi}{m}\omega_0$ (c) $\frac{2\pi}{\omega_0}m$ (d) $2\pi m\omega_0$

6. The fundamental period of a sinusoidal sequence is $N =$
(a) $2\pi m$ (b) $\frac{\omega_0}{2\pi m}$ (c) $m\omega_0$ (d) $\frac{2\pi}{\omega_0}m$

7. A signal is an energy signal if
(a) $E = 0$, $P = 0$ (b) $E = \infty$, $P =$ finite
(c) $E =$ finite, $P = 0$ (d) $E =$ finite, $P = \infty$

8. A signal is a power signal if
(a) $P =$ finite, $E = 0$ (b) $P =$ finite, $E = \infty$
(c) $P =$ finite, $E =$ finite (d) $P = \infty$, $E = \infty$

9. The signal $\alpha^n u(n)$ is an energy signal if
(a) $|\alpha| < 1$ (b) $|\alpha| > 1$ (c) $|\alpha| = 1$ (d) $|\alpha| = 0$

10. The signal $\alpha^n u(n)$ is a power signal if
(a) $|\alpha| < 1$ (b) $|\alpha| > 1$ (c) $|\alpha| = 1$ (d) $|\alpha| = 0$

11. A system whose output depends on future inputs is a
(a) static system (b) dynamic system
(c) non-causal system (d) both (b) and (c)

12. A non-anticipative system is a
(a) static system (b) dynamic system
(c) causal system (d) non-causal system

13. $y(n) = x(n + 2)$ is for a
(a) linear system (b) dynamic system
(c) both linear and dynamic system (d) non-linear system

14. $y(n) = x(2n)$ is for a
(a) time-invariant system (b) time varying, dynamic system
(c) linear, time varying, dynamic system (d) linear, time-invariant, static system

15. $y(n) = x(-n)$ is for a
(a) non-causal system
(b) linear, causal, time-invariant system
(c) linear, non-causal, time-invariant system
(d) linear, non-causal, time varying, dynamic system

16. $y(n) = x(n) + nx(n - 1)$ is for a
(a) dynamic system (b) causal system
(c) linear system (d) all of these

17. $y(n) = x(n)\,u(n)$ is for a
(a) static, linear system
(b) causal, time-invariant system
(c) both (a) and (b)
(d) none of these

18. A system which has a unique relation between its input and output is called
(a) linear system
(b) causal system
(c) time-invariant system
(d) invertible system

19. A system which does not have a unique relation between its input and output is called
(a) non-linear system
(b) non-causal system
(c) time-variant system
(d) non-invertible system

PROBLEMS

1. Evaluate the following:

(a) $\sum_{n=-\infty}^{\infty} e^{2n}\,\delta(n-2)$

(b) $\sum_{n=-\infty}^{\infty} n^2\delta(n-3)$

(c) $\sum_{n=-\infty}^{\infty} \delta(n)\,5^n$

(d) $\sum_{n=-\infty}^{\infty} \delta(n)\sin 2n$

2. Sketch the following signals:
(a) $u(n + 2) - u(n)$
(b) $u(-n + 2) - u(-n - 2)$

3. Determine whether the following signals are periodic or not. If periodic, determine the fundamental period.

(a) $\cos\,(0.04\pi n)$

(b) $\sin\frac{4\pi n}{3} + \cos\frac{2n}{3}$

(c) $e^{j(\pi/3)n}$

(d) $\sin\left(\frac{n}{2}\right)\sin\left(\frac{n\pi}{2}\right)$

4. Find which of the following signals are energy signals, power signals, neither energy nor power signals. Calculate the power and energy in each case.

(a) $\left(\frac{1}{3}\right)^n u(n)$

(b) $e^{j[(\pi/2)n + \pi/2]}$

(c) $u(n) - u(n - 4)$

5. Find which of the following signals are causal or non-causal.
(a) $u(-2n)$
(b) $u(n + 3) - u(n + 1)$
(c) e^{3n}

6. Find the even and odd components of the following signals:

(a) $\left\{-2, 4, \underset{\uparrow}{1}, 3, 6\right\}$ (b) $\left\{3, -2, \underset{\uparrow}{4}, 5\right\}$

(c) $\left\{\underset{\uparrow}{2}, 1, 4, 3, 5\right\}$

7. Find whether the following signal is even or odd:

$$u(-n + 2)u(n + 2)$$

8. Find whether the following systems are dynamic or not:

(a) $y(n) = nx^2(n)$ (b) $y(n) = x(n) + x(n + 2)$

(c) $y(n) = nx(2n)$

9. Check whether the following systems are causal or not:

(a) $y(n) = x(n) + \dfrac{1}{2x(n - 2)}$ (b) $y(n) = x(-2n)$

(c) $y(n) = \displaystyle\sum_{k=-\infty}^{n+2} x(k)$

10. Check whether the following systems are linear or not:

(a) $y(n) = Ax(n) + B$ (b) $y(n) = 2x(n) + \dfrac{1}{x(n-3)}$

(c) $y(n) = n^2 x(2n)$

11. Determine whether the following systems are time-invariant or not:

(a) $y(n) = x(n) + nx(n-3)$ (b) $y(n) = x^2(n-2)$

(c) $y(n) = \sin[x(n)]$

12. Determine whether the following systems are stable or not:

(a) $y(n) = 8x(n-4)$ (b) $y(n) = nu(n) + \delta(n-2)$

(c) $h(n) = 2^{-n} u(n)$

13. Check whether the following systems are:

(i) Static or dynamic (ii) Linear or non-linear

(iii) Causal or non-causal (iv) Time-invariant or time-variant

(a) $y(n) = \displaystyle\sum_{k=-\infty}^{n+4} x(k)$ (b) $y(n) = |x(n)|$

(c) $y(n) = 2x(n+2) - x(n-2)$

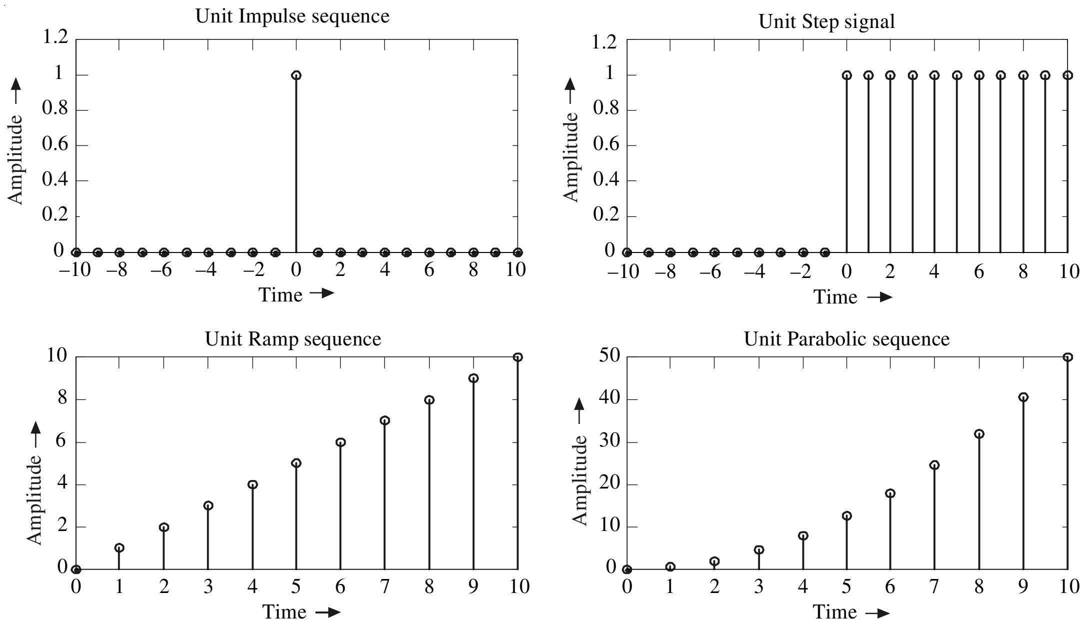
MATLAB PROGRAMS

Program 1.1

% Generation of Elementary signals in Discrete-time

```
clc; close all; clear all;
% Unit Impulse sequence
n=–10:1:10;
impulse=[zeros(1,10),ones(1,1),zeros(1,10)];
subplot(2,2,1);stem(n,impulse);
xlabel('Discrete time n  ——>');ylabel('Amplitude ——>');
title('Unit Impulse sequence');
axis([–10 10 0 1.2]);
% Unit Step sequence
n=–10:1:10;
step=[zeros(1,10),ones(1,11)];
subplot(2,2,2);stem(n,step);
xlabel(''Discrete time n  ——>');ylabel('Amplitude ——>');
title('Unit Step sequence');
axis([–10 10 0 1.2]);
% Unit Ramp sequence
n=0:1:10;
ramp=n;
subplot(2,2,3);stem(n,ramp);
xlabel(''Discrete time n  ——>');ylabel('Amplitude ——>');
title('Unit Ramp sequence');
% Unit Parabolic sequence
n=0:1:10;
parabola=0.5*(n.^2);
subplot(2,2,4);stem(n,parabola);
xlabel(''Discrete time n  ——>');ylabel('Amplitude ——>');
title('Unit Parabolic sequence');
```

Output:



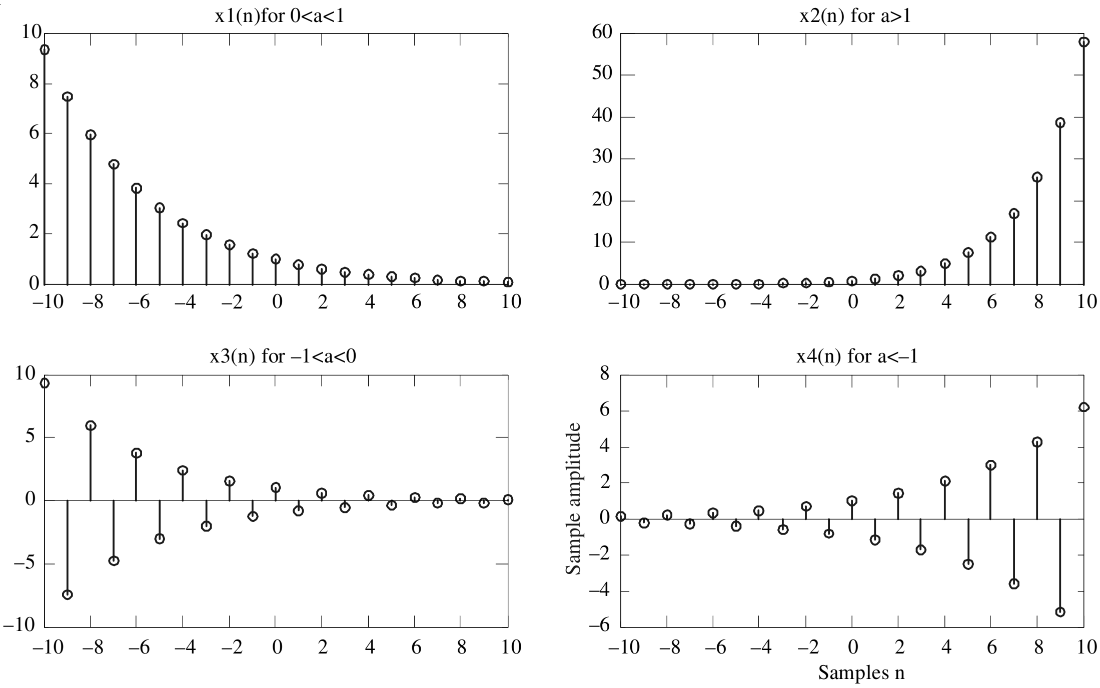
Program 1.2

% Generation of a Discrete-time exponential sequence

```
clc;close all;clear all;
n=–10:1:10;
% for 0<a<1
a=0.8;
x1=a.^n;
subplot(2,2,1);stem(n,x1);
title('x1(n)for 0<a<1');
% for a>1
a=1.5;
x2=a.^n;
subplot(2,2,2);stem(n,x2);
title('x2(n) for a>1');
% for –1<a<0
a=-0.8;
x3=a.^n;
```

```
subplot(2,2,3);stem(n,x3);
title('x3(n) for –1<a<0');
% for a<–1
a=–1.2;
x4=a.^n;
subplot(2,2,4);stem(n,x4);
title('x4(n) for a<–1');
xlabel('samples n');ylabel('sample amplitude')
```

Output:



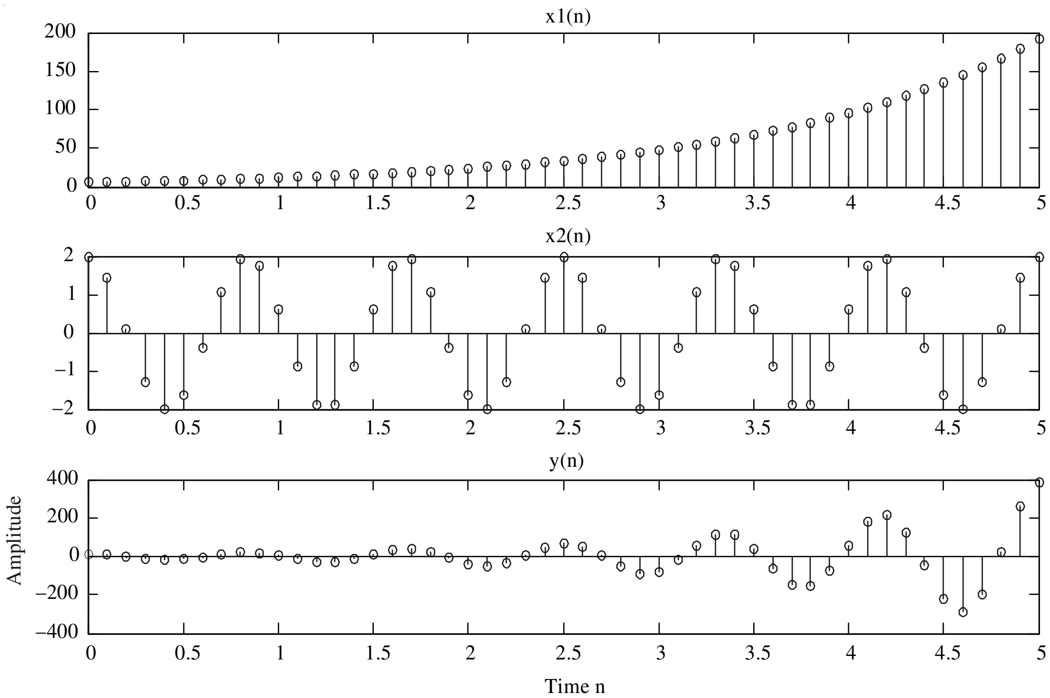
Program 1.3

% Multiplication of Discrete-time signals

```
clc;close all;clear all;
% x1(n)=6*a^n;
n=0:0.1:5;
a=2;
x1=6*(a.^n);
```

```
subplot(3,1,1);stem(n,x1);
title('x1(n)');
% x2(n)=2*cos(wn)
f=1.2;
x2=2*cos(2*pi*f*n);
subplot(3,1,2);stem(n,x2);
title('x2(n)');
% multiplication of two sequences
y=x1.*x2;
subplot(3,1,3);stem(n,y);
xlabel('time n');ylabel('amplitude');
title('y(n)');
```

Output:

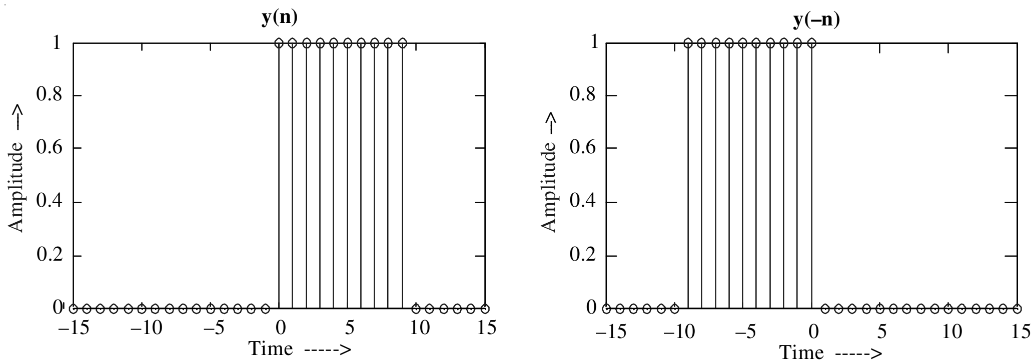


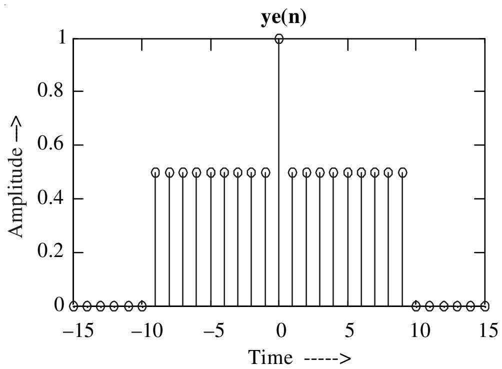
Program 1.4

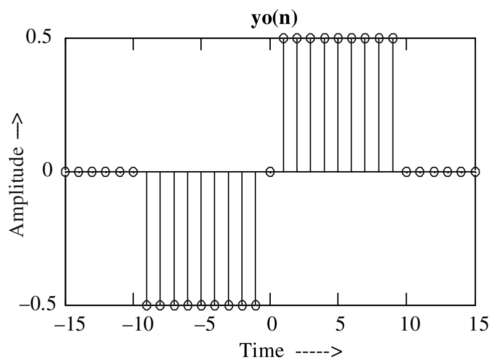
% Even and Odd components of the sequence y(n)=u(n)–u(n–10)

```
n=–15:1:15;
y1=[zeros(1,15),ones(1,10),zeros(1,6)];
y2=fliplr(y1);
ye=0.5*(y1+y2);
yo=0.5*(y1–y2);
subplot(2,2,1);stem(n,y1);
xlabel('time ——>');ylabel('Amplitude ——>');
title('y(n)');
subplot(2,2,2);stem(n,y2);
xlabel('time ——>');ylabel('Amplitude ——>');
title('y(–n)');
subplot(2,2,3);stem(n,ye);
xlabel('time ——>');ylabel('Amplitude ——>');
title('ye(n)');
subplot(2,2,4);stem(n,yo);
xlabel('time ——>');ylabel('Amplitude ——>');
title('yo(n)');
```

Output:





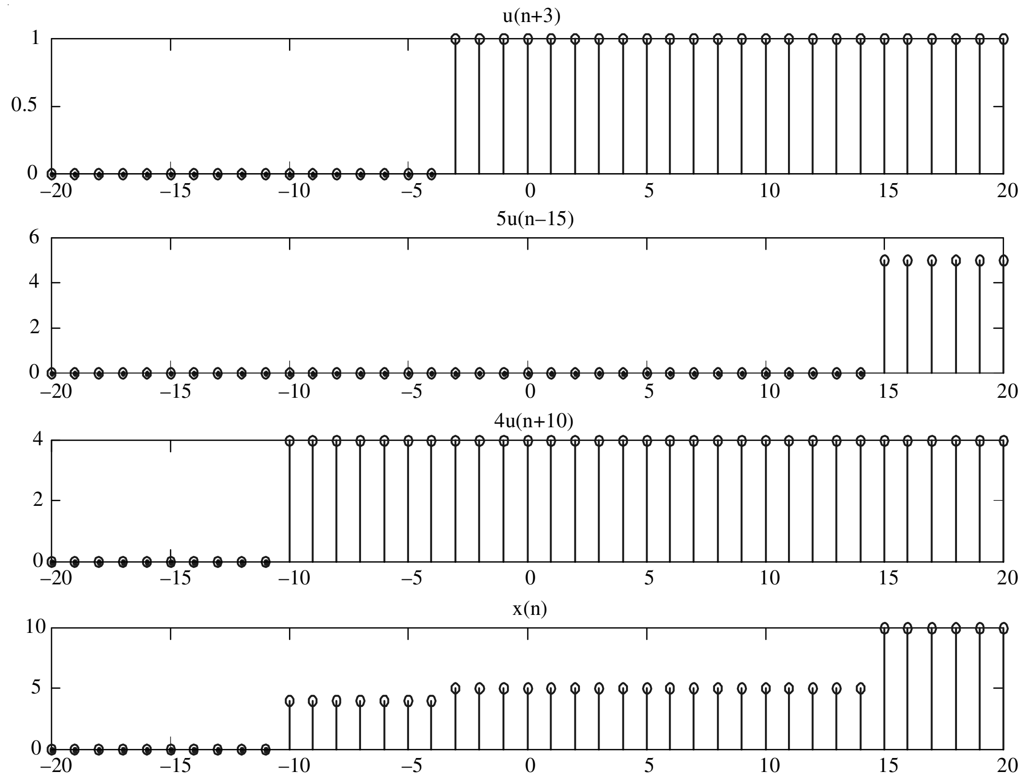


Program 1.5

% Generation of the composite sequence x(n)=u(n+3)+5u(n-15)+4u(n+10)

```
clc;close all;clear all;
n=–20:1:20;
u=[zeros(1,20),ones(1,21)];
u1=[zeros(1,17),ones(1,24)];
u2=[zeros(1,35),ones(1,6)];u2=5*u2;
u3=[zeros(1,10),ones(1,31)];u3=4*u3;
x=u1+u2+u3;
subplot(4,1,1);stem(n,u1);
title('u(n+3)');
subplot(4,1,2);stem(n,u2);
title('5u(n–15)');
subplot(4,1,3);stem(n,u3);
title('4u(n+10)');
subplot(4,1,4);stem(n,x);
title('x(n)');
```

Output:

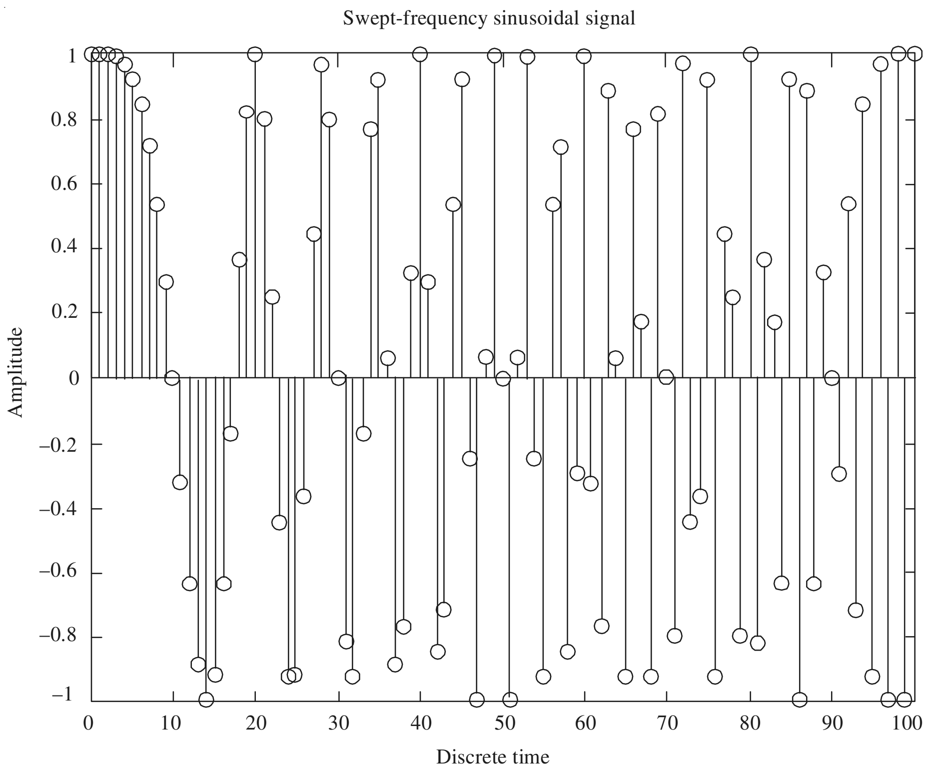


Program 1.6

% Generation of swept frequency sinusoidal signal

```
clc; clear all; close all;
n=0:100;
a=pi/2/100;
b=0;
arg=a*n.*n+b*n;
x=cos(arg);
stem(n,x)
xlabel('Discrete time')
ylabel('Amplitude')
title('Swept-frequency sinusoidal signal')
```

Output:



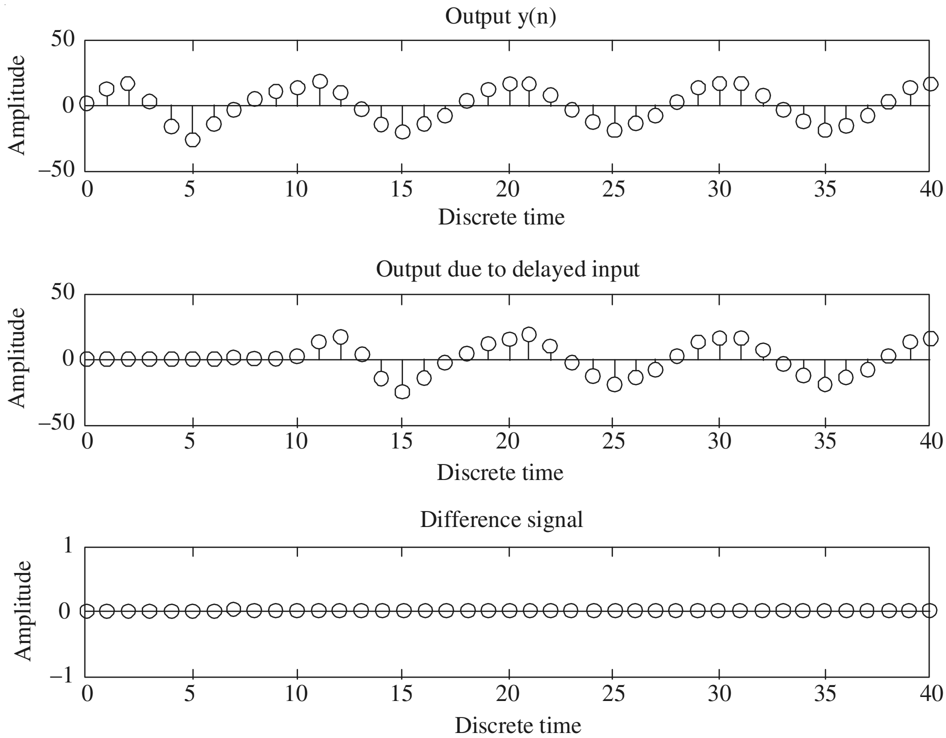
Program 1.7

% Checking the Time-invariance property

```
clc; clear all; close all;
n=0:40;D=10;
x=3*cos(2*pi*0.1*n)–2*cos(2*pi*0.4*n);
xd=[zeros(1,D) x];
num=[2.2403 2.4908 2.2403];
den=[1 –0.4 0.75];
ic=[0 0];
y=filter(num,den,x,ic);
yd=filter(num,den,xd,ic);
d=y–yd(1+D:41+D);
subplot(3,1,1),stem(n,y);
xlabel('Discrete time')
ylabel('Amplitude')
title('output y[n]')
```

```
subplot(3,1,2),stem(n,yd(1:41));
xlabel('Discrete time')
ylabel('Amplitude')
title('output due to delayed input')
subplot(3,1,3),stem(n,d);
xlabel('Discrete time');ylabel('Amplitude')
title('difference signal')
```

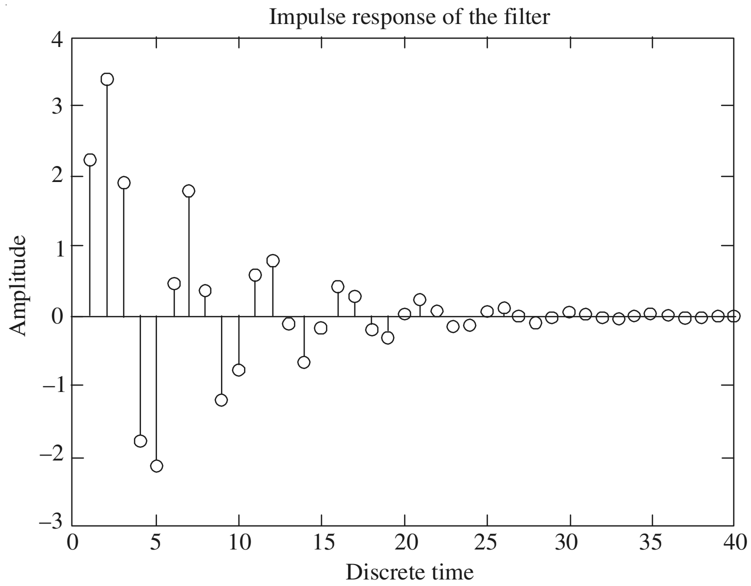
Output:



Program 1.8

```
% Computation of impulse response
clc; clear all; close all;
N=40;
num=[2.2403 2.4908 2.2403];
den=[1 –0.4 0.75];
y=impz(num,den,N);
stem(y);
xlabel('Discrete time')
ylabel('Amplitude')
title('Impulse response of the filter')
```

Output:



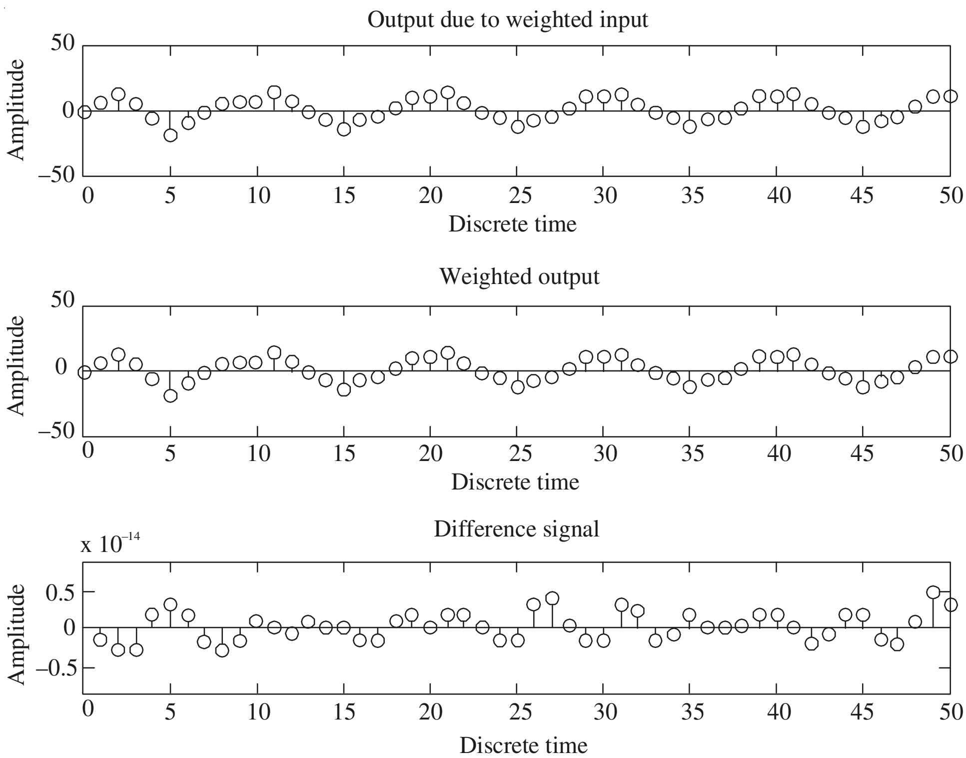
Program 1.9

% Checking the linearity of a system

```
clc; clear all; close all;
n=0:50;a=2;b=–3;
x1=cos(2*pi*0.1*n);
x2=cos(2*pi*0.4*n);
x=a*x1+b*x2;
num=[2.2403 2.4908 2.2403];
den=[1 –0.4 0.75];
ic=[0 0];
y1=filter(num,den,x1,ic);
y2=filter(num,den,x2,ic);
y=filter(num,den,x,ic);
yt=a*y1+b*y2;
d=y–yt;
subplot(3,1,1);stem(n,y);
xlabel('Discrete time')
ylabel('Amplitude')
```

```
title('output due to weighted input')
subplot(3,1,2);stem(n,yt);
xlabel('Discrete time')
ylabel('Amplitude')
title('Weighted output')
subplot(3,1,3);stem(n,d);
xlabel('Discrete time')
ylabel('Amplitude')
title('difference signal')
```

Output:

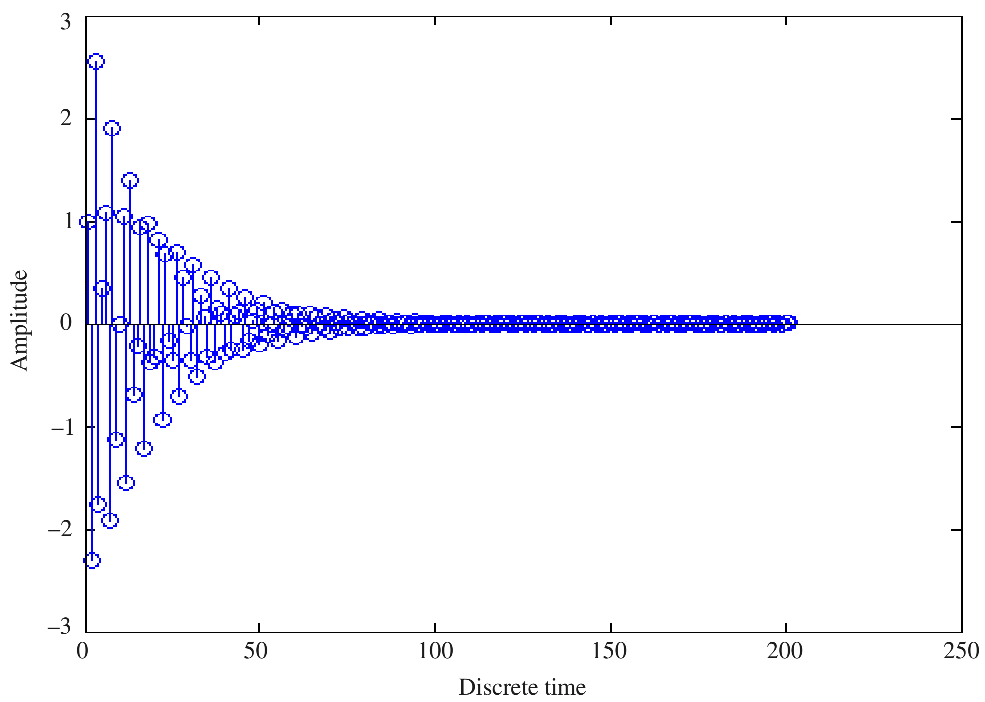


Program 1.10

% Testing the stability of a system

```
clc; clear all; close all;
num=[1 –0.8];
den=[1 1.5 0.9];
N=200;
h=impz(num,den,N+1);
parsum=0;
for k=N+1
    parsum=parsum+abs(h(k));
    if abs(h(k))<10^(–6)
        break
    end
end
stem(h);
xlabel('Discrete time')
ylabel('Amplitude')
disp('Value=')
disp(abs(h(k)))
```

Output:



2

Discrete Convolution and Correlation

2.1 INTRODUCTION

Convolution is a mathematical operation equivalent to finite impulse response (FIR) filtering. It is a method of finding the zero-state response of relaxed linear time-invariant systems. It is important in digital signal processing because convolving two sequences in time domain is equivalent to multiplying the sequences in frequency domain. It is based on the concepts of linearity and time invariance and assumes that the system information is known in terms of impulse response. Correlation is a measure of similarity between two signals and is found using a process similar to convolution. There are two types of correlation: cross correlation and autocorrelation. In this chapter, various methods of evaluating the discrete convolution of finite duration sequences, and periodic sequences are discussed. The discrete cross correlation and autocorrelation and evaluation of these are also discussed.

2.2 IMPULSE RESPONSE AND CONVOLUTION SUM

A discrete-time system performs an operation on an input signal based on a predefined criterion to produce a modified output signal. The input signal $x(n)$ is the system excitation and the output signal $y(n)$ is the system response. This transform operation is shown in Figure 2.1.

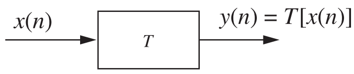


Figure 2.1 A discrete-time system.

If the input to the system is a unit impulse, i.e. $x(n) = \delta(n)$, then the output of the system is known as impulse response denoted by $h(n)$ where

$$h(n) = T[\delta(n)]$$

The system is assumed to be initially relaxed, i.e. the system has zero initial conditions.

We know that any arbitrary sequence $x(n)$ can be represented as a weighted sum of discrete impulses as:

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

So the system response $y(n)$ is given by

$$y(n) = T[x(n)] = T\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right]$$

For a linear system, the above equation for $y(n)$ reduces to

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)T[\delta(n-k)]$$

The response due to shifted impulse sequence $\delta(n - k)$ can be denoted by $h(n, k)$, i.e.

$$h(n, k) = T[\delta(n - k)]$$

For a shift-invariant system,

Delayed output = Output due to delayed input

i.e. $$h(n - k) = h(n, k)$$

Therefore, $$T[\delta(n - k)] = h(n - k)$$

Therefore, the equation for $y(n)$ reduces to

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

So we can conclude that:

For a linear time invariant system, if the input sequence $x(n)$ and the impulse response $h(n)$ are given, the output sequence $y(n)$ can be found using the equation:

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k)$$

This is known as convolution sum and is represented as

$$y(n) = x(n) * h(n) = h(n) * x(n)$$

where $*$ denotes the convolution operation.

This is an extremely powerful and useful result that allows us to compute the system output for any input signal excitation.

2.3 ANALYTICAL EVALUATION OF DISCRETE CONVOLUTION

If $x(n)$ and $h(n)$ are described by simple enough analytical expressions, the convolution sum can be implemented quite readily to obtain closed-form results. While evaluating the convolution sum, it is useful to keep in mind that $x(k)$ and $h(n-k)$ are functions of the summation variable k. For causal signals of the form $x(n)u(n)$ and $h(n)u(n)$, the summation involves step functions of the form $u(k)$ and $u(n-k)$. Since $u(k) = 0$ for $k < 0$ and $u(n-k) = 0$ for $k > n$, these can be used to simplify the lower and upper summation limits to $k = 0$ and $k = n$, respectively.

The limits in the convolution sum can be modified according to the type of sequence and system.

For a causal system, $h(n) = 0$ for $n < 0$

and for a causal input, $x(n) = 0$ for $n < 0$

If $h(n)$ is the impulse response and $x(n)$ is the input, then
For a non-causal system excited by a non-causal input,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

For a non-causal system excited by a causal input,

$$y(n) = \sum_{k=0}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{n} h(k)x(n-k)$$

For a causal system excited by a non-causal input,

$$y(n) = \sum_{k=-\infty}^{n} x(k)h(n-k) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

For a causal system excited by a causal input,

$$y(n) = \sum_{k=0}^{n} x(k)h(n-k) = \sum_{k=0}^{n} h(k)x(n-k)$$

Properties of convolution

1. Commutative property: $x(n) * h(n) = h(n) * x(n)$
2. Associative property: $[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$
3. Distributive property: $x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$
4. Shifting property:
   If $x(n) * h(n) = y(n)$, then $x(n-k) * h(n-m) = y(n-k-m)$
5. Convolution with an impulse: $x(n) * \delta(n) = x(n)$

EXAMPLE 2.1 Find the convolution of two finite duration sequences:

$$h(n) = a^n u(n) \quad \text{for all } n$$
$$x(n) = b^n u(n) \quad \text{for all } n$$

(i) When $a \neq b$
(ii) When $a = b$

Solution: The impulse response $h(n)$ and the input $x(n)$ are zero for $n < 0$, i.e. both $h(n)$ and $x(n)$ are causal.

$$\therefore \qquad y(n) = \sum_{k=0}^{n} x(k)h(n-k)$$

$$= \sum_{k=0}^{n} b^k a^{(n-k)} = a^n \sum_{k=0}^{n} \left(\frac{b}{a}\right)^k$$

$$= a^n \left[\frac{1 - \left(\frac{b}{a}\right)^{n+1}}{1 - \left(\frac{b}{a}\right)}\right] \qquad [\text{when } a \neq b]$$

When $a = b$

$$y(n) = a^n[1 + 1 + 1 + \cdots + n + 1 \text{ terms}] = a^n(n + 1)$$

EXAMPLE 2.2 Find $y(n)$ if $x(n) = n + 3 \quad \text{for } 0 \le n \le 2$

$$h(n) = a^n u(n) \quad \text{for all } n$$

Solution: We have

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

Given

$$x(n) = n + 3 \quad \text{for } 0 \le n \le 2$$
$$h(n) = a^n u(n) \quad \text{for all } n$$

$h(n) = 0$ for $n < 0$, so the system is causal. $x(n)$ is a causal finite duration sequence whose value is zero for $n > 2$. Therefore,

$$y(n) = \sum_{k=0}^{2} x(k)h(n-k)$$

$$= \sum_{k=0}^{2} (k + 3)a^{n-k}u(n-k)$$

$$= 3a^n u(n) + 4a^{n-1}u(n-1) + 5a^{n-2}u(n-2)$$

EXAMPLE 2.3 Determine the response of the system characterized by the impulse response $h(n) = (1/3)^n u(n)$ to the input signal $x(n) = 3^n u(n)$.

Solution: Given $\qquad x(n) = 3^n u(n) \quad \text{and} \quad h(n) = \left(\dfrac{1}{3}\right)^n u(n)$

A causal signal is applied to a causal system

$$\therefore \qquad y(n) = \sum_{k=0}^{n} x(k)h(n-k) = \sum_{k=0}^{n} 3^k \left(\frac{1}{3}\right)^{n-k}$$

$$= \left(\frac{1}{3}\right)^n \sum_{k=0}^{n} 3^k \times 3^k$$

$$= \left(\frac{1}{3}\right)^n \sum_{k=0}^{n} (3^2)^k$$

$$= \left(\frac{1}{3}\right)^n \left[\frac{1 - (3^2)^{n+1}}{1-3^2}\right]$$

$$= \left(\frac{1}{3}\right)^n \left[\frac{9^{n+1} - 1}{9 - 1}\right] = \left(\frac{1}{3}\right)^n \left[\frac{9^{n+1} - 1}{8}\right]$$

EXAMPLE 2.4 Determine the response of the system characterized by the impulse response $h(n) = 2^n u(n)$ for an input signal $x(n) = 3^n u(n)$.

Solution: Given $\qquad x(n) = 3^n u(n) \quad \text{and} \quad h(n) = 2^n u(n)$

Since both $x(n)$ and $h(n)$ are causal, we have

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=0}^{n} x(k)h(n-k)$$

$$= \sum_{k=0}^{n} 3^k \cdot 2^{n-k} = 2^n \sum_{k=0}^{n} \left(\frac{3}{2}\right)^k = 2^n \left[\frac{1 - \left(\dfrac{3}{2}\right)^{n+1}}{1 - \dfrac{3}{2}}\right]$$

$$= 2^n \left[\frac{\left(\dfrac{3}{2}\right)^{n+1} - 1}{\dfrac{1}{2}}\right] = 2^{n+1}\left[\left(\frac{3}{2}\right)^{n+1} - 1\right]$$

EXAMPLE 2.5 Find the convolution of

$$x(n) = \cos n\pi \, u(n), \;\; h(n) = \left(\frac{1}{2}\right)^n u(n)$$

Solution: Given $x(n) = \cos n\pi \, u(n) = (-1)^n u(n), \, h(n) = \left(\frac{1}{2}\right)^n u(n)$

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} (-1)^k u(k) \left(\frac{1}{2}\right)^{n-k} u(n-k)$$

$$= \sum_{k=0}^{n} (-1)^k \left(\frac{1}{2}\right)^{n-k} = \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} (-1)^k (2)^k$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} (-2)^k = \left(\frac{1}{2}\right)^n \frac{1 - (-2)^{n+1}}{1 - (-2)}$$

$$= \left(\frac{1}{2}\right)^n \left[\frac{1 + 2(-2)^n}{3}\right] = \frac{1}{3}\left(\frac{1}{2}\right)^n + \frac{2}{3}(-1)^n \quad \text{for } n > 0$$

$$= \frac{1}{3}\left(\frac{1}{2}\right)^n u(n) + \frac{2}{3}(-1)^n u(n)$$

EXAMPLE 2.6 Find the convolution of

$$x(n) = u(n), \quad h(n) = u(n-3)$$

Solution: Given $x(n) = u(n), \quad h(n) = u(n-3)$

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} u(k)\, u(n-3-k)$$

$$= \sum_{k=0}^{n-3} (1)(1) = n - 2$$

EXAMPLE 2.7 Consider a system with unit sample response

$$h(n) = \begin{cases} a^n, & n \ge 0 \\ 0, & n < 0 \end{cases}, \text{ or equivalently } h(n) = a^n u(n).$$

Find the response to an input $x(n) = u(n) - u(n-N)$.

Solution: We know that the output $y(n)$ is given by

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

From this we can observe that to obtain the nth value of the output sequence we must form the product $x(k)\,h(n-k)$ and sum the values of the resulting sequence. The two component sequences are shown in Figure 2.2 as a function of k, with $h(n-k)$ shown for several values of n.

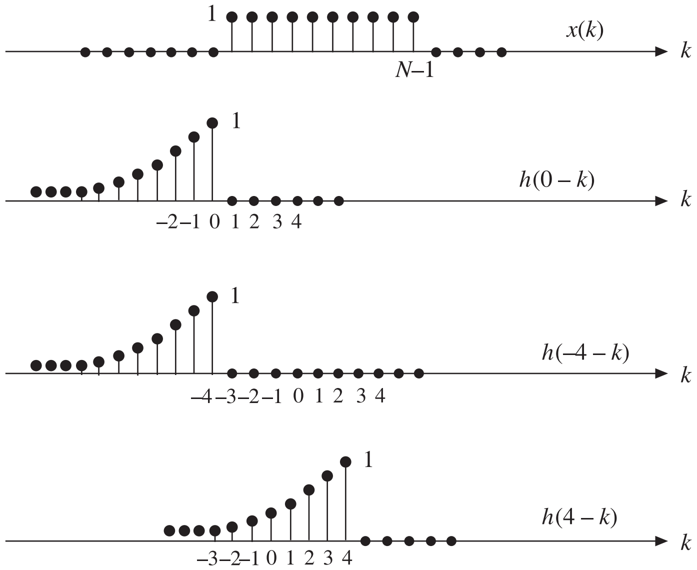


Figure 2.2 Component sequences in evaluating the convolution sum with $h(n-k)$ shown for several values of n.

As we see in Figure 2.2, for $n < 0$, $h(n-k)$ and $x(k)$ have no nonzero samples that overlap, and consequently $y(n) = 0$, $n < 0$. For n greater than or equal to zero but less than N, $h(n-k)$ and $x(k)$ have nonzero samples that overlap from $k = 0$ to $k = n$; thus for $0 \le n < N$,

$$y(n) = \sum_{k=0}^{n} a^{n-k} = a^n \frac{1 - a^{-(n+1)}}{1 - a^{-1}} = \frac{1 - a^{n+1}}{1 - a}, \quad 0 \le n < N$$

For $N - 1 \le n$, the nonzero samples that overlap extend from $k = 0$ to $k = N - 1$ and thus

$$y(n) = \sum_{k=0}^{N-1} a^{n-k} = a^n \frac{1 - a^{-N}}{1 - a^{-1}} = a^{n-(N-1)} \left[\frac{1 - a^N}{1 - a}\right], \quad N \le n$$

The response $y(n)$ is sketched in Figure 2.3.

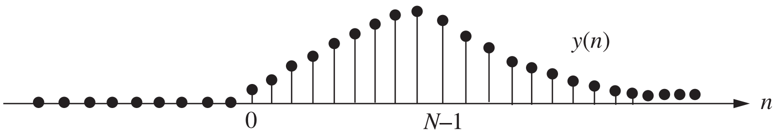


Figure 2.3 Response of the system with unit sample response $h(n) = a^n u(n)$ to the input $u(n) - u(n - N)$.

EXAMPLE 2.8 Find the convolution of

$$x(n) = \left(\frac{1}{2}\right)^n u(n), \quad h(n) = u(n) - u(n-10)$$

***Solution*:** Given $x(n) = \left(\frac{1}{2}\right)^n u(n), \quad h(n) = u(n) - u(n-10)$

For $n < 10$, i.e., for $n \le 9$,

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u(k)\,(1) \quad \text{for } 0 \le n \le 9$$

$$= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \qquad 0 \le n \le 9$$

$$= \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} = \left(\frac{1}{2}\right)^n \left[\frac{1 - \left(\frac{1}{2}\right)^{-(n+1)}}{1 - \left(\frac{1}{2}\right)^{-1}}\right]$$

$$= \left(\frac{1}{2}\right)^n [2^{n+1} - 1] \qquad 0 \le n \le 9$$

For $n \ge 10$, i.e., for $n > 9$,

$$y(n) = \sum_{k=n-9}^{n} \left(\frac{1}{2}\right)^k$$

$$= \left(\frac{1}{2}\right)^{n-9} + \left(\frac{1}{2}\right)^{n-9+1} + \cdots + \left(\frac{1}{2}\right)^{n-9+9}$$

$$= \left(\frac{1}{2}\right)^{n-9} \left[1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^9\right]$$

$$= \left(\frac{1}{2}\right)^{n-9} \left[\frac{1 - \left(\frac{1}{2}\right)^{9+1}}{1 - \frac{1}{2}}\right] \qquad \text{for } n > 9$$

$$= 2\left(\frac{1}{2}\right)^{n-9} - 2\left(\frac{1}{2}\right)^{n+1}$$

$$= \left(\frac{1}{2}\right)^{n} [2^{10} - 1]$$

EXAMPLE 2.9 Find the convolution of

$$x(n) = \left(\frac{1}{3}\right)^{-n} u(-n - 1) \quad \text{and} \quad h(n) = u(n - 1)$$

Solution: Let $y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$

$x(k) = \left(\frac{1}{3}\right)^{-k} u(-k - 1)$ and $h(k) = u(k-1)$ are plotted as shown in Figure 2.4. From Figure 2.4 [(a) and (b)] we can find that $x(k) = 0$ for $k > -1$ and $h(n - k) = 0$ for $k > n - 1$.

For $n - 1 < - 1$, i.e., for $n < 0$, the interval of summation is from $k = -\infty$ to $n - 1$.

$$\therefore \qquad y(n) = \sum_{k=-\infty}^{n-1} \left(\frac{1}{3}\right)^{-k} \quad \text{for } n-1 \le -1 \text{ or for } n \le 0$$

$$= \left(\frac{1}{3}\right)^{-(n-1)} + \left(\frac{1}{3}\right)^{-(n-2)} + \cdots$$

$$= \left(\frac{1}{3}\right)^{-(n-1)} \left[1 + \frac{1}{3} + \left(\frac{1}{3}\right)^{2} + \cdots\right]$$

$$= 3^{n-1} \left[\frac{1}{1 - 1/3}\right] = 0.5(3)^{n}$$

For $n - 1 > -1$, i.e. for $n > 0$, the interval of summation is from $k = -\infty$ to $k = - 1$.

$$\therefore \qquad y(n) = \sum_{k=-\infty}^{-1} \left(\frac{1}{3}\right)^{-k} = \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^{k} = \frac{1/3}{1 - 1/3} = 0.5$$

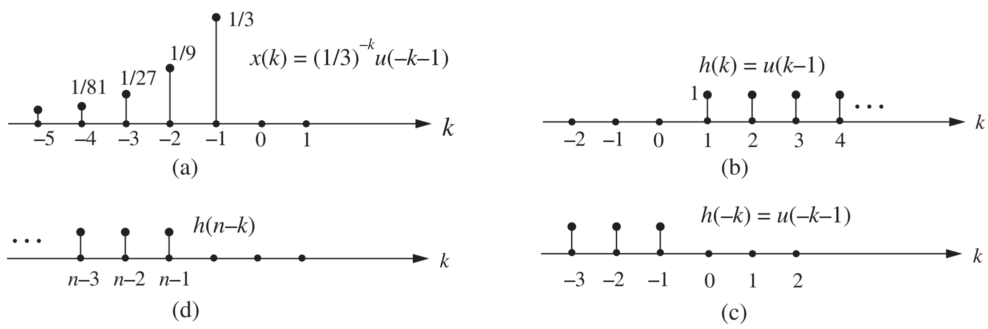


Figure 2.4 Component sequences in evaluating the convolution sum.

Figure 2.5 shows the plot of $y(n)$ for all values of n.

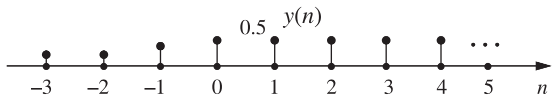


Figure 2.5 Plot of $y(n) = x(n) * h(n)$.

Unit step response

The unit step response can be obtained by exciting the input of the system by a unit step sequence, i.e. $x(n) = u(n)$. It can be easily expressed in terms of the impulse response using convolution sum. Let the impulse response of the discrete-time system be $h(n)$. Then the step response $s(n)$ can be obtained using the convolution sum:

$$s(n) = h(n) * u(n) = \sum_{k=-\infty}^{\infty} h(k)\,u(n-k)$$

Since $u(n-k) = 0$ for $k > n$ and $u(n-k) = 1$ for $k \le n$,

$$s(n) = \sum_{k=-\infty}^{n} h(k)$$

That is the step response is the running sum of the impulse response.

For a causal system,

$$s(n) = \sum_{k=0}^{n} h(k)$$

EXAMPLE 2.10 Evaluate the step response for the LTI system represented by the following impulse response:

(a) $h(n) = \delta(n) - \delta(n-2)$

(b) $h(n) = \left(\frac{1}{4}\right)^n u(n)$

(c) $h(n) = nu(n)$

(d) $h(n) = u(n)$

Solution:

(a) Given $\quad h(n) = \delta(n) - \delta(n-2)$

The step response $\quad s(n) = h(n) * u(n)$

$$= [\delta(n) - \delta(n-2)] * u(n)$$

$$= \delta(n) * u(n) - \delta(n-2) * u(n)$$

$$= u(n) - u(n-2)$$

(b) Given $\quad h(n) = \left(\frac{1}{4}\right)^n u(n)$

$$s(n) = \left(\frac{1}{4}\right)^n u(n) * u(n) = \sum_{k=-\infty}^{\infty} u(k) \left(\frac{1}{4}\right)^{n-k} u(n-k)$$

$$= \sum_{k=0}^{n} \left(\frac{1}{4}\right)^{n-k} = \left(\frac{1}{4}\right)^n \sum_{k=0}^{n} \left(\frac{1}{4}\right)^{-k}$$

$$= \left(\frac{1}{4}\right)^n \sum_{k=0}^{n} 4^k$$

$$= \left(\frac{1}{4}\right)^n \left[\frac{1-4^{n+1}}{1-4}\right]$$

(c) Given $\quad h(n) = nu(n)$

$$s(n) = h(n) * u(n)$$

$$= nu(n) * u(n)$$

$$= \sum_{k=0}^{n} k\, u(k) u(n-k)$$

$$= \sum_{k=0}^{n} k$$

(d) Given $\quad h(n) = u(n)$

$$s(n) = u(n) * u(n) = \sum_{k=-\infty}^{\infty} u(k) u(n-k)$$

$$= \sum_{k=0}^{n} 1 = n+1$$

$\therefore \qquad s(n) = (n + 1)$

2.4 CONVOLUTION OF FINITE SEQUENCES

In practice, we often deal with sequences of finite length, and their convolution may be found by several methods. The convolution $y(n)$ of two finite-length sequences $x(n)$ and $h(n)$ is also of finite length and is subject to the following rules, which serve as useful consistency checks:

1. The starting index of $y(n)$ equals the sum of the starting indices of $x(n)$ and $h(n)$.
2. The ending index of $y(n)$ equals the sum of the ending indices of $x(n)$ and $h(n)$.
3. The length L_y of $y(n)$ is related to the lengths L_x and L_h of $x(n)$ and $h(n)$ by $L_y = L_x + L_h - 1$.

2.5 METHODS TO COMPUTE THE CONVOLUTION SUM OF TWO SEQUENCES *x*(*n*) AND *h*(*n*)

2.5.1 Method 1 Linear Convolution Using Graphical Method

Step 1: Choose the starting time n for evaluating the output sequence $y(n)$. If $x(n)$ starts at $n = n_1$ and $h(n)$ starts at $n = n_2$, then $n = n_1 + n_2$ is a good choice.

Step 2: Express both the sequences $x(n)$ and $h(n)$ in terms of the index k.

Step 3: Fold $h(k)$ about $k = 0$ to obtain $h(-k)$ and shift by n to the right if n is positive and to the left if n is negative to obtain $h(n - k)$.

Step 4: Multiply the two sequences $x(k)$ and $h(n - k)$ element by element and sum the products to get $y(n)$.

Step 5: Increment the index n, shift the sequence $h(n - k)$ to the right by one sample and perform Step 4.

Step 6: Repeat Step 5 until the sum of products is zero for all remaining values of n.

2.5.2 Method 2 Linear Convolution Using Tabular Array

Let $x_1(n)$ and $x_2(n)$ be the given N sample sequences. Let $x_3(n)$ be the N sample sequence obtained by linear convolution of $x_1(n)$ and $x_2(n)$. The following procedure can be used to obtain one sample of $x_3(n)$ at $n = q$:

Step 1: Change the index from n to k, and write $x_1(k)$ and $x_2(k)$.

Step 2: Represent the sequences $x_1(k)$ and $x_2(k)$ as two rows of tabular array.

Step 3: Fold one of the sequences. Let us fold $x_2(k)$ to get $x_2(-k)$.

Step 4: Shift the sequence $x_2(-k)$, q times to get the sequence $x_2(q-k)$. If q is positive, then shift the sequence to the right and if q is negative, then shift the sequence to the left.

Step 5: The sample of $x_3(n)$ at $n = q$ is given by

$$x_3(q) = \sum_{k=0}^{N-1} x_1(k)\,x_2(q-k)$$

Determine the product sequence $x_1(k)x_2(q-k)$ for one period.

Step 6: The sum of the samples of the product sequence gives the sample $x_3(q)$ [i.e. $x_3(n)$ at $n = q$].

The above procedure is repeated for all possible values of n to get the sequence $x_3(n)$.

2.5.3 Method 3 Linear Convolution Using Tabular Method

Given $\quad x(n) = \{x_1, x_2, x_3, x_4\}$, $h(n) = \{h_1, h_2, h_3, h_4\}$

The convolution of $x(n)$ and $h(n)$ can be computed as per the following procedure.

Step 1: Write down the sequences $x(n)$ and $h(n)$ as shown in Table 2.1.

Step 2: Multiply each and every sample in $h(n)$ with the samples of $x(n)$ and tabulate the values.

Step 3: Group the elements in the table by drawing diagonal lines as shown in table.

Step 4: Starting from the left sum all the elements in each strip and write down in the same order.

$$\begin{aligned} y(n) = {} & x_1h_1,\ x_1h_2 + x_2h_1,\ x_1h_3 + x_2h_2 + x_3h_1,\ x_1h_4 + x_2h_3 + x_3h_2 \\ & + x_4h_1,\ x_2h_4 + x_3h_3 + x_4h_2,\ x_3h_4 + x_4h_3,\ x_4h_4 \end{aligned}$$

Step 5: Mark the symbol $\uparrow$ at time origin ($n = 0$).

TABLE 2.1 Table for Computing $y(n)$

$h(n)$ \ $x(n)$	x_1	x_2	x_3	x_4
h_1	x_1h_1	x_2h_1	x_3h_1	x_4h_1
h_2	x_1h_2	x_2h_2	x_3h_2	x_4h_2
h_3	x_1h_3	x_2h_3	x_3h_3	x_4h_3
h_4	x_1h_4	x_2h_4	x_3h_4	x_4h_4

2.5.4 Method 4 Linear Convolution Using Matrices

If the number of elements in $x(n)$ are N_1 and in $h(n)$ are N_2, then to find the convolution of $x(n)$ and $h(n)$ form the following matrices:

1. Matrix H of order $(N_1 + N_2 - 1) \times N_1$ with the elements of $h(n)$
2. A column matrix X of order $(N_1 \times 1)$ with the elements of $x(n)$
3. Multiply the matrices H and X to get a column matrix Y of order $(N_1 + N_2 - 1)$ that has the elements of $y(n)$, the convolution of $x(n)$ and $h(n)$.

$$\begin{bmatrix} h(0) & 0 & \cdots & 0 \\ h(1) & h(0) & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ h(N_2-1) & h(N_2-2) & \cdots & h(0) \\ 0 & h(N_2-1) & \cdots & h(1) \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & h(N_2-1) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ x(N_1-1) \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ y(N_1+N_2-1) \end{bmatrix}$$

$$H \qquad X = Y$$

2.5.5 Method 5 Linear Convolution Using the Sum-by Column Method

This method is based on the idea that the convolution $y(n)$ equals the sum of the (shifted) impulse responses due to each of the impulses that make up the input $x(n)$. To find the convolution, we set up a row of index values beginning with the starting index of the convolution and $h(n)$ and $x(n)$ below it. We regard $x(n)$ as a sequence of weighted shifted impulses. Each element (impulse) of $x(n)$ generates a shifted impulse response [Product with $h(n)$], starting at its index (to indicate the shift). Summing the response (by columns) gives the discrete convolution. Note that none of the sequences is flipped. It is better (if only to save paper) to let $x(n)$ be the shorter sequence. The starting index (and the marker location corresponding to $n = 0$) for the convolution $y(n)$ is found from the starting indices of $x(n)$ and $h(n)$.

2.5.6 Method 6 Linear Convolution Using the Flip, Shift, Multiply, and Sum Method

The convolution sum may also be interpreted as follows. We flip $x(n)$ to generate $x(-n)$ and shift the flipped signal $x(-n)$ to line up its last element with the first element of $h(n)$. We then successively shift $x(-n)$ (to the right) past $h(n)$, one index at a time, and find the convolution at each index as the sum of the pointwise products of the overlapping samples. One method of computing $y(n)$ is to list the values of the flipped function on a strip of paper and slide it along the stationary function, to better visualize the process. This technique has

prompted the name sliding strip method. We simulate this method by showing the successive positions of the stationary and flipped sequence along with the resulting products, the convolution sum, and the actual convolution.

EXAMPLE 2.11 Determine the convolution sum of two sequences:

$$x(n) = \{4, 2, 1, 3\}, \qquad h(n) = \left\{\begin{matrix} 1, 2, 2, 1 \\ \uparrow \end{matrix}\right\}$$

Solution:
$x(n)$ starts at $n_1 = 0$ and $h(n)$ starts at $n_2 = -1$. Therefore, the starting sample of $y(n)$ is at

$$n = n_1 + n_2 = 0 - 1 = -1$$

$x(n)$ has 4 samples, $h(n)$ has 4 samples. Therefore, $y(n)$ will have $N = 4 + 4 - 1 = 7$ samples, i.e., from $n = -1$ to $n = 5$.

Method 1 Graphical method

We know that $$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

From Figure 2.6, we get

For $n = -1$ $$y(-1) = \sum_{k=-\infty}^{\infty} x(k)\,h(-1-k) = 4 \cdot 1 = 4$$

For $n = 0$ $$y(0) = \sum_{k=-\infty}^{\infty} x(k)\,h(-k) = 4 \cdot 2 + 2 \cdot 1 = 10$$

For $n = 1$ $$y(1) = \sum_{k=-\infty}^{\infty} x(k)\,h(1-k) = 4 \cdot 2 + 2 \cdot 2 + 1 \cdot 1 = 13$$

For $n = 2$ $$y(2) = \sum_{k=-\infty}^{\infty} x(k)\,h(2-k) = 4 \cdot 1 + 2 \cdot 2 + 1 \cdot 2 + 3 \cdot 1 = 13$$

For $n = 3$ $$y(3) = \sum_{k=-\infty}^{\infty} x(k)\,h(3-k) = 2 \cdot 1 + 1 \cdot 2 + 3 \cdot 2 = 10$$

For $n = 4$ $$y(4) = \sum_{k=-\infty}^{\infty} x(k)\,h(4-k) = 1 \cdot 1 + 3 \cdot 2 = 7$$

For $n = 5$ $\qquad y(5) = \sum_{k=-\infty}^{\infty} x(k)h(5-k) = 3 \cdot 1 = 3$

$$\therefore \qquad y(n) = \left\{4, \underset{\uparrow}{10}, 13, 13, 10, 7, 3\right\}$$

To check the correctness of the result sum all the samples in $x(n)$ and multiply with the sum of all samples in $h(n)$. This value must be equal to sum of all samples in $y(n)$.

In the given problem, $\sum_n x(n) = 10$, $\sum_n h(n) = 6$ and $\sum_n y(n) = 60$

This shows $\sum_n x(n) \cdot \sum_n h(n) = \sum_n y(n)$ (proved). Therefore, the result is correct.

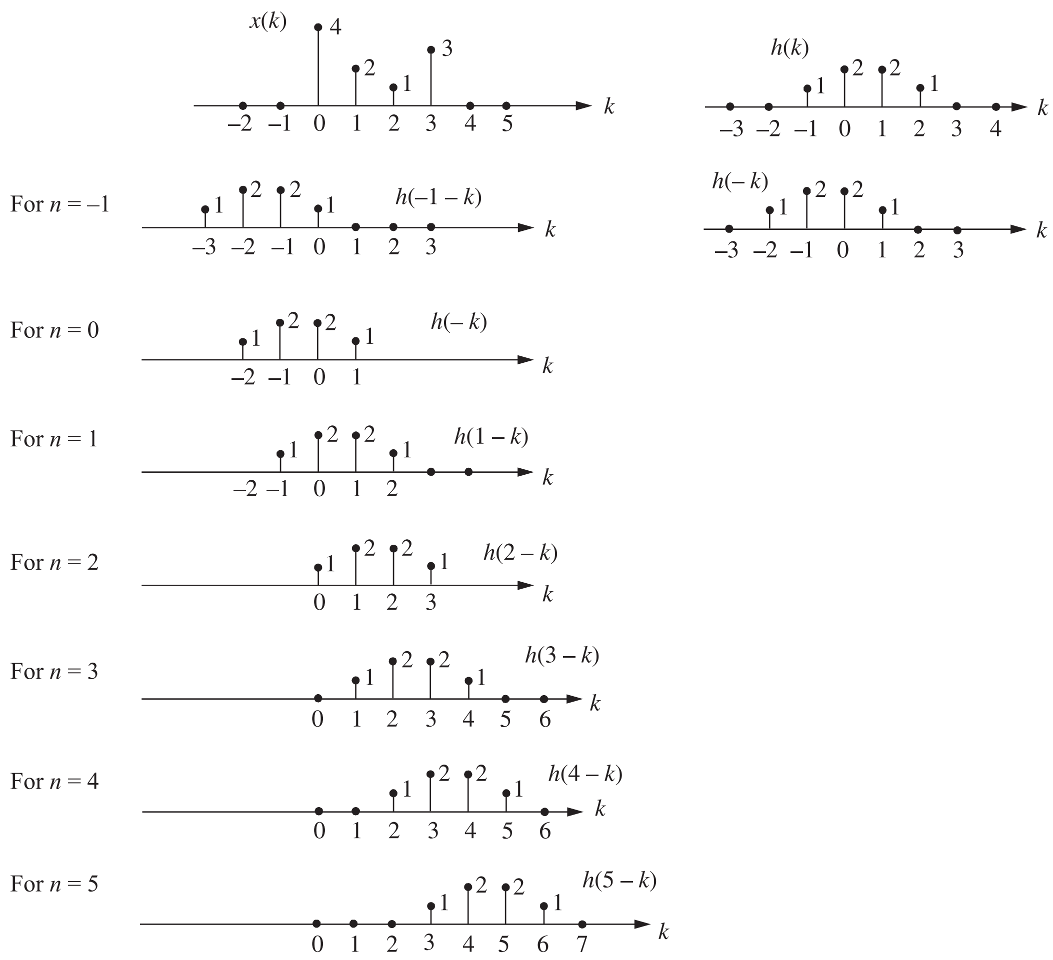


Figure 2.6 Operation on signals $x(n)$ and $h(n)$ to compute convolution.

Method 2 *Tabular array*

Tabulate the sequence $x(k)$ and shifted version of $h(k)$ as shown in Table 2.2.

TABLE 2.2 Table for computing $y(n)$.

k		–4	–3	–2	–1	0	1	2	3	4	5	6	7
$x(k)$		–	–	–	–	4	2	1	3	–	–	–	–
$h(-k)$		–	–	1	2	2	1	–	–	–	–	–	–
$n = -1$	$h(-1-k)$	–	1	2	2	1	–	–	–	–	–	–	–
$n = 0$	$h(-k)$	–	–	1	2	2	1	–	–	–	–	–	–
$n = 1$	$h(1-k)$	–	–	–	1	2	2	1	–	–	–	–	–
$n = 2$	$h(2-k)$	–	–	–	–	1	2	2	1	–	–	–	–
$n = 3$	$h(3-k)$	–	–	–	–	–	1	2	2	1	–	–	–
$n = 4$	$h(4-k)$	–	–	–	–	–	–	1	2	2	1	–	–
$n = 5$	$h(5-k)$	–	–	–	–	–	–	–	1	2	2	1	–

The starting value of $n = -1$. From the table, we can see that

For $n = -1$ $\quad y(-1) = \sum_{k=-\infty}^{\infty} x(k)h(-1-k) = 4 \cdot 1 = 4$

For $n = 0$ $\quad y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k) = 4 \cdot 2 + 2 \cdot 1 = 10$

For $n = 1$ $\quad y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k) = 4 \cdot 2 + 2 \cdot 2 + 1 \cdot 1 = 13$

For $n = 2$ $\quad y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k) = 4 \cdot 1 + 2 \cdot 2 + 1 \cdot 2 + 3 \cdot 1 = 13$

For $n = 3$ $\quad y(3) = \sum_{k=-\infty}^{\infty} x(k)h(3-k) = 2 \cdot 1 + 1 \cdot 2 + 3 \cdot 2 = 10$

For $n = 4$ $\quad y(4) = \sum_{k=-\infty}^{\infty} x(k)h(4-k) = 1 \cdot 1 + 3 \cdot 2 = 7$

For $n = 5$ $\quad y(5) = \sum_{k=-\infty}^{\infty} x(k)h(5-k) = 3 \cdot 1 = 3$

$$\therefore \qquad y(n) = \left\{ \begin{matrix} 4, & 10, 13, 13, 10, 7, 3 \\ & \uparrow \end{matrix} \right\}$$

Method 3 *Tabular method*

Given $x(n) = \{4, 2, 1, 3\}, \quad h(n) = \left\{\begin{matrix}1, 2, 2, 1\\ \uparrow\end{matrix}\right\}$

The convolution of $x(n)$ and $h(n)$ can be computed as shown in Table 2.3.

TABLE 2.3 Table for computing $y(n)$.

$h(n)$ \ $x(n)$	4	2	1	3
1	4	2	1	3
2	8	4	2	6
2	8	4	2	6
1	4	2	1	3

$$y(n) = 4,\ 8 + 2,\ 8 + 4 + 1,\ 4 + 4 + 2 + 3,\ 2 + 2 + 6,\ 1 + 6,\ 3$$
$$= 4,\ 10,\ 13,\ 13,\ 10,\ 7,\ 3$$

The starting value of n is equal to –1, mark the symbol $\uparrow$ at time origin (n = 0).

$$\therefore \qquad y(n) = \left\{\begin{matrix}4, 10, 13, 13, 10, 7, 3\\ \ \ \uparrow \qquad\qquad\qquad\end{matrix}\right\}$$

Method 4 *Matrices method*

The given sequences are: $x(n) = \{x(0), x(1), x(2), x(3)\} = \{4, 2, 1, 3\}$

and $h(n) = \{h(0), h(1), h(2), h(3)\} = \{1, \underset{\uparrow}{2}, 2, 1\}$

The sequence $x(n)$ is starting at n = 0 and the sequence $h(n)$ is starting at n = –1. So the sequence $y(n)$ corresponding to the linear convolution of $x(n)$ and $h(n)$ will start at $n = 0 + (–1) = –1$. $x(n)$ is of length 4 and $h(n)$ is also of length 4. So length of $y(n) = 4 + 4 – 1 = 7$. Substituting the sequence values in matrix form and multiplying as shown below, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 13 \\ 13 \\ 10 \\ 7 \\ 3 \end{bmatrix}$$

$\therefore \qquad y(n) = x(n) * h(n) = [4 \;\; \underset{\uparrow}{10} \;\; 13 \;\; 13 \;\; 10 \;\; 7 \;\; 3]$

EXAMPLE 2.12 Find the convolution of the signals

$$x(n) = \begin{cases} 2 & n = -2, 0, 1 \\ 3 & n = -1 \\ 0 & \text{elsewhere} \end{cases}$$

$$h(n) = \delta(n) - 2\delta(n-1) + 3\delta(n-2) - \delta(n-3)$$

Solution: Given $\quad x(n) = \left\{2, 3, \underset{\uparrow}{2}, 2\right\}; \quad h(n) = \{1, -2, 3, -1\}$

$x(n)$ starts at $n_1 = -2$ and $h(n)$ starts at $n_2 = 0$. The starting sample of $y(n)$ is at $n = n_1 + n_2 = -2 + 0 = -2$. Since number of samples in $x(n)$ is 4, and the number of samples in $h(n)$ is 4, the number of samples in $y(n)$ will be $4 + 4 - 1 = 7$. So $y(n)$ exists from –2 to 4.

Method 1 *Graphical method*

We know that $\qquad y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$

From Figure 2.7, we get

For $n = -2$ $\qquad y(-2) = \sum_{k=-\infty}^{\infty} x(k)h(-2-k) = 2\cdot1 = 2$

For $n = -1$ $\qquad y(-1) = \sum_{k=-\infty}^{\infty} x(k)h(-1-k) = 2(-2) + 3\cdot1 = -1$

For $n = 0$ $\qquad y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k) = 2\cdot3 + 3(-2) + 2\cdot1 = 2$

For $n = 1$ $\qquad y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k) = 2(-1) + 3(3) + 2(-2) + 2\cdot1 = 5$

For $n = 2$ $\qquad y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k) = 3(-1) + 2\cdot3 + 2(-2) = -1$

For $n = 3$ $\qquad y(3) = \sum_{k=-\infty}^{\infty} x(k)h(3-k) = 2(-1) + 2\cdot3 = 4$

For $n = 4$ $\quad y(4) = \sum_{k=-\infty}^{\infty} x(k)\,h(4-k) = 2(-1) = -2$

$$\therefore \qquad y(n) = \left\{\begin{matrix} 2, -1, 2, 5, -1, 4, -2 \\ \uparrow \end{matrix}\right\}$$

To check the correctness of the result sum all the samples in $x(n)$ and multiply with the sum of all samples in $h(n)$. This value must be equal to sum of all samples in $y(n)$.

In the given problem, $\sum_n x(n) = 9$, $\sum_n h(n) = 1$, and $\sum_n y(n) = 9$.

This shows $\sum_n x(n) \cdot \sum_n h(n) = \sum_n y(n)$ (proved).

Therefore, the result is correct.

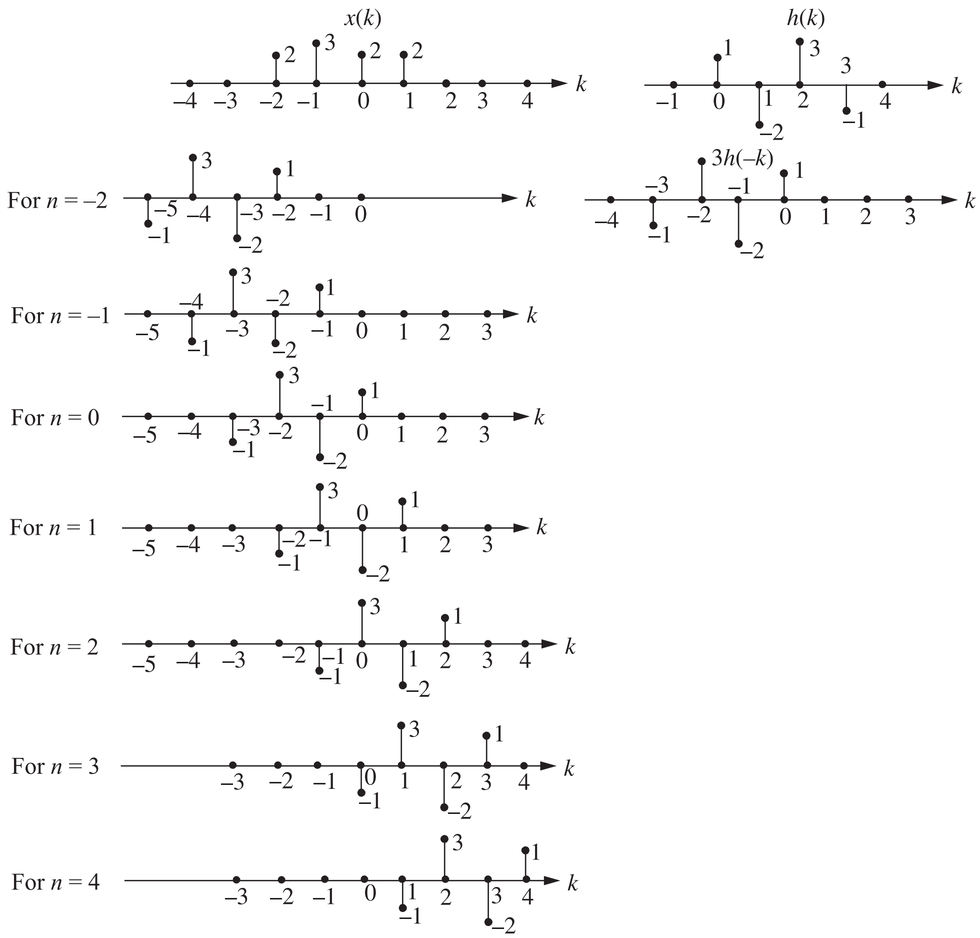


Figure 2.7 Operation on signals $x(n)$ and $h(n)$ to compute convolution.

Method 2 *Tabular array*

Tabulate the sequence $x(k)$ and shifted version of $h(k)$ as shown in Table 2.4.

TABLE 2.4 Table for computing $y(n)$.

k		–5	–4	–3	–2	–1	0	1	2	3	4	5	6
$x(k)$		–	–	–	2	3	2	2	–	–	–	–	–
$h(-k)$		–	–	–1	3	–2	1	–	–	–	–	–	–
$n = -2$	$h(-2 - k)$	–1	3	–2	1	–	–	–	–	–	–	–	–
$n = -1$	$h(-1 - k)$	–	–1	3	–2	1	–	–	–	–	–	–	–
$n = 0$	$h(- k)$	–	–	–1	3	–2	1	–	–	–	–	–	–
$n = 1$	$h(1 - k)$	–	–	–	–1	3	–2	1	–	–	–	–	–
$n = 2$	$h(2 - k)$	–	–	–	–	–1	3	–2	1	–	–	–	–
$n = 3$	$h(3 - k)$	–	–	–	–	–	–1	3	–2	1	–	–	–
$n = 4$	$h(4 - k)$	–	–	–	–	–	–	–1	3	–2	1	–	–

The starting sample of $y(n)$ is at $n = -2$. $y(n)$ is calculated as shown below. From the table, we can see that

For $n = -2$ $$y(-2) = \sum_{k=-\infty}^{\infty} x(k)h(-2-k) = 2 \cdot 1 = 2$$

For $n = -1$ $$y(-1) = \sum_{k=-\infty}^{\infty} x(k)h(-1-k) = 2(-2) + 3 \cdot 1 = -1$$

For $n = 0$ $$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k) = 2 \cdot 3 + 3(-2) + 2 \cdot 1 = 2$$

For $n = 1$ $$y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k) = 2(-1) + 3(3) + 2(-2) + 2 \cdot 1 = 5$$

For $n = 2$ $$y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k) = 3(-1) + 2 \cdot 3 + 2(-2) = -1$$

For $n = 3$ $$y(3) = \sum_{k=-\infty}^{\infty} x(k)h(3-k) = 2(-1) + 2 \cdot 3 = 4$$

For $n = 4$ $$y(4) = \sum_{k=-\infty}^{\infty} x(k)h(4-k) = 2(-1) = -2$$

$\therefore$ $$y(n) = \left\{\begin{matrix} 2, -1, 2, 5, -1, 4, -2 \\ \uparrow \end{matrix}\right\}$$

Method 3 *Tabular method*

$y(n) = x(n) * h(n)$ is computed by tabular method as shown in Table 2.5.

TABLE 2.5 Table for computing $y(n)$.

		$x(n)$			
		2	3	2	2
$h(n)$	1	2	3	2	2
	−2	−4	−6	−4	−4
	3	6	9	6	6
	−1	−2	−3	−2	−2

$$\therefore \quad y(n) = 2,\ -4+3,\ 6-6+2,\ -2+9-4+2,\ -3+6-4,\ -2+6,\ -2$$

$$= \{2, -1, \underset{\uparrow}{2}, 5, -1, 4, -2\}$$

Method 4 *Matrices method*

The given sequences are: $x(n) = \{x(0), x(1), x(2), x(3)\} = \{2, \underset{\uparrow}{3}, 2, 2\}$

and $\qquad h(n) = \{h(0), h(1), h(2), h(3)\} = \{1, \underset{\uparrow}{-2}, 3, -1\}$

The sequence $x(n)$ is starting at $n = -1$ and the sequence $h(n)$ is also starting at $n = -1$. So the sequence $y(n)$ corresponding to the linear convolution of $x(n)$ and $y(n)$ will start at $n = -1 + (-1) = -2$. $x(n)$ is of length 4 and $h(n)$ is also of length 4. So length of $y(n) = 4 + 4 - 1 = 7$. Substituting the sequence values in matrix form and multiplying, we get the convolution of $x(n)$ and $h(n)$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -1 & 3 & -2 & 1 \\ 0 & -1 & 3 & -2 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \\ 5 \\ -1 \\ 4 \\ -2 \end{bmatrix}$$

$$\therefore \qquad y(n) = x(n) * h(n) = \{2,\ -1,\ \underset{\uparrow}{2},\ 5,\ -1,\ 4,\ -2\}$$

Check

$$\sum x(n) \cdot \sum h(n) = \sum y(n)$$

$$9.1 = 9$$

From the above examples we find that if the length of the sequence $x(n)$ is N_1, and the length of the sequence $h(n)$ is N_2, then the convolution of these two sequences produces a sequence $y(n)$ whose length is equal to $N_1 + N_2 - 1$, and if the first element of $x(n)$ is at n_1 and the first element of $h(n)$ is at n_2, the first element of $y(n)$ is at $n_1 + n_2$.

EXAMPLE 2.13 Find the convolution of the following sequences:

$$x(n) = 3\delta(n+1) - 2\delta(n) + \delta(n-1) + 4\delta(n-2)$$

$$h(n) = 2\delta(n-1) + 5\delta(n-2) + 3\delta(n-3)$$

Solution: Given $x(n) = \{3,\ \underset{\uparrow}{-2},\ 1,\ 4\}$, i.e. $x(n)$ starts at $n_1 = -1$

$$h(n) = \{2,\ 5,\ 3\}, \text{ i.e. } h(n) \text{ starts at } n_2 = 1$$

$\therefore$ $y(n) = x(n) * h(n)$ starts at $n = n_1 + n_2 = -1 + 1 = 0$. $y(n)$ is computed using the tabular method as shown below.

From Table 2.6, we can observe that $y(n)$ is given by

$$y(n) = \{6,\ 15-4,\ 9-10+2,\ -6+5+8,\ 3+20,\ 12\}$$
$$= \{6,\ 11,\ 1,\ 7,\ 23,\ 12\}$$

TABLE 2.6 Table for computing $y(n)$.

		$x(n)$: 3	−2	1	4
$h(n)$	2	6	−4	2	8
	5	15	−10	5	20
	3	9	−6	3	12

EXAMPLE 2.14 Find the discrete convolution of

$$u(n) * u(n-2)$$

Solution: Given $\qquad y(n) = u(n) * u(n-2)$

$\therefore \qquad x(n) = u(n) = 1, 1, 1, \ldots$

$\qquad h(n) = u(n-2) = 0, 0, 1, 1, 1, \ldots$

$y(n)$ can be computed from Table 2.7.

TABLE 2.7 Table for computing $y(n)$.

$h(n)$ \ $x(n)$	1	1	1	1	1	–	–
0	0	0	0	0	0	–	–
0	0	0	0	0	0	–	–
1	1	1	1	1	1	–	–
1	1	1	1	1	1	–	–
1	1	1	1	1	1	–	–

Therefore, $\qquad y(n) = 0, 0, 1, 2, 3, \ldots$

i.e. $\qquad y(0) = 0, y(1) = 0, y(2) = 1, y(3) = 2, y(4) = 3, \ldots$

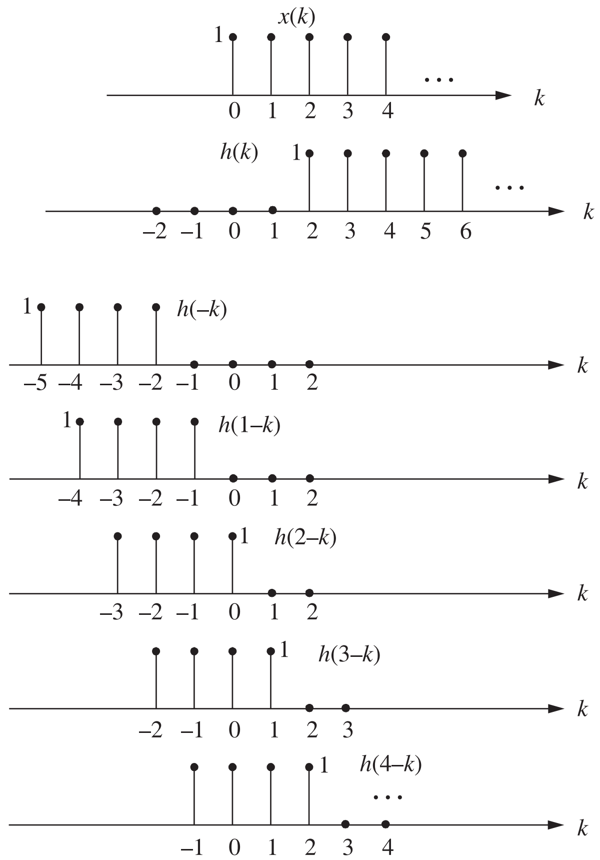
$\therefore \qquad y(n) = n - 1$

$\therefore \qquad u(n) * u(n-2) = n - 1$

or
$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) = \sum_{k=-\infty}^{\infty} u(k)u[(n-2)-k]$$
$$= \sum_{k=0}^{n-2} 1 = n - 1$$

or
$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k) = \sum_{k=-\infty}^{\infty} u(k-2)\,u(n-k)$$
$$= \sum_{k=2}^{n} 1 = n - 2 + 1 = n - 1$$

The convolution can be performed graphically as shown in Figure 2.8.



$$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k) = 0$$

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k) = 0$$

$$y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k) = 1$$

$$y(3) = \sum_{k=-\infty}^{\infty} x(k)h(3-k) = 2$$

$$y(4) = \sum_{k=-\infty}^{\infty} x(k)h(4-k) = 3$$

$$\therefore\ y(n) = n-1$$

Figure 2.8 Graphical convolution of Example 2.14.

EXAMPLE 2.15 Find the discrete convolution of

(a) $y(n) = \sin\left(\dfrac{n\pi}{2}\right)u(n) * u(n-2)$ (b) $y(n) = 3^n u(-n+3) * u(n-2)$

Solution:

(a) Given $$y(n) = \sin\left(\frac{n\pi}{2}\right)u(n) * u(n-2)$$

Let $x(n) = \sin\left(\dfrac{n\pi}{2}\right)u(n)$ and $h(n) = u(n-2)$

The given sequences are:

$$x(n) = \underset{\uparrow}{0}, 1, 0, -1, 0, 1, 0, -1, \ldots$$

$$h(n) = 0, 0, 1, 1, 1, 1, 1, 1, \ldots$$
$$\uparrow$$

Formulate the following table (Table 2.8):

TABLE 2.8 Table for computing $y(n)$.

$x(n)$	0	1	0	–1	0	1	0	–1	–	–	–
$h(n)$ 0	0	0	0	0	0	0	0	0	–	–	
0	0	0	0	0	0	0	0	0	–	–	
1	0	1	0	–1	0	1	0	–1	–	–	
1	0	1	0	–1	0	1	0	–1	–	–	
1	0	1	0	–1	0	1	0	–1	–	–	
1	0	1	0	–1	0	1	0	–1	–	–	
1	0	1	0	–1	0	1	0	–1	–	–	
1	0	1	0	–1	0	1	0	–1	–	–	
1	0	1	0	–1	0	1	0	–1	–	–	
1	0	1	0	–1	0	1	0	–1	–	–	

$$y(n) = \{0, 0, 0, 1, 1, 0, 0, 1, 1, \ldots\}$$

or
$$y(n) = x(n) * h(n) = \sum_{k=0}^{\infty} x(k)\, h(n-k)$$

Given $\quad x(k) = \sin\left(\dfrac{k\pi}{2}\right)u(k) \quad$ and $\quad h(n - k) = u(n - 2 - k)$

$$x(k) = 0 \qquad \text{for} \quad k < 0$$
$$h(n - k) = 0 \qquad \text{for} \quad k > n - 2$$

Therefore, $y(n) = \displaystyle\sum_{k=0}^{n-2} \sin\frac{k\pi}{2}$

$$= \text{Imaginary part of } \left[\sum_{k=0}^{n-2} e^{j\frac{k\pi}{2}}\right]$$

$$= \text{Im}\left[1 + e^{\frac{j\pi}{2}} + e^{j\pi} + e^{j\frac{3\pi}{2}} + \cdots + (n-1)\right] \text{ terms}$$

$$= \text{Im}\left[\frac{1-e^{j\pi\frac{(n-1)}{2}}}{1-e^{j\frac{\pi}{2}}}\right] = \text{Im}\left[\frac{1-e^{j\pi\frac{(n-1)}{2}}}{1-j}\right]$$

$$= \text{Im}\left[\frac{\left(1-e^{j\pi\frac{(n-1)}{2}}\right)(1+j)}{2}\right]$$

$$= \text{Im}\left[\frac{1-e^{\frac{j\pi}{2}(n-1)}+j-je^{\frac{j\pi}{2}(n-1)}}{2}\right]$$

$$= \frac{1}{2}\,\text{Im}\left[1 - \cos\frac{\pi}{2}(n-1) - j\sin\frac{\pi}{2}(n-1)+j-j\left[\cos\frac{\pi}{2}(n-1)+j\sin\frac{\pi}{2}(n-1)\right]\right]$$

$$= \frac{1}{2}\left[1-\sin\frac{\pi}{2}(n-1)-\cos\frac{\pi}{2}(n-1)\right]$$

(b) Given $y(n) = 3^n u(-n + 3) * u(n - 2)$

Let $x(n) = 3^n u(-n + 3)$

and $h(n) = u(n - 2)$

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} 3^k u(-k+3)u(n-2-k)$$

Figure 2.9 shows the components of $x(k)$ and $h(k)$.

For $-\infty \le n \le 5$,

$$y(n) = \sum_{k=-\infty}^{n-2} 3^k = [3^{n-2} + 3^{n-3} + 3^{n-4} + \cdots]$$

$$= 3^{n-2}[1 + 3^{-1} + 3^{-2} + 3^{-3} + \cdots] = 3^{n-2}\left[\frac{1}{1-\frac{1}{3}}\right] = \frac{3^{n-1}}{2}$$

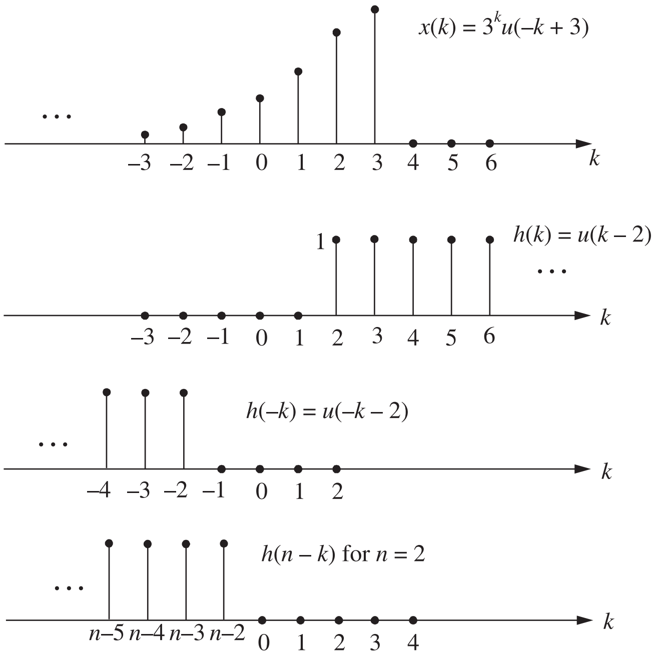


Figure 2.9 Example 2.15(b).

For $n > 5$,

$$y(n) = \sum_{k=-\infty}^{3} 3^k = [3^3 + 3^2 + 3 + 1 + \frac{1}{3} + \frac{1}{9} + \cdots]$$

$$= 3^3[1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots] = 27\left[\frac{1}{1-\frac{1}{3}}\right] = 40.5$$

EXAMPLE 2.16 (a) An FIR (finite impulse response) filter has an impulse response given by $h(n) = \{2, 1, 2, 1\}$. Find its response $y(n)$ to the input $x(n) = \{1, -2, 4\}$. Assume that both $x(n)$ and $h(n)$ start at $n = 0$.

(b) Find response of the system if $h(n) = \{3, 7, 0, 5\}$ and $x(n) = \{2, 3, 4\}$.

Solution: (a) The paper-and-pencil method expresses the input as $x(n) = \delta(n) - 2\delta(n - 1) + 4\delta(n - 2)$ and tabulates the response to each impulse and the total response as follows:

This is called sum-by-column methods.

$h(n)$		=	2	1	2	1		
$x(n)$		=	1	–2	4			
Input	Response							
$\delta(n)$	$h(n)$	=	2	1	2	1		
$-2\delta(n-1)$	$-2h(n-1)$	=		– 4	–2	– 4	–2	
$4\delta(n-2)$	$4h(n-2)$	=			8	4	8	4
sum = $x(n)$	sum = $y(n)$	=	2	–3	8	1	6	4

So
$$y(n) = x(n) * h(n) = \{\underset{\uparrow}{2}, -3, 8, 1, 6, 4\}$$

$$= 2\delta(n) - 3\delta(n-1) + 8\delta(n-2) + \delta(n-3) + 6\delta(n-4) + 4\delta(n-5)$$

(b) We note that the convolution starts at $n = (-2) + (-1) = -3$ and use this to set up the index array and generate the convolution using the sum-by-column method as follows:

n	–3	–2	–1	0	1	2
$h(n)$	3	7	0	5		
$x(n)$	2	3	4			
	6	14	0	10		
		9	21	0	15	
			12	28	0	20
$y(n)$	6	23	33	38	15	20

The convolution starts at $n = -3$ and we get

$$x(n) * h(n) = y(n) = \left\{6, 23, 33, \underset{\uparrow}{38}, 15, 20\right\}$$

The convolution process is illustrated graphically in Figure 2.10

EXAMPLE 2.17 Find the convolution of the sequences by the sliding strip method:

$$h(n) = \{3, 7, 0, 5\} \quad \text{and} \quad x(n) = \{2, 3, 4\}$$

Solution: Since both sequences start at $n = 0$, the flipped sequence is:

$$x(-k) = \left\{4, 3, \underset{\uparrow}{2}\right\}$$

We line up the flipped sequence below $h(n)$ to begin overlap and shift it successively, summing the product sequence as we go, to obtain the discrete convolution. The results are computed in Figure 2.11.

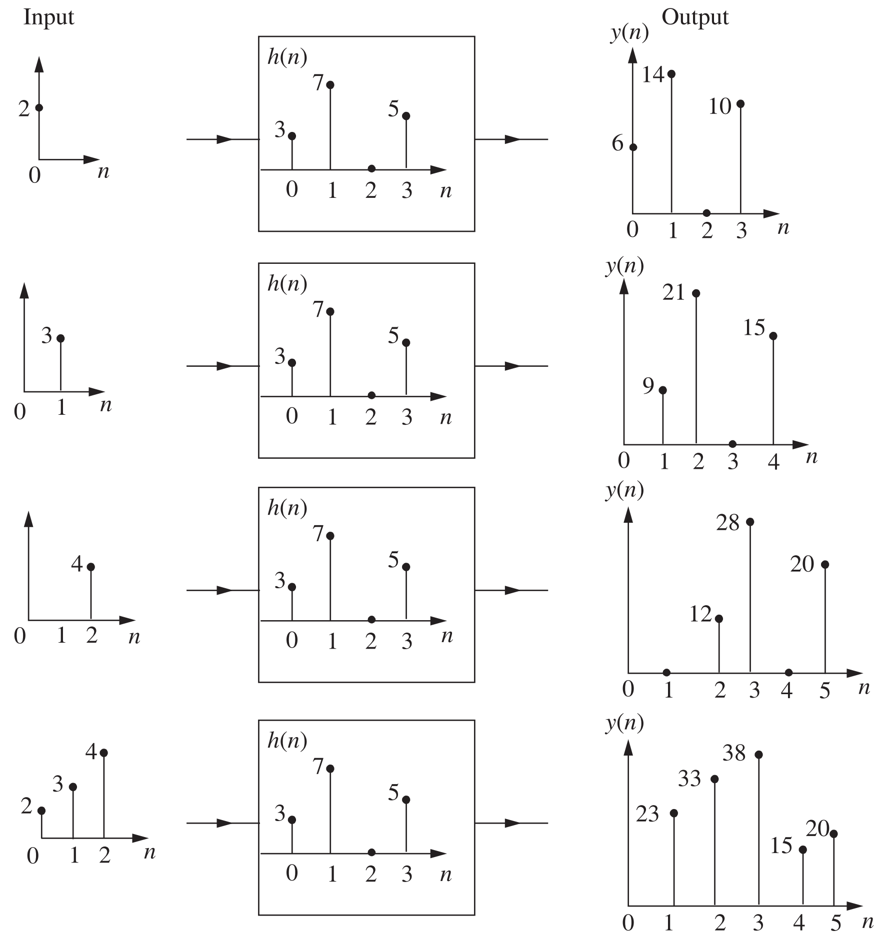


Figure 2.10 Convolution process for Example 2.16.

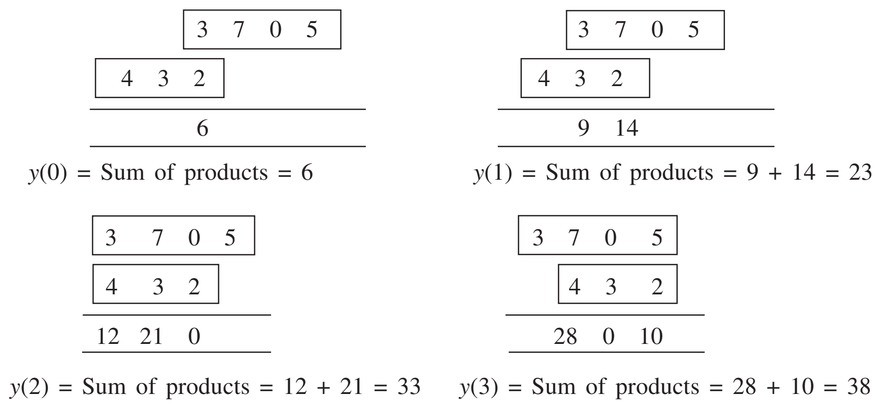


Figure 2.11 *Contd.*

$\boxed{3 \quad 7 \quad 0 \quad 5}$

$\qquad\quad \boxed{4 \quad 3 \quad 2}$

$\qquad\quad 0 \quad 15$

$y(4)$ = Sum of products = 15

$\boxed{3 \quad 7 \quad 0 \quad 5}$

$\qquad\qquad\quad \boxed{4 \quad 3 \quad 2}$

$\qquad\qquad 20$

$y(5)$ = Sum of products = 20

Figure 2.11 Convolution by sliding-strip method.

The discrete convolution is $y(n) = \{6, 23, 33, 38, 15, 20\}$.

Discrete convolution, multiplication, and zero insertion

The discrete convolution of two finite-length sequences $x(n)$ and $h(n)$ is entirely equivalent to the multiplication of two polynomials whose coefficients are described by the arrays $x(n)$ and $h(n)$ (in ascending or descending order). The convolution sequence corresponds to the coefficients of the product polynomial. Based on this result, if we insert N zeros between each pair of adjacent samples of each sequence to be convolved, their convolution corresponds to the original convolution sequence with N zeros inserted between each pair of its adjacent samples.

If we append zeros to one of the convolved sequences, the convolution result also will show as many appended zeros at the corresponding location. For example, leading zeros appended to a sequence will appear as leading zeros in the convolution result. Similarly, trailing zeros appended to a sequence will show up as trailing zeros in the convolution.

EXAMPLE 2.18 Given the sequence $h(n) = \{3, 7, 0, 5\}$ and $x(n) = \{2, 3, 7\}$.

(a) Find the convolution of the sequences by multiplication of the sequences by multiplication of polynomials.
(b) Insert a zero between each sample of $h(n)$ and $x(n)$ and find their convolution.
(c) Insert two zeros at the end of sequence $h(n)$ and one zero at the end of $x(n)$ and find their convolution.

Solution:

(a) Given $h(n) = \{3, 7, 0, 5\}$ and $x(n) = \left\{\begin{matrix} 2, 3, 4 \\ \uparrow \quad\quad \end{matrix}\right\}$

To find their convolution, we set up the polynomials:

$$H(z) = 3z^3 + 7z^2 + 0z + 5 \text{ and } X(z) = 2z^2 + 3z + 4$$

Their product is $Y(z) = (3z^3 + 7z^2 + 5)\,(2z^2 + 3z + 4)$

$$= 6z^5 + 23z^4 + 33z^3 + 38z^2 + 15z + 20$$

The convolution is thus $y(n) = x(n) * h(n) = \left\{\begin{matrix} 6, 23, 33, 38, 15, 20 \\ \uparrow \qquad\qquad\qquad\qquad \end{matrix}\right\}$

(b) Zero insertion of each convolved sequence gives

$$h_1(n) = \begin{Bmatrix} 3, 0, 7, 0, 0, 0, 5 \\ \uparrow \end{Bmatrix} \text{ and } x_1(n) = \begin{Bmatrix} 2, 0, 3, 0, 4 \\ \uparrow \end{Bmatrix}$$

To find their convolution, we set up the polynomials

$$H_1(z) = 3z^6 + 7z^4 + 0z^2 + 5 \quad \text{and} \quad X_1(z) = 2z^4 + 3z^2 + 4$$

Their product is $Y_1(z) = (3z^6 + 7z^4 + 5)\,(2z^4 + 3z^2 + 4)$

$$= 6z^{10} + 23z^8 + 33z^6 + 38z^4 + 15z^2 + 20$$

The convolution is then $y_1(n) = \begin{Bmatrix} 6, 0, 22, 0, 33, 0, 38, 0, 15, 0, 20 \\ \uparrow \end{Bmatrix}$

This result is just $y(n)$ with zeros inserted between adjacent samples.

(c) *Zero-padding:* If we pad the first sequence by two zeros and the second by one zero, we get

$$h_2(n) = \{3, 7, 0, 5, 0, 0\} \quad \text{and} \quad x_2(n) = \{2, 3, 4, 0\}$$

To find their convolution, we set up the polynomials:

$$H_2(z) = 3z^5 + 7z^4 + 5z^2 \quad \text{and} \quad X_2(z) = 2z^3 + 3z^2 + 4z$$

Their product is $Y_2(z) = 6z^8 + 23z^7 + 33z^6 + 38z^5 + 15z^4 + 20z^3$.

The convolution is then $y_1(n) = \begin{Bmatrix} 6, 23, 33, 38, 15, 20, 0, 0, 0 \\ \uparrow \end{Bmatrix}$.

This result is just $y(n)$ with three zeros appended at the end.

EXAMPLE 2.19 Find the convolution

$$y(n) = \{1, 2, 1\} * \{2, 0, 1\}$$

Solution: Writing the product of corresponding polynomials, we have

$$(x^2 + 2x + 1)\{2x^2 + 1\} = 2x^4 + 4x^3 + 3x^2 + 2x + 1$$

Taking the coefficients of the polynomial, we have

$$x(n) * h(n) = y(n) = \{2, 4, 3, 2, 1\}$$

EXAMPLE 2.20 Find the convolution

$$y_1(n) = \{1, 0, 2, 0, 1\} * \{2, 0, 0, 0, 1\}$$

Solution: We have from Example 2.19,

$$\{1, 2, 1\} * \{2, 0, 1\} = \{2, 4, 3, 2, 1\}$$

The given sequences are same as sequences in Example 2.19, except that a zero is inserted between two elements in both the sequences.

Since zero insertion of both sequences leads to zero insertion of the convolution, we have

$$y_1(n) = \{1, 0, 2, 0, 1\} * \{2, 0, 0, 0, 1\} = \{2, 0, 4, 0, 3, 0, 2, 0, 1\}$$

EXAMPLE 2.21 Find the convolution

$$\{0, 0, 1, 2, 1, 0, 0\} * \{2, 0, 1, 0\}$$

Solution: Here two zeros are added at the start and end of the first sequence and one zero is added at the end of second sequence of Example 2.19. Since zero padding of one sequence leads to zero padding of the convolution, we have

$$\{0, 0, 1, 2, 1, 0, 0\} * \{2, 0, 1, 0\} = \{0, 0, 2, 4, 3, 2, 1, 0, 0, 0\}$$

2.6 DECONVOLUTION

2.6.1 Deconvolution using *Z*-transform

Deconvolution is the process of finding the input $x(n)$ [or impulse response $h(n)$] applied to the system once the output $y(n)$ and the impulse response $h(n)$ [or the input $x(n)$] of the system are known. The Z-transform also can be used for deconvolution operation.

We know that

$$Y(z) = X(z)H(z) \quad \text{or} \quad X(z) = \frac{Y(z)}{H(z)}$$

where $Y(z)$, $X(z)$ and $H(z)$ are the Z-transforms of output, input and impulse response respectively. If $y(n)$ and $h(n)$ are given, we can determine their Z-transforms $Y(z)$ and $H(z)$. Knowing $Y(z)$ and $H(z)$, we can determine $X(z)$ and knowing $X(z)$, we can determine the input $x(n)$. Thus the deconvolution is reduced to the procedure of evaluating an inverse Z-transform.

EXAMPLE 2.22 Find the input $x(n)$ of the system, if the impulse response $h(n)$ and the output $y(n)$ are given as:

$$h(n) = \{2, 1, 0, -1, 3\}; \quad y(n) = \{2, -5, 1, 1, 6, -11, 6\}$$

Solution: Given $h(n) = \{2, 1, 0, -1, 3\}$

$$\therefore \quad H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n} = \sum_{n=0}^{4} h(n)z^{-n}$$

$$= 2 + z^{-1} - z^{-3} + 3z^{-4}$$

$$y(n) = \{2, -5, 1, 1, 6, -11, 6\}$$

$$\therefore \qquad Y(z) = \sum_{n=-\infty}^{\infty} y(n)\, z^{-n} = \sum_{n=0}^{6} y(n)\, z^{-n}$$

$$= 2 - 5z^{-1} + z^{-2} + z^{-3} + 6z^{-4} - 11z^{-5} + 6z^{-6}$$

$$X(z) = \frac{Y(z)}{H(z)} = \frac{2 - 5z^{-1} + z^{-2} + z^{-3} + 6z^{-4} - 11z^{-5} + 6z^{-6}}{2 + z^{-1} - z^{-3} + 3z^{-4}}$$

$$\begin{array}{r|l}
 & 1 - 3z^{-1} + 2z^{-2} \\ \hline
2 + z^{-1} - z^{-3} + 3z^{-4} & 2 - 5z^{-1} + z^{-2} + z^{-3} + 6z^{-4} - 11z^{-5} + 6z^{-6} \\
 & 2 + z^{-1} - z^{-3} + 3z^{-4} \\ \hline
 & \quad - 6z^{-1} + z^{-2} + 2z^{-3} + 3z^{-4} - 11z^{-5} + 6z^{-6} \\
 & \quad - 6z^{-1} - 3z^{-2} + 3z^{-4} - 9z^{-5} \\ \hline
 & \qquad\quad 4z^{-2} + 2z^{-3} - 2z^{-5} + 6z^{-6} \\
 & \qquad\quad 4z^{-2} + 2z^{-3} - 2z^{-5} + 6z^{-6} \\ \hline
 & \qquad\qquad\qquad 0
\end{array}$$

$$\therefore \qquad X(z) = 1 - 3z^{-1} + 2z^{-2}$$

Taking inverse Z-transform, we have input

$$x(n) = \delta(n) - 3\delta(n-1) + 2\delta(n-2)$$

i.e. $\qquad x(n) = \{1, -3, 2\}$

2.6.2 Deconvolution by Recursion

The process of recovering $x(n)$ or $h(n)$ from $y(n) = x(n) * h(n)$ is known as deconvolution.

We have $\qquad y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$

Assuming $y(n)$ and $h(n)$ are one sided sequences, we have

$$y(n) = \sum_{k=0}^{n} x(k)\, h(n-k)$$

From which we get

$$\begin{aligned}
y(0) &= x(0)\, h(0) \\
y(1) &= x(0)\, h(1) + x(1)\, h(0) \\
y(2) &= x(0)\, h(2) + x(1)\, h(1) + x(2)\, h(0) \\
&\vdots
\end{aligned}$$

In matrix form,

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ - \\ - \\ - \end{bmatrix} = \begin{bmatrix} h(0) & 0 & 0 & - & - & 0 \\ h(1) & h(0) & 0 & - & - & 0 \\ h(2) & h(1) & h(0) & - & - & 0 \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ - & - & - & - & - & - \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ - \\ - \\ - \end{bmatrix}$$

From the above matrix form, we can find

$$y(0) = h(0)x(0)$$

$$\therefore \qquad x(0) = \frac{y(0)}{h(0)}$$

$$y(1) = x(0)h(1) + x(1)h(0)$$

$$\therefore \qquad x(1) = \frac{y(1) - x(0)h(1)}{h(0)}$$

$$y(2) = x(0)h(2) + x(1)h(1) + x(2)h(0)$$

$$\therefore \qquad x(2) = \frac{y(2) - x(0)h(2) - x(1)h(1)}{h(0)}$$

In general,

$$x(n) = \frac{y(n) - \sum_{k=0}^{n-1} x(k)h(n-k)}{h(0)}$$

If all goes well, we need to evaluate $x(n)$ [or $h(n)$] only at $M - N + 1$ points where M and N are the lengths of $y(n)$ and $h(n)$ [or $x(n)$] respectively.

Naturally, problems arise if a remainder is involved. This may well happen in the presence of noise, which could modify the values in the output sequence even slightly. In other words, the approach is quite susceptible to noise or round off error and not very practical.

EXAMPLE 2.23 What is the input signal $x(n)$ that will generate the output sequence $y(n) = \{1, 1, 2, 0, 2, 1\}$ for a system with impulse response $h(n) = \{1, -1, 1\}$.

Solution: Given $y(n) = \{1, 1, 2, 0, 2, 1\}$ and $h(n) = \{1, -1, 1\}$

Let N_1 be the number of samples in $x(n)$, and let N_2 be the number of samples in $h(n)$. The number of samples in $y(n) = N_1 + N_2 - 1 = 6$, therefore, $N_1 = 6 - N_2 + 1 = 6 - 3 + 1 = 4$.

Let $$x(n) = \{x(0),\ x(1),\ x(2),\ x(3)\}$$

We have $$x(n) = \frac{y(n) - \sum_{k=0}^{n-1} x(k)\,h(n-k)}{h(0)}$$

For $n = 0$, $$x(0) = \frac{y(0)}{h(0)} = \frac{1}{1} = 1$$

For $n = 1$, $$x(1) = \frac{y(1) - x(0)h(1)}{h(0)} = \frac{1 - 1(-1)}{1} = 2$$

For $n = 2$, $$x(2) = \frac{y(2) - x(0)h(2) - x(1)h(1)}{h(0)} = \frac{2 - 1\cdot1 - 2(-1)}{1} = 3$$

For $n = 3$, $$x(3) = \frac{y(3) - x(0)h(3) - x(1)h(2) - x(2)h(1)}{h(0)} = \frac{3 - 1\cdot1 - 2(-1) - 3\cdot1}{1} = 1$$

$\therefore$ $$x(n) = \{1,\ 2,\ 3,\ 1\}$$

2.6.3 Deconvolution using Tabular Method

To perform deconvolution using tabular method, i.e. given $x(n)$ and $y(n)$ to find $h(n)$ assume $h(n) = \{a, b, c, d, ...,\}$, formulate the convolution table using $x(n)$ and assumed $h(n)$, and then obtain the convolution in terms of $a, b, c, d, ...$. Equate the obtained convolution to given $y(n)$ and solve for $a, b, c, d, ...$. This gives the deconvolution.

EXAMPLE 2.24 Find the input signal $x(n)$ that will generate $y(n) = \{1, 1, 2, 0, 2, 1\}$ for a system with impulse response $h(n) = \{1, -1, 1\}$ using the tabular method.

Solution: Given the impulse response $h(n) = \{1, -1, 1\}$ and the output $y(n) = \{1, 1, 2, 0, 2, 1\}$. We know that $y(n) = x(n) * h(n)$. Since $h(n)$ has 3 samples and $y(n)$ has 6 samples, $x(n)$ will have $6 - 3 + 1 = 4$ samples. Let $x(n) = \{a, b, c, d\}$. The input $x(n)$ can be determined using the tabular method (Table 2.9) as follows:

TABLE 2.9 Table for computing $y(n)$.

		$h(n)$: 1	-1	1
$x(n)$	a	a	$-a$	a
	b	b	$-b$	b
	c	c	$-c$	c
	d	d	$-d$	d

From Table 2.9 $y(n) = \{a, b - a, c - b + a, d - c + b, -d + c, d\}$. Comparing this $y(n)$ with given $y(n) = \{1, 1, 2, 0, 2, 1\}$, we get

$$\begin{array}{lll} a = 1 & & \\ b - a = 1 & \therefore & b = 1 + a = 1 + 1 = 2 \\ c - b + a = 2 & \therefore & c = 2 + b - a = 2 + 2 - 1 = 3 \\ d - c + b = 0 & \therefore & d = c - b = 3 - 2 = 1 \\ \therefore & & x(n) = \{1, 2, 3, 1\} \end{array}$$

EXAMPLE 2.25 Find the impulse response $h(n)$ that will generate $y(n) = \{1, 1, 2, 0, 2, 1\}$ for a system with input signal $x(n) = \{1, 2, 3, 1\}$.

Solution: Given the input $x(n) = \{1, 2, 3, 1\}$ and the output $y(n) = \{1, 1, 2, 0, 2, 1\}$. We know that $y(n) = x(n) * h(n)$. Since $x(n)$ has 4 samples and $y(n)$ has 6 samples, $h(n)$ will have $6 - 4 + 1 = 3$ samples. Let $h(n) = \{a, b, c\}$. The impulse response $h(n)$ can be determined using the tabular method (Table 2.10) as follows:

TABLE 2.10 Table for computing $y(n)$.

		$h(n)$	
$x(n)$	a	b	c
1	a	b	c
2	$2a$	$2b$	$2c$
3	$3a$	$3b$	$3c$
1	a	b	c

From Table 2.10, $y(n) = [a, 2a + b, 3a + 2b + c, a + 3b + 2c, b + 3c, c\}$. Comparing this $y(n)$ with given $y(n) = \{1, 1, 2, 0, 2, 1\}$, we get

$$\begin{array}{lll} a = 1 & & \\ 2a + b = 1 & \therefore & b = 1 - 2a = 1 - 2 = -1 \\ 3a + 2b + c = 2 & \therefore & c = 2 - 3a - 2b = 2 - 3 + 2 = 1 \\ \therefore & h(n) = \{1, -1, 1\} & \end{array}$$

2.7 INTERCONNECTION OF LTI SYSTEMS

2.7.1 Parallel Connection of Systems

Consider two LTI systems with impulse responses $h_1(n)$ and $h_2(n)$ connected in parallel as shown in Figure 2.12.

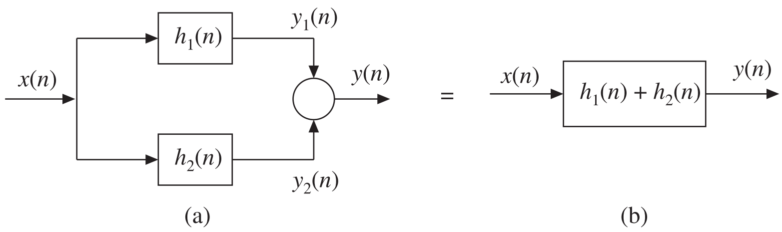


Figure 2.12 (a) Parallel connection of two systems (b) Equivalent system.

From Figure 2.12(a), the output of system 1 is:

$$y_1(n) = x(n) * h_1(n)$$

and the output of system 2 is:

$$y_2(n) = x(n) * h_2(n)$$

The output of the system $y(n)$ is:

$$\begin{aligned} y(n) &= y_1(n) + y_2(n) \\ &= x(n) * h_1(n) + x(n) * h_2(n) \\ &= \sum_{k=-\infty}^{\infty} x(k)h_1(n-k) + \sum_{k=-\infty}^{\infty} x(k)h_2(n-k) \\ &= \sum_{k=-\infty}^{\infty} x(k)[h_1(n-k) + h_2(n-k)] \\ &= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\ &= x(n) * h(n) \end{aligned}$$

where $$h(n) = h_1(n) + h_2(n)$$

Thus if two systems are connected in parallel, the overall impulse response is equal to the sum of two impulse responses.

2.7.2 Cascade Connection of Systems

Consider two LTI systems with impulse responses $h_1(n)$ and $h_2(n)$ connected in cascade as shown in Figure 2.13.

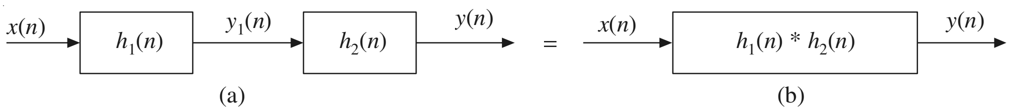


Figure 2.13 (a) Cascade connection of two systems (b) Equivalent system.

Let $y_1(n)$ be the output of the first system. Then

$$y_1(k) = x(k) * h_1(k)$$

$$= \sum_{k=-\infty}^{\infty} x(v) h_1(k-v)$$

The output

$$y(n) = y_1(k) * h_2(k)$$

$$= \left[\sum_{v=-\infty}^{\infty} x(v) h_1(k-v)\right] * h_2(k)$$

$\therefore$
$$y(n) = \sum_{k=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} x(v) h_1(k-v) h_2(n-k)$$

Let $k - v = p$

$\therefore$
$$y(n) = \sum_{v=-\infty}^{\infty} x(v) \sum_{p=-\infty}^{\infty} h_1(p) h_2(n-v-p)$$

$$= \sum_{v=-\infty}^{\infty} x(v) h(n-v)$$

$$= x(n) * h(n)$$

where
$$h(n) = \sum_{k=-\infty}^{\infty} h_1(k) h_2(n-k) = h_1(n) * h_2(n)$$

EXAMPLE 2.26 An interconnection of LTI systems is shown in Figure 2.14. The impulse responses are $h_1(n) = (1/2)^n\,[u(n) - u(n-4)]$; $h_2(n) = \delta(n)$ and $h_3(n) = u(n-2)$. Let the impulse response of the overall system from $x(n)$ to $y(n)$ be denoted as $h(n)$.

(a) Express $h(n)$ in terms of $h_1(n)$, $h_2(n)$ and $h_3(n)$.
(b) Evaluate $h(n)$.

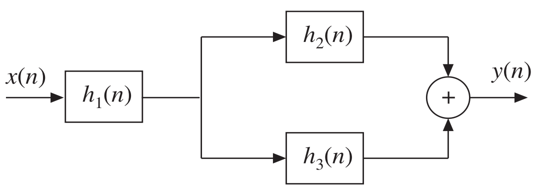


Figure 2.14 System for Example 2.26.

Solution: The systems with impulse responses $h_2(n)$ and $h_3(n)$ are connected in parallel. They can be replaced by a system with impulse response $h_2(n) + h_3(n)$. So the system can be represented as shown in Figure 2.15.

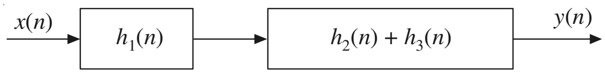


Figure 2.15 Equivalent of system in Figure 2.14.

Now, systems with impulse responses $h_1(n)$ and $h_2(n) + h_3(n)$ are in cascade. So the overall impulse response is:

$$h(n) = h_1(n) * [h_2(n) + h_3(n)]$$
$$= h_1(n) * h_2(n) + h_1(n) * h_3(n)$$

Given
$$h_1(n) = \left(\frac{1}{2}\right)^n [u(n) - u(n-4)]$$
$$h_2(n) = \delta(n)$$
$$h_3(n) = u(n-2)$$

$$h_1(n) * h_2(n) = \left[\left(\frac{1}{2}\right)^n [u(n) - u(n-4)]\right] * \delta(n)$$

$$= \left(\frac{1}{2}\right)^n [u(n) - u(n-4)] \qquad [\because x(n) * \delta(n) = x(n)]$$

$$h_1(n) * h_3(n) = \left[\left(\frac{1}{2}\right)^n [u(n) - u(n-4)]\right] * u(n-2)$$

$$= \left(\frac{1}{2}\right)^n u(n) * u(n-2) - \left(\frac{1}{2}\right)^n u(n-4) * u(n-2) \,]$$

Let
$$y_1(n) = \left(\frac{1}{2}\right)^n u(n) * u(n-2)$$

$\therefore$
$$y_1(n) = \sum_{k=0}^{n-2} \left(\frac{1}{2}\right)^k \qquad \text{for } n \ge 2$$
$$= 0 \qquad \text{for } n < 2$$

$\therefore$
$$y_1(n) = \frac{1-(1/2)^{n-1}}{1 - 1/2} = 2\left[1-\left(\frac{1}{2}\right)^{n-1}\right]$$

i.e. $$y_1(n) = 2\left[1-\left(\frac{1}{2}\right)^{n-1}\right] \quad \text{for } n \geq 2$$

$$= 0 \quad \text{for } n < 2$$

Therefore, $$y_1(n) = 2\left[1-\left(\frac{1}{2}\right)^{n-1}\right]u(n-1)$$

$$y_2(n) = \left(\frac{1}{2}\right)^n u(n-4) * u(n-2)$$

$$= \sum_{k=4}^{n-2}\left(\frac{1}{2}\right)^k \quad \text{for } n \geq 6$$

$$= 0 \quad \text{for } n < 6$$

i.e. $$y_2(n) = \sum_{k=4}^{n-2}\left(\frac{1}{2}\right)^{k-4}\left(\frac{1}{2}\right)^4 \quad \text{for } n \geq 6$$

$$= \frac{1}{16}\sum_{p=0}^{n-6}\left(\frac{1}{2}\right)^p \quad \text{for } n \geq 6$$

$$= \frac{1}{16}\left[\frac{1-(1/2)^{n-5}}{1 - 1/2}\right] \quad \text{for } n \geq 6$$

$$= \left[\frac{1}{8}-\frac{1}{16}\left(\frac{1}{2}\right)^n (2)^5\right] \quad \text{for } n \geq 6$$

$$= \frac{1}{8}u(n-6) - 2\left(\frac{1}{2}\right)^n u(n-6)$$

$\therefore$ $$h(n) = \left(\frac{1}{2}\right)^n [u(n)-u(n-4)] + 2\left[1-\left(\frac{1}{2}\right)^{n-1}\right]u(n-2) + \left[\frac{1}{8} - 2\left(\frac{1}{2}\right)^n\right]u(n-6)$$

2.8 CIRCULAR SHIFT AND CIRCULAR SYMMETRY

The defining relation for the DFT requires signal values for $0 \leq n \leq N-1$. By implied periodicity these values correspond to one period of a periodic signal. If we wish to find the DFT of a time shifted signal $x[n - n_0]$, its value must also be selected over $(0, N - 1)$ from

its periodic extension. This concept is called circular shifting. To generate $x_p(n - n_0)$ we delay $x(n)$ by n_0, create the periodic extension of the shifted signal and pick N samples over $(0, N - 1)$. This is equivalent to moving the last n_0 samples of $x(n)$ to the beginning of the sequence. Similarly to generate $x_p(n + n_0)$ we move the first n_0 samples to the end of the sequence. Circular flipping generates the signal $x(-n)$ from $x(n)$. We flip $x(n)$, create the periodic extension of the flipped signal and pick N samples of the periodic extension over $(0, N - 1)$.

The circular shift and circular symmetry of a sequence are illustrated below.

Consider a finite duration sequence $x(n) = \{4, 3, 2, 1\}$ shown in Figure 2.16(a). The periodic extension can be expressed as $x_p(n) = x(n + N)$ where N is the period. Let $N = 4$.

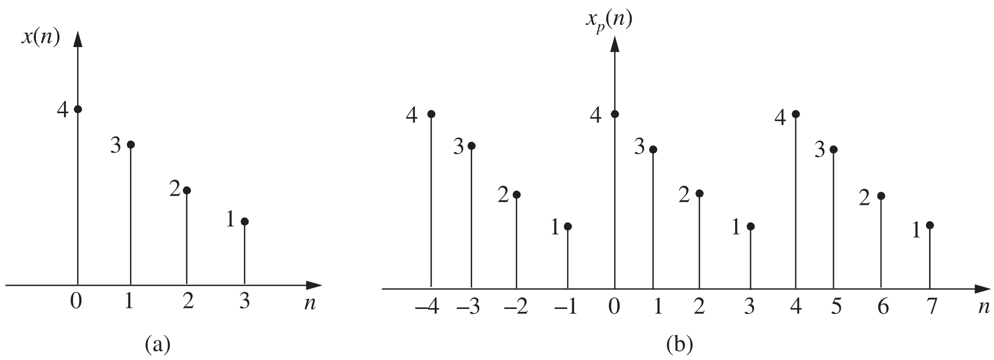


Figure 2.16 (a) $x(n) = \{4, 3, 2, 1\}$ (b) $x_p(n) = x(n+N)$ for $N = 4$.

Let us shift the sequence $x_p(n)$ by two units of time to the right as shown in Figure 2.17(a). Let us denote one period of that shifted sequence by $x'(n)$. One period of the shifted sequence is shown in Figure 2.17(b).

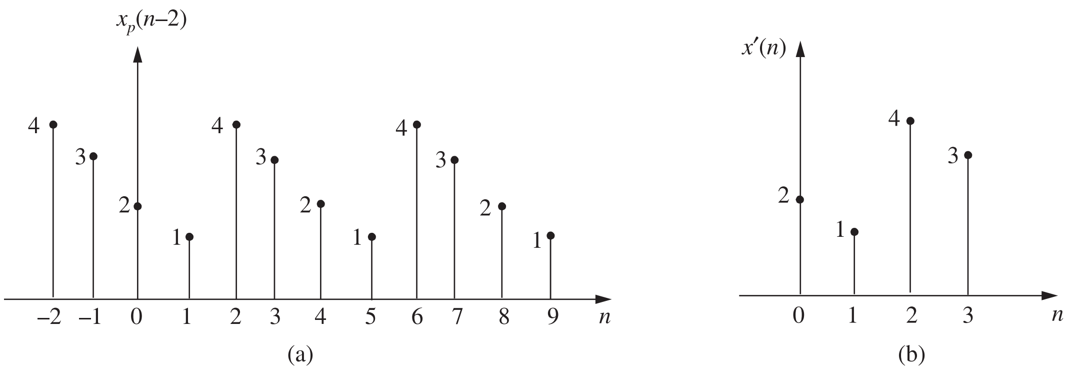


Figure 2.17 (a) $x_p(n-2) = x(n-2, (\text{mod } 4))$ (b) one period of $x_p(n-2)$.

The sequence $x'(n)$ can be represented by $x(n-2, (\text{mod } 4))$, where mod 4 indicates that the sequence repeats after 4 samples. The relation between the original sequence $x(n)$ and one period of the shifted sequence $x'(n)$ are shown below.

$$x'(n) = x(n-2, (\text{mod } 4))$$

$$\therefore \quad x'(0) = x(-2, (\text{mod } 4)) = x(2) = 2$$

$$x'(1) = x(-1, (\text{mod } 4)) = x(3) = 1$$

$$x'(2) = x(0, (\text{mod } 4)) = x(0) = 4$$

$$x'(3) = x(1, (\text{mod } 4)) = x(1) = 3$$

The sequences $x(n)$ and $x'(n)$ can be represented as points on a circle as shown in Figure 2.18[(a) and (b)]. From Figure 2.18[(a) and (b)], we can say that, $x'(n)$ is simply $x(n)$ shifted circularly by two units in time, where the counter clockwise direction has been arbitrarily selected as the positive direction. Thus we conclude that a circular shift of an N-point sequence is equivalent to a linear shift of its periodic extension and vice versa.

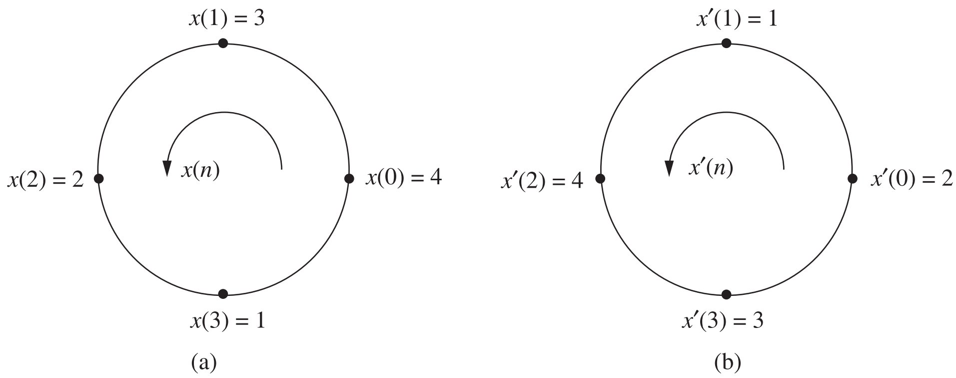


Figure 2.18 (a) Circular representation of $x(n)$ (b) Circular representation of $x'(n)$.

If a non-periodic N-point sequence is represented on the circumference of a circle, then it becomes a periodic sequence of periodicity N. When the sequence is shifted periodically, the samples repeat after N shift. This is similar to modulo N-operation. Hence in general the circular shift may be represented by the index mod-N.

Let $x(n)$ be a N-point sequence represented on a circle and $x'(n)$ be the sequence $x(n)$ shifted by k units of time.

Now, $$x'(n) = x(n-k, (\text{mod } N)) = x(n-k)_N$$

The circular representation of a sequence and the resulting periodicity give rise to new definitions for even symmetry, odd symmetry and the time reversal of the sequence.

An N-point sequence is called even, if it is symmetric about the point zero on the circle. This implies that,

$$x(n-N) = x(n); \quad \text{for } 0 \le n \le N-1$$

An N-point sequence is called odd, if it is antisymmetric about the point zero on the circle. This implies that,

$$x(n - N) = -x(n); \quad \text{for } 0 \le n \le N - 1$$

The time reversal of a N-point sequence is obtained by reversing its sample about the point zero on the circle. Thus the sequence $x(-n, (\text{mod } N))$ is simply given as:

$$x(-n, (\text{mod } N)) = x(n - N); \quad \text{for } 0 \le n \le N - 1$$

This time reversal is equivalent to plotting $x(n)$ in clockwise direction on a circle. An 8-point sequence and its folded version are shown in Figure 2.19[(a) and (b)].

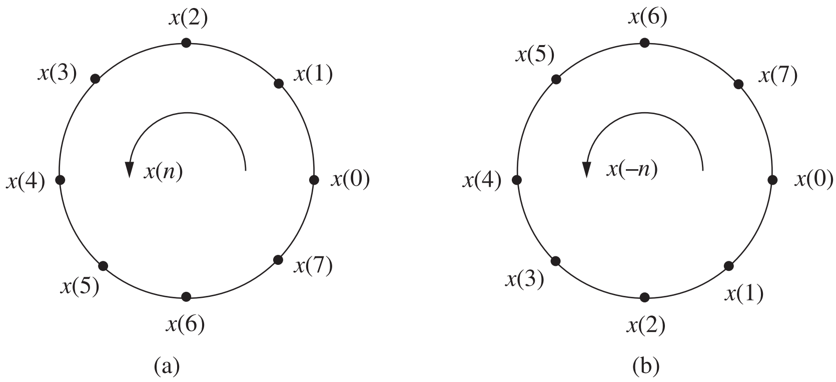


Figure 2.19 (a) An 8-point sequence (b) Its folded version.

EXAMPLE 2.27 Let $y(n) = \{2, 3, 4, 5, 6, 0, 0, 7\}$.

(a) Find one period of the circularly shifted signal $f(n) = y(n - 2)$.
(b) Find one period of the circularly shifted signal $g(n) = y(n + 2)$.
(c) Find one period of the circularly flipped signal $h(n) = y(-n)$.
(d) Plot $y(n), y(n-2), y(n+2)$ and $y(-n)$ on circles.

Solution:

(a) To create $f(n) = y(n - 2)$, we move the last two samples to the beginning. So

$$f(n) = y(n - 2) = \{0, 7, 2, 3, 4, 5, 6, 0\}$$

(b) To create $g(n) = y(n + 2)$, we move the first two samples to the end. So

$$g(n) = y(n + 2) = \{4, 5, 6, 0, 0, 7, 2, 3\}$$

(c) To create $h(n) = y(-n)$, we flip $y(n)$ to $\{7, 0, 0, 6, 5, 4, 3, 2\}$ and create its periodic extension to get

$$h(n) = y(-n) = \{2, 7, 0, 0, 6, 5, 4, 3\}$$

(d) The plots of $y(n), y(n-2), y(n+2)$ and $y(-n)$ are shown in Figure 2.20[(a), (b), (c), and (d)].

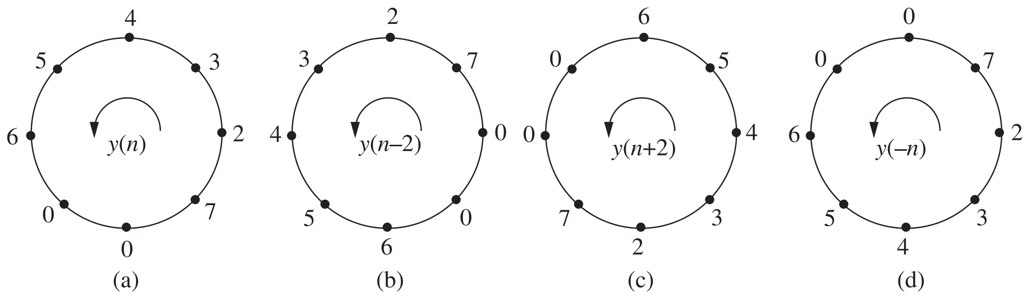


Figure 2.20 Plots of (a) $y(n) = \{2, 3, 4, 5, 6, 0, 0, 7\}$ (b) $y(n-2)$ (c) $y(n+2)$ (d) $y(-n)$.

2.9 PERIODIC OR CIRCULAR CONVOLUTION

The regular (or linear) convolution of two signals, both of which are periodic, does not exist. For this reason we resort to periodic convolution by using averages. If both $x_p(n)$ and $h_p(n)$ are periodic with identical period N, their periodic convolution generates a convolution result $y_p(n)$ that is also periodic with the same period N. The periodic convolution or circular convolution or cyclic convolution $y_p(n)$ of $x_p(n)$ and $h_p(n)$ is denoted as $y_p(n) = x_p(n) \oplus h_p(n)$. Over one period ($n = 0, 1, ..., N-1$), it is defined as

$$y_p(n) = x_p(n) \oplus h_p(n) = h_p(n) \oplus x_p(n) = \sum_{k=0}^{N-1} x_p(k) h_p(n-k) = \sum_{k=0}^{N-1} h_p(k) x_p(n-k)$$

An averaging factor of $1/N$ is sometimes included with the summation.

2.10 METHODS OF PERFORMING PERIODIC OR CIRCULAR CONVOLUTION

The circular convolution of two sequences requires that atleast one of the two sequences should be periodic. If both the sequences are non-periodic, then periodically extend one of the sequences and then perform circular convolution.

The circular convolution can be performed only if both the sequences consists of the same number of samples. If the sequences have different number of samples, then convert the smaller size sequence to the size of larger size sequence by appending zeros. The circular convolution produces a sequence whose length is same as that of input sequences.

Circular convolution basically involves the same four steps as linear convolution namely folding one sequence, shifting the folded sequence, multiplying the two sequences and finally summing the value of the product sequences.

The difference between the two is that in circular convolution the folding and shifting (rotating) operations are performed in a circular fashion by computing the index of one of the sequences by modulo-N operation. In linear convolution there is no modulo-N operation.

In circular convolution, any one of the sequence is folded and rotated without changing the result of circular convolution.

$$\therefore \qquad x_3(n) = \sum_{k=0}^{N-1} x_1(k)\, x_2(n-k)$$

$$= \sum_{k=0}^{N-1} x_2(k)\, x_1(n-k); \quad \text{for } n = 0, 1, ..., (N-1)$$

or
$$x_3(n) = x_1(n) \oplus x_2(n) = x_2(n) \oplus x_1(n)$$

The circular convolution of two sequences can be performed by the following methods:

2.10.1 Method 1 Graphical Method (Concentric Circle Method)

In graphical method, also called concentric circle method, the given sequences are represented on concentric circles. Given two sequences $x_1(n)$ and $x_2(n)$ the circular convolution of these two sequences, $x_3(n) = x_1(n) \oplus x_2(n)$ can be found using the following steps:

Step 1: Graph N samples of $x_1(n)$, as equally spaced points around an outer circle in anticlockwise direction.

Step 2: Starting at the same point as $x_1(n)$, graph N samples of $x_2(n)$ as equally spaced points around an inner circle in the clockwise direction. This corresponds to $x_2(-n)$.

Step 3: Multiply the corresponding samples on the two circles and sum the products to produce output, $x_3(0) = \sum_{n=0}^{N-1} x_1(n)\, x_2(-n)$

Step 4: Rotate the inner circle one sample at a time in anticlockwise direction and go to Step 3 to obtain the next value of output. If it is rotated by q samples, $x_3(q) = \sum_{n=0}^{N-1} x_1(n)\, x_2(q-n)$.

Step 5: Repeat Step 4 until the inner circle first sample lines up with the first sample of the outer circle once again.

2.10.2 Method 2 Circular Convolution Using Tabular Array

Let $x_1(n)$ and $x_2(n)$ be the given N-sample sequences. Let $x_3(n)$ be the N-sample sequence obtained by circular convolution of $x_1(n)$ and $x_2(n)$. The following procedure can be used to obtain one sample of $x_3(n)$ at $n = q$:

Step 1: Change the index from n to k, and write $x_1(k)$ and $x_2(k)$.

Step 2: Represent the sequences $x_1(k)$ and $x_2(k)$ as two rows of tabular array.

Step 3: Fold one of the sequences. Let us fold $x_2(k)$ to get $x_2(-k)$.

Step 4: Periodically extend the sequence $x_2(-k)$. Here the periodicity is N, where N is the length of the given sequences.

Step 5: Shift the sequence $x_2(-k)$, q times to get the sequence $x_2(q-k)$. If q is positive, then shift the sequence to the right and if q is negative, then shift the sequence to the left.

Step 6: The sample of $x_3(n)$ at $n = q$ is given by

$$x_3(q) = \sum_{k=0}^{N-1} x_1(k)\, x_2(q-k)$$

Determine the product sequence $x_1(k)\, x_2(q-k)$ for one period.

Step 7: The sum of the samples of the product sequence gives the sample $x_3(q)$ [i.e. $x_3(n)$ at $n = q$].

The above procedure is repeated for all possible values of n to get the sequence $x_3(n)$.

2.10.3 Method 3 Circular Convolution Using Matrices

For circular convolution, the length of the sequences must be same. If it is not same, it should be made same by appending with zeros.

Let $x_1(n)$ and $x_2(n)$ be the given N-sample sequences. The circular convolution of $x_1(n)$ and $x_2(n)$ yields another N-sample sequence $x_3(n)$.

In this method, an $N \times N$ matrix is formed using one of the sequence as shown below. Another sequence is arranged as a column vector (column matrix) of order $N \times 1$. The product of the two matrices gives the resultant sequence $x_3(n)$.

$$\begin{bmatrix} x_2(0) & x_2(N-1) & x_2(N-2) & \cdots & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(N-1) & \cdots & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & \cdots & x_2(4) & x_2(3) \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ x_2(N-2) & x_2(N-3) & x_2(N-4) & \cdots & x_2(0) & x_2(N-1) \\ x_2(N-1) & x_2(N-2) & x_2(N-3) & \cdots & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ \vdots \\ x_1(N-2) \\ x_1(N-1) \end{bmatrix} = \begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ \vdots \\ x_3(N-2) \\ x_3(N-1) \end{bmatrix}$$

EXAMPLE 2.28 Find the circular convolution of two finite duration sequences:

$$x_1(n) = \{1, 2, -1, -2, 3, 1\}, \quad x_2(n) = \{3, 2, 1\}$$

Solution: Let $x_3(n)$ be the circular convolution of $x_1(n)$ and $x_2(n)$. To find the circular convolution, both sequences must be of same length. Here $x_1(n)$ is of length 6 and $x_2(n)$ is of length 3. Therefore, we append three zeros to the sequence $x_2(n)$ and use concentric circle method to find circular convolution. So we have

$$x_1(n) = \{1, 2, -1, -2, 3, 1\}, \quad x_2(n) = \{3, 2, 1, 0, 0, 0\}$$

Graph all points of $x_1(n)$ on the outer circle in the anticlockwise direction. Starting at the same point as $x_1(n)$, graph all points of $x_2(n)$ on the inner circle in clockwise direction as shown in Figure 2.21(a). Multiply corresponding samples on the circles and add to obtain

$$x_3(0) = (1)(3) + (2)(0) + (-1)(0) + (-2)(0) + (3)(1) + (1)(2) = 8$$

Rotate the inner circle in anticlockwise direction by one sample, and multiply the corresponding samples and add the products to obtain $y(1)$ as shown in Figure 2.21(b).

Continue it to obtain $y(2)$, $y(3)$, $y(4)$ and $y(5)$ as shown in Figure 2.21[(c), (d), (e) and (f)].

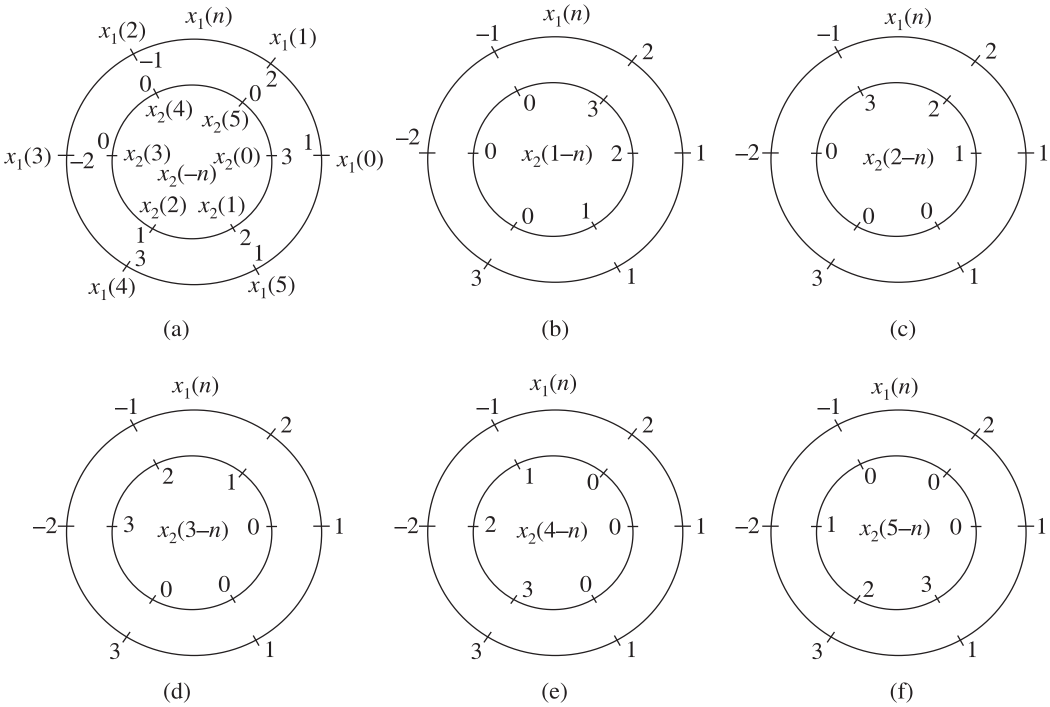


Figure 2.21 Computation of circular convolution by graphical method.

From Figure 2.21(b), $x_3(1) = (1)(2) + (2)(3) + (-1)(0) + (-2)(0) + (3)(0) + (1)(1) = 9$
From Figure 2.21(c), $x_3(2) = (1)(1) + (2)(2) + (-1)(3) + (-2)(0) + (3)(0) + (1)(0) = 2$
From Figure 2.21(d), $x_3(3) = (1)(0) + (2)(1) + (-1)(2) + (-2)(3) + (3)(0) + (1)(0) = -6$
From Figure 2.21(e), $x_3(4) = (1)(0) + (2)(0) + (-1)(1) + (-2)(2) + (3)(3) + (1)(0) = 4$
From Figure 2.21(f), $x_3(5) = (1)(0) + (2)(0) + (-1)(0) + (-2)(1) + (3)(2) + (1)(3) = 7$

Therefore, the circular convolution of $x_1(n)$ and $x_2(n)$ is:

$$x_3(n) = x_1(n) \oplus x_2(n) = \{8, 9, 2, -6, 4, 7\}$$

EXAMPLE 2.29 Find the circular convolution of $x_1(n) = \{1, 2, 1, 2\}$ and $x_2(n) = \{4, 3, 2, 1\}$ by the graphical method.

Solution: Let $x_3(n)$ be the circular convolution of $x_1(n)$ and $x_2(n)$. Here $x_1(n)$ and $x_2(n)$ are of the same length. So no padding of zeros is required. The computation of the circular convolution $x_3(n)$ of the given sequences $x_1(n) = \{1, 2, 1, 2\}$ and $x_2(n) = \{4, 3, 2, 1\}$ by the concentric circles method is done as shown in Figure 2.22[(a), (b), (c) and (d)].

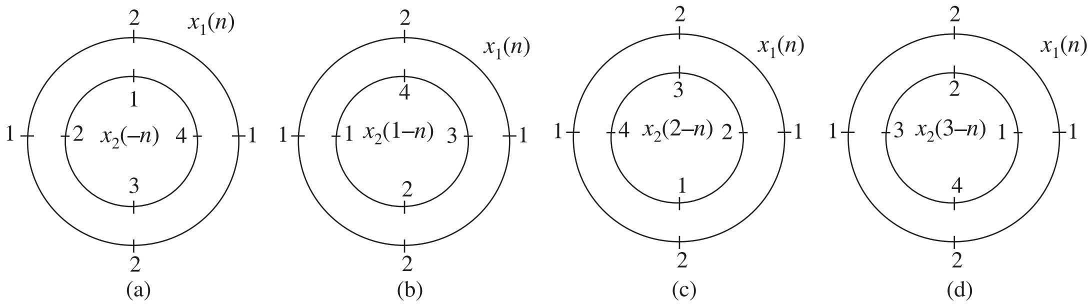


Figure 2.22 Computation of circular convolution by graphical method.

From Figure 2.22(a), $x_3(0) = (1)(4) + (2)(1) + (1)(2) + (2)(3) = 14$
From Figure 2.22(b), $x_3(1) = (1)(3) + (2)(4) + (1)(1) + (2)(2) = 16$
From Figure 2.22(c), $x_3(2) = (1)(2) + (2)(3) + (1)(4) + (2)(1) = 14$
From Figure 2.22(d), $x_3(3) = (1)(1) + (2)(2) + (1)(3) + (2)(4) = 16$

Therefore, the circular convolution of $x_1(n)$ and $x_2(n)$ is:

$$x_3(n) = x_1(n) \oplus x_2(n) = \{14, 16, 14, 16\}$$

EXAMPLE 2.30 Find the circular convolution of

$$x_1(n) = \{1, 2, 1, 2\} \quad \text{and} \quad x_2(n) = \{4, 3, 2, 1\}$$

using the matrices method.

Solution: Let $x_3(n)$ be the circular convolution of $x_1(n)$ and $x_2(n)$.

Given $\qquad x_1(n) = \{x_1(0), x_1(1), x_1(2), x_1(3)\} = \{1, 2, 1, 2\}$

and $\qquad x_2(n) = \{x_2(0), x_2(1), x_2(2), x_2(3)\} = \{4, 3, 2, 1\}$

To find the circular convolution using the matrix method, form an $N \times N$ (4×4) matrix using the elements of $x_2(n)$ and arrange the sequence $x_1(n)$ as a column vector of order $N \times 1$ (4×1) as shown below. The product of the two matrices gives the sequence $x_3(n)$, the circular convolution of $x_1(n)$ and $x_2(n)$.

$$\begin{bmatrix} x_2(0) & x_2(3) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & x_2(3) \\ x_2(3) & x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \end{bmatrix} = \begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ x_3(3) \end{bmatrix}$$

Using the given samples, we have

$$\begin{bmatrix} 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ x_3(3) \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \\ 14 \\ 16 \end{bmatrix}$$

Therefore, the circular convolution of $x_1(n)$ and $x_2(n)$ is:

$$x_3(n) = x_1(n) \oplus x_2(n) = \{14, 16, 14, 16\}$$

EXAMPLE 2.31 Perform the circular convolution of the two sequences $x_1(n)$ and $x_2(n)$ where

$$x_1(n) = \{1.6, 1.4, 1.2, 1.0, 0.8, 0.6, 0.4, 0.2\}$$

and $$x_2(n) = \{1.5, 1.3, 1.1, 0.9, 0.7, 0.5, 0.3, 0.1\}$$

by the matrices method.

Solution: Let $x_3(n)$ be the circular convolution of $x_1(n)$ and $x_2(n)$. The circular convolution of the given sequences:

$$\begin{aligned} x_1(n) &= \{x_1(0), x_1(1), x_1(2), x_1(3), x_1(4), x_1(5), x_1(6), x_1(7)\} \\ &= \{1.6, 1.4, 1.2, 1.0, 0.8, 0.6, 0.4, 0.2\} \end{aligned}$$

and $$\begin{aligned} x_2(n) &= \{x_2(0), x_2(1), x_2(2), x_2(3), x_2(4), x_2(5), x_2(6), x_2(7)\} \\ &= \{1.5, 1.3, 1.1, 0.9, 0.7, 0.5, 0.3, 0.1\} \end{aligned}$$

can be determined using matrices as follows:

$$\begin{bmatrix} 1.5 & 0.1 & 0.3 & 0.5 & 0.7 & 0.9 & 1.1 & 1.3 \\ 1.3 & 1.5 & 0.1 & 0.3 & 0.5 & 0.7 & 0.9 & 1.1 \\ 1.1 & 1.3 & 1.5 & 0.1 & 0.3 & 0.5 & 0.7 & 0.9 \\ 0.9 & 1.1 & 1.3 & 1.5 & 0.1 & 0.3 & 0.5 & 0.7 \\ 0.7 & 0.9 & 1.1 & 1.3 & 1.5 & 0.1 & 0.3 & 0.5 \\ 0.5 & 0.7 & 0.9 & 1.1 & 1.3 & 1.5 & 0.1 & 0.3 \\ 0.3 & 0.5 & 0.7 & 0.9 & 1.1 & 1.3 & 1.5 & 0.1 \\ 0.1 & 0.3 & 0.5 & 0.7 & 0.9 & 1.1 & 1.3 & 1.5 \end{bmatrix} \begin{bmatrix} 1.6 \\ 1.4 \\ 1.2 \\ 1.0 \\ 0.8 \\ 0.6 \\ 0.4 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 5.2 \\ 6 \\ 6.48 \\ 6.64 \\ 6.48 \\ 6 \\ 5.2 \\ 4.08 \end{bmatrix}$$

Therefore, the circular convolution of $x_1(n)$ and $x_2(n)$ is:

$$x_3(n) = x_1(n) \oplus x_2(n) = \{5.2, 6, 6.48, 6.64, 6.48, 6, 5.2, 4.08\}$$

EXAMPLE 2.32 Find the circular convolution of $x_1(n) = \{1, 2, 1, 2\}$ and $x_2(n) = \{4, 3, 2, 1\}$ by the tabular method.

Solution: Changing the index from n to k, the given sequences are:

$$x_1(k) = \{1, 2, 1, 2\} \quad \text{and} \quad x_2(k) = \{4, 3, 2, 1\}$$

The given sequences can be represented in the tabular array as shown below. Here the folded sequences $x_2(-k)$ is periodically extended with a periodicity of $N = 4$.

Let $x_3(n)$ be the sequence obtained by convolution of $x_1(n)$ and $x_2(n)$. Each sample of $x_3(n)$ is given by the equation:

$$x_3(n) = \sum_{k=0}^{N-1} x_1(k)\, x_2(n-k)$$

To determine a sample of $x_3(n)$ at $n = q$, multiply the sequences $x_1(k)$ and $x_2(q - k)$ to get a product sequence $x_1(k)\, x_2(q - k)$, i.e. multiply the corresponding elements of the rows $x_1(k)$ and $x_2(q - k)$. The sum of all the samples of the product sequence gives $x_3(q)$.

k	–3	–2	–1	0	1	2	3
$x_1(k)$	–	–	–	1	2	1	2
$x_2(k)$	–	–	–	4	3	2	1
$x_2(-k)$	1	2	3	4	1	2	3
$x_2(1-k)$	–	1	2	3	4	1	2
$x_2(2-k)$	–	–	1	2	3	4	1
$x_2(3-k)$	–	–	–	1	2	3	4

When $n = 0$, $x_3(0) = \sum_{k=0}^{3} x_1(k)\, x_2(-k)$

$$= x_1(0)\, x_2(0) + x_1(1)\, x_2(-1) + x_1(2)\, x_2(-2) + x_1(3)\, x_2(-3)$$

$$= (1)(4) + (2)(1) + (1)(2) + (2)(3) = 14$$

When $n = 1$, $x_3(1) = \sum_{k=0}^{3} x_1(k)\, x_2(1-k)$

$$= x_1(0)\, x_2(1) + x_1(1)\, x_2(0) + x_1(2)\, x_2(-1) + x_1(3)\, x_2(-2)$$

$$= (1)(3) + (2)(4) + (1)(1) + (2)(2) = 16$$

When $n = 2$, $x_3(2) = \sum_{k=0}^{3} x_1(k)\, x_2(2-k)$

$$= x_1(0)\, x_2(2) + x_1(1)\, x_2(1) + x_1(2)\, x_2(0) + x_1(3)\, x_2(-1)$$

$$= (1)(2) + (2)(3) + (1)(4) + (2)(1) = 14$$

When $n = 3$, $x_3(3) = \sum_{k=0}^{3} x_1(k)\, x_2(3-k)$

$$= x_1(0)\, x_2(3) + x_1(1)\, x_2(2) + x_1(2)\, x_2(1) + x_1(3)\, x_2(0)$$

$$= (1)(1) + (2)(2) + (1)(3) + (2)(4) = 16$$

Therefore, the circular convolution of $x_1(n)$ and $x_2(n)$ is:

$$x_3(n) = x_1(n) \oplus x_2(n) = \{14, 16, 14, 16\}$$

EXAMPLE 2.33 Find the circular convolution of $x(n) = \{1, 0.5\}$; $h(n) = \{0.5, 1\}$ by all the methods.

Solution: (a) *Graphical method*
Let $y(n)$ be the circular convolution of $x(n)$ and $h(n)$. The computation of circular convolution of $x(n)$ and $h(n)$ by the graphical method is shown in Figure 2.23.

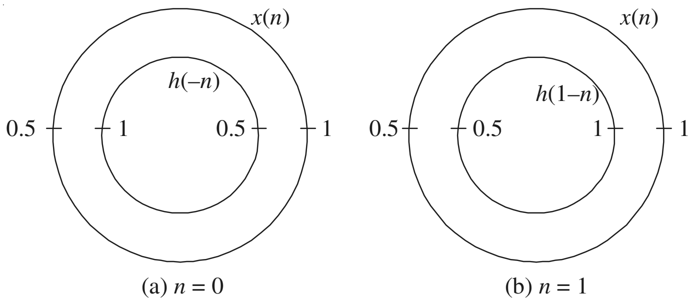


Figure 2.23 Graphical method (a) $n = 0$, (b) $n = 1$.

$$y(p) = \sum_{n=0}^{N-1} x(n)\, h(p-n)$$

From Figure 2.23(a), when $y(0) = (1)(0.5) + (0.5)(1) = 1$
From Figure 2.23(b), when $y(1) = (1)(1) + (0.5)(0.5) = 1.25$
Therefore, the circular convolution of $x(n)$ and $h(n)$ is

$$\therefore \qquad y(n) = x(n) \oplus h(n) = \{1, 1.25\}$$

(b) *By Tabular array*
Changing the index from n to k, we have $x(k) = \{1, 0.5\}$ and $h(k) = (0.5, 1)$. The computation of circular convolution of $x(n)$ and $h(n)$ by tabular array is shown below.
Here the folded sequence $h(-k)$ is periodically extended with a periodicity of $N = 2$. Let $y(n)$ be the sequence obtained by convolution of $x(n)$ and $h(n)$. Each sample of $y(n)$ is given by the equation

$$y(n) = \sum_{k=0}^{N-1} x(k)\, h(n-k)$$

To determine a sample of $y(n)$ at $n = p$, multiply the sequences $x(k)$ and $h(p - k)$ to get a product sequence $x(k)\, h(p - k)$. The sum of all the samples of the product sequence gives $y(p)$.

k	-1	0	1
$x(k)$	–	1	0.5
$h(k)$	–	0.5	1
$h(-k)$	1	0.5	1
$h(1-k)$	–	1	0.5

$$y(n) = x(n) \oplus h(n) = \sum_{k=0}^{N-1} x(k)\, h(n-k)$$

When $n = 0$, $y(0) = \sum_{k=0}^{1} x(k)\; h(-k) = x(0)\,h(0) + x(1)\,h(-1)$

$$= (1)(0.5) + (0.5)(1) = 1$$

When $n = 1$, $y(1) = \sum_{k=0}^{1} x(k)\; h(1-k) = x(0)\,h(1) + x(1)\,h(0)$

$$= (1)(1) + (0.5)(0.5) = 1.25$$

$\therefore$ $$y(n) = x(n) \oplus h(n) = \{y(0), y(1)\} = \{1, 1.25\}$$

(c) *By Matrix method*

The circular convolution $y(n)$ of $x(n)$ and $h(n)$ is computed using matrices as follows:

$$\begin{pmatrix} h(0) & h(1) \\ h(1) & h(0) \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \end{pmatrix} = \begin{pmatrix} y(0) \\ y(1) \end{pmatrix}$$

$$\begin{pmatrix} 0.5 & 1 \\ 1 & 0.5 \end{pmatrix} \begin{pmatrix} 1 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1.25 \end{pmatrix}$$

$\therefore$ $$y(n) = x(n) \oplus h(n) = \{y(0), y(1)\} = \{1, 1.25\}$$

EXAMPLE 2.34 Compute the circular convolution of the following sequences and compare it with linear convolution

$\therefore$ $$x(n) = \{1, -1, 1, -1\}; \quad h(n) = \{1, 2, 3, 4\}$$

Solution: Given the sequences $x(n) = \{1, -1, 1, -1\}$ and $h(n) = \{1, 2, 3, 4\}$, the linear convolution $y(n)$ of $x(n)$ and $h(n)$ can be determined using tabular method (Table 2.11) as shown below.

TABLE 2.11 Table for computing $y(n)$

		$h(n)$			
		1	2	3	4
	1	1	2	3	4
	−1	−1	−2	−3	−4
$x(n)$	1	1	2	3	4
	1	−1	−2	−3	−4

From Table 2.11, we observe that the linear convolution $y(n)$ of $x(n)$ and $h(n)$ is:

$$y(n) = x(n) * h(n) = \{1,\ -1+2,\ 1-2+3,\ -1+2-3+4,\ -2+3-4,\ -3+4,\ -4\}$$
$$= \{1, 1, 2, 2, -3, 1, -4\}$$

The circular convolution $y(n)$ of the given sequences $x(n)$ and $h(n)$ can be determined using the concentric circles method as shown in Figure 2.24.

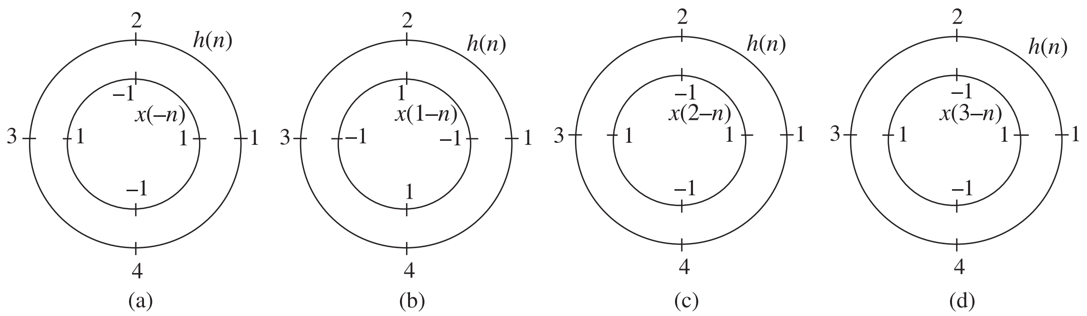


Figure 2.24 Circular convolution of $x(n)$ and $h(n)$.

From Figure 2.24(a), $y(0) = (1)(1) + (4)(-1) + (3)(1) + (2)(-1) = -2$
From Figure 2.24(b), $y(1) = (1)(-1) + (4)(1) + (3)(-1) + (2)(1) = 2$
From Figure 2.24(c), $y(2) = (1)(1) + (4)(-1) + (3)(1) + (2)(-1) = -2$
From Figure 2.24(d), $y(3) = (1)(-1) + (4)(1) + (3)(-1) + (2)(1) = 2$

Therefore, the circular convolution $y(n)$ of $x(n)$ and $h(n)$ is:

$$y(n) = x(n) \oplus h(n) = \{-2, 2, -2, 2\}$$

Comparing the linear and circular convolutions, we observe that the circular convolution of two sequences of length N each results in a sequence of length N only but the linear convolution of two sequences of length N each yields a sequence of length $N + N - 1$ and also $\Sigma y(n)$ in both the cases is same.

EXAMPLE 2.35 Find the circular convolution of the following sequences and compare it with linear convolution:

$$x_1(n) = \{1, 2, 0, 1\} \quad \text{and} \quad x_2(n) = \{2, 2, 1, 1\}$$

Solution: Given the sequences $x_1(n) = \{1, 2, 0, 1\}$ and $x_2(n) = \{2, 2, 1, 1\}$ the linear convolution $y(n)$ of $x_1(n)$ and $x_2(n)$ can be determined using tabular method as shown below.

TABLE 2.12 Table for computing $y(n)$

$x_1(n)$ \ $x_2(n)$	2	2	1	1
1	2	2	1	1
2	4	4	2	2
0	0	0	0	0
1	2	2	1	1

From Table 2.12, we observe that the linear convolution of $x_1(n)$ and $x_2(n)$ is:

$$y(n) = x_1(n) * x_2(n) = \{2, 4 + 2, 0 + 4 + 1, 2 + 0 + 2 + 1, 2 + 0 + 2, 1 + 0, 1\}$$

$$= \{2, 6, 5, 5, 4, 1, 1\}$$

The circular convolution $x(n)$ of the given sequences $x_1(n)$ and $x_2(n)$ can be determined using the concentric circles method as shown in Figure 2.25.

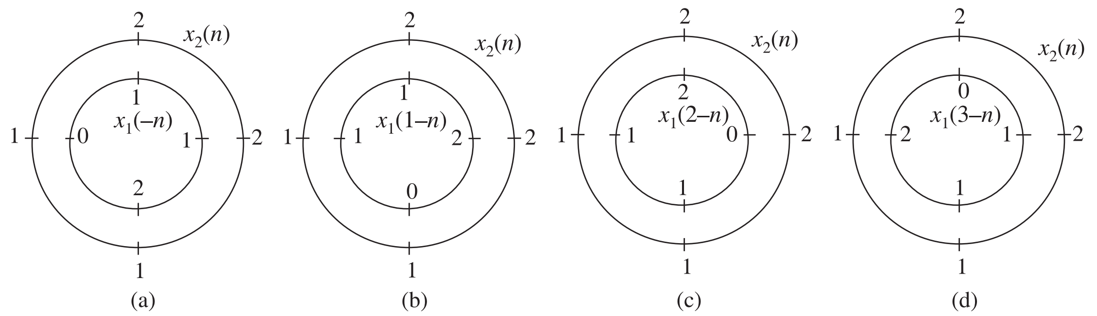


Figure 2.25 Circular convolution of $x_1(n)$ and $x_2(n)$.

From Figure 2.25(a), $x(0) = (2)(1) + (2)(1) + (1)(0) + (1)(2) = 6$
From Figure 2.25(b), $x(1) = (2)(2) + (2)(1) + (1)(1) + (1)(0) = 7$
From Figure 2.25(c), $x(2) = (2)(0) + (2)(2) + (1)(1) + (1)(1) = 6$
From Figure 2.25(d), $x(3) = (2)(1) + (2)(0) + (1)(2) + (1)(1) = 5$

Therefore, the circular convolution of $x_1(n)$ and $x_2(n)$ is:

$$x(n) = x_1(n) \oplus x_2(n) = \{6, 7, 6, 5\}$$

Comparing the linear and circular convolutions, we observe that the circular convolution of two sequences of length N each results in a sequence of length N only, whereas linear convolution of two sequences of length N each results in a sequence of length $N + N - 1$. The sum of the samples of the sequences of linear convolution $\Sigma y(n)$ and of circular convolution $\Sigma x(n)$ is same.

EXAMPLE 2.36 The input $x(n)$ and the impulse response $h(n)$ of a LTI system are given by

$$x(n) = \{\underset{\uparrow}{-2}, 1, -1, 2\}; \quad h(n) = \{0.25, \underset{\uparrow}{2}, 1, -1, 0.5\}$$

Determine the response of the system using

(a) Linear convolution (b) Circular convolution

Solution: (a) *Response of LTI system using linear convolution*
Let $y(n)$ be the response of LTI system. By convolution sum formula,

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

The sequence $x(n)$ starts at $n_1 = 0$, and the sequence $h(n)$ starts at $n_2 = -1$. Hence the sequence $y(n)$ starts at $n = n_1 + n_2 = 0 + (-1) = -1$.

The length of $x(n)$ is $N_1 = 4$ and the length of $h(n)$ is $N_2 = 5$. Hence the length of $y(n)$ is $N = N_1 + N_2 - 1 = 4 + 5 - 1 = 8$. So $y(n)$ extends from -1 to 6 (8 samples). The linear convolution of $x(n)$ and $h(n)$ is computed using the tabular method (Table 2.13) as shown below.

TABLE 2.13 Table for computing $y(n)$

		$x(n)$			
		–2	1	–1	2
	0.25	–0.5	0.25	–0.25	0.5
	2	–4	2	–2	4
$h(n)$	1	–2	1	–1	2
	–1	2	–1	1	–2
	0.5	–1	0.5	–0.5	1

From Table 2.13, we get the response:

$$y(n) = \{-0.5, -4 + 0.25, -2 + 2 - 0.25, 2 + 1 - 2 + 0.5, -1 - 1 - 1 + 4, 0.5 + 1 + 2, -0.5 - 2, 1\}$$
$$= \{-0.5, \underset{\uparrow}{-3.75}, -0.25, 1.5, 1, 3.5, -2.5, 1\}$$

(b) *Response of LTI system using circular convolution*

The response of LTI system is given by the linear convolution of $x(n)$ and $h(n)$. Let $y(n)$ be the response sequence. To get the result of linear convolution from circular convolution, both the sequences should be converted to the size of $y(n)$ and circular convolution is to be performed on these converted sequences.

The length of $x(n)$ is 4 and the length of $h(n)$ is 5. Hence the length of $y(n)$ is 4 + 5 – 1 = 8. Therefore, both the sequences should be converted to 8-point sequences by appending with zeros.

The $x(n)$ starts at $n = 0$ and $h(n)$ starts at $n = -1$. So $y(n)$ will start at $n = 0 + (-1) = -1$ and it extends from –1 to 6.

$$x(n) = \{\underset{\uparrow}{-2}, 1, -1, 2, 0, 0, 0, 0\} \quad \text{and} \quad h(n) = \{0.25, \underset{\uparrow}{2}, 1, -1, 0.5, 0, 0, 0\}$$

The circular convolution of $x(n)$ and $h(n)$ using matrices is performed as follows:

$$\begin{bmatrix} 0.25 & 0 & 0 & 0 & 0.5 & -1 & 1 & 2 \\ 2 & 0.25 & 0 & 0 & 0 & 0.5 & -1 & 1 \\ 1 & 2 & 0.25 & 0 & 0 & 0 & 0.5 & -1 \\ -1 & 1 & 2 & 0.25 & 0 & 0 & 0 & 0.5 \\ 0.5 & -1 & 1 & 2 & 0.25 & 0 & 0 & 0 \\ 0 & 0.5 & -1 & 1 & 2 & 0.25 & 0 & 0 \\ 0 & 0 & 0.5 & -1 & 1 & 2 & 0.25 & 0 \\ 0 & 0 & 0 & 0.5 & -1 & 1 & 2 & 0.25 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ -1 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.5 \\ -3.75 \\ -0.25 \\ 1.5 \\ 1 \\ 3.5 \\ -2.5 \\ 1 \end{bmatrix}$$

We can observe that the response $y(n)$ obtained by the circular convolution of $x(n)$ and $h(n)$ is:

$$y(n) = x(n) \oplus h(n) = \{-0.5, \underset{\uparrow}{-3.75}, -0.25, 1.5, 1, 3.5, -2.5, 1\}$$

which is same as that obtained earlier by the linear convolution.

EXAMPLE 2.37 The input $x(n)$ and the impulse response $h(n)$ of a LTI system are given by

$$x(n) = \{1, \underset{\uparrow}{2}, 0, 1\}; \quad h(n) = \{2, \underset{\uparrow}{2}, 1, 1\}$$

Determine the response of the system using
(a) Linear convolution (b) Circular convolution

Solution: (a) *Response of LTI system using linear convolution*
Let $y(n)$ be the response of LTI system. By convolution sum formula,

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

The sequence $x(n)$ starts at $n_1 = -1$, and the sequence $h(n)$ starts at $n_2 = -1$. Hence the sequence $y(n)$ starts at $n = n_1 + n_2 = -1 + (-1) = -2$.

The length of $x(n)$ is $N_1 = 4$ and the length of $h(n)$ is $N_2 = 4$. Hence the length of $y(n)$ is $N = N_1 + N_2 - 1 = 4 + 4 - 1 = 7$. So $y(n)$ extends from –2 to 4 (7 samples). The linear convolution of $x(n)$ and $h(n)$ is computed using tabular method as shown below.

TABLE 2.14 Table for computing $y(n)$

$h(n)$ \ $x(n)$	1	2	0	1
2	2	4	0	2
2	2	4	0	2
1	1	2	0	1
1	1	2	0	1

From Table 2.14, we get the response:

$$\begin{aligned} y(n) &= 2,\ 2+4,\ 1+4+0,\ 1+2+0+2,\ 2+0+2,\ 0+1,\ 1 \\ &= \{2, 6, \underset{\uparrow}{5}, 5, 4, 1, 1\} \end{aligned}$$

(b) *Response of LTI system using circular convolution*
The response of LTI system is given by linear convolution of $x(n)$ and $h(n)$. Let $y(n)$ be the response sequence. To get the result of linear convolution from circular convolution, both the sequences should be converted to the size of $y(n)$ and circular convolution is to be performed on these converted sequences.

The length of $x(n)$ is 4 and the length of $h(n)$ is 4. Hence the length of $y(n)$ is $4 + 4 - 1 = 7$. Therefore, both the sequences should be converted to 7-point sequences by appending with zeros.

The $x(n)$ starts at $n = -1$ and $h(n)$ starts at $n = -1$. So $y(n)$ will start at $n = (-1) + (-1) = -2$ and it extends from –2 to 4.

$$x(n) = \{1, \underset{\uparrow}{2}, 0, 1, 0, 0, 0\} \text{ and } h(n) = \{2, \underset{\uparrow}{2}, 1, 1, 0, 0, 0\}$$

The circular convolution of $x(n)$ and $h(n)$ using matrices is given as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 2 \\ 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 5 \\ 5 \\ 4 \\ 1 \\ 1 \end{bmatrix}$$

We can observe that the response $y(n)$ obtained by the circular convolution of $x(n)$ and $h(n)$ is:

$$y(n) = x(n) \oplus h(n) = \{2, 6, \underset{\uparrow}{5}, 5, 4, 1, 1\}$$

which is same as that obtained earlier by the linear convolution.

2.11 LINEAR (REGULAR) CONVOLUTION FROM PERIODIC CONVOLUTION

The linear convolution of $x(n)$ (with length N_1) and $h(n)$ (with length N_2) may also be found using the periodic convolution. Since the linear convolution of $x(n)$ and $h(n)$ yields a sequence of length $N_1 + N_2 - 1$, convert the sequences $x(n)$ and $h(n)$ to length $N_1 + N_2 - 1$ by padding with zeros and then perform circular convolution. The regular convolution of the original unpadded sequences equals the periodic convolution of the zero-padded sequences.

EXAMPLE 2.38 Find the regular convolution of the following sequences using periodic convolution:

$$x(n) = \{3, -2, 1, 4\} \quad \text{and} \quad h(n) = (2, 5, 3\}$$

Solution: Given $x(n) = \{3, -2, 1, 4\}$ and $h(n) = (2, 5, 3\}$, their regular convolution has $4 + 3 - 1 = 6$ samples. Using trailing zeros, we create the padded sequences:

$$x_p(n) = \{3, -2, 1, 4, 0, 0\} \text{ and } h_p(n) = \{2, 5, 3, 0, 0, 0\}$$

The periodic convolution of these padded sequences is obtained using the matrices method.

$$\begin{bmatrix} 3 & 0 & 0 & 4 & 1 & -2 \\ -2 & 3 & 0 & 0 & 4 & 1 \\ 1 & -2 & 3 & 0 & 0 & 4 \\ 4 & 1 & -2 & 3 & 0 & 0 \\ 0 & 4 & 1 & -2 & 3 & 0 \\ 0 & 0 & 4 & 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 1 \\ 7 \\ 23 \\ 12 \end{bmatrix}$$

$$\therefore \quad y(n) = \{3, -2, 1, 4\} * \{2, 5, 3\} = \{3, -2, 1, 4, 0, 0\} \circledast \{2, 5, 3, 0, 0, 0\}$$
$$= \{6, 11, 1, 7, 23, 12\}$$

EXAMPLE 2.39 Let $x(n) = \{1, 3, 0, 2, 1\}$ and $h(n) = \{2, 3\}$.

(a) How many zeros must be appended to $x(n)$ and $h(n)$ in order to generate their regular convolution from the periodic convolution of the zero-padded sequences?
(b) What is the regular convolution of the original sequences?
(c) What is the circular convolution of the zero-padded sequences?
(d) What is the regular convolution of the zero-padded sequences?

Solution:

(a) Since the length of $x(n)$ is 5 and the length of $h(n)$ is 2, the length of $y(n)$, the linear convolution of $x(n)$ and $h(n)$ is $5 + 2 - 1 = 6$. Therefore, one zero must be appended to $x(n)$ and four zeros must be appended to $h(n)$ in order to generate their regular convolution from the periodic convolution of the zero-padded sequences.
(b) The regular convolution of original sequences $x(n)$ and $h(n)$ is obtained using matrix method as follows:

$$\begin{array}{cc|ccccc} & & & & x(n) & & \\ & & 1 & 3 & 0 & 2 & 1 \\ \hline h(n) & 2 & 2 & 6 & 0 & 4 & 2 \\ & 3 & 3 & 9 & 0 & 6 & 3 \end{array}$$

$$\therefore \quad x(n) * h(n) = \{2, 3 + 6, 9 + 0, 0 + 4, 6 + 2, 3\} = \{2, 9, 9, 4, 8, 3\}$$

(c) The circular convolution of the zero-padded sequences $x(n)$ and $h(n)$ is same as the regular convolution of original sequences.

$$\{1, 3, 0, 2, 1, 0\} \circledast \{2, 3, 0, 0, 0, 0\} = \{1, 3, 0, 2, 1\} * \{2, 3\} = \{2, 9, 9, 4, 8, 3\}$$

(d) The regular convolution of the zero-padded sequences is obtained by padding $1 + 4 = 5$ trailing zeros to the regular convolution.

$$\{1, 3, 0, 2, 1, 0\} * \{2, 3, 0, 0, 0, 0\} = \{2, 9, 9, 4, 8, 3, 0, 0, 0, 0, 0\}$$

2.12 PERIODIC CONVOLUTION FROM LINEAR CONVOLUTION

Periodic convolution can be implemented using wrap-around. We find the linear convolution of one period of $x_p(n)$ and $h_p(n)$ which will have $(2N - 1)$ samples. We then extend its length to $2N$ (by appending a zero), slice it into two halves (of length N each), line up the second half with the first, and add the two halves to get the periodic convolution.

EXAMPLE 2.40 Find the periodic convolution of the following sequences using linear convolution.

(a) $x_1(n) = \{1, 2, -1, -2, 3, 1\}$ $\quad x_2(n) = \{3, 2, 1\}$
(b) $x_1(n) = \{1, 2, 1, 2\}$ $\quad x_2(n) = \{4, 3, 2, 1\}$
(c) $x(n) = \{1, -1, 1, -1\}$ $\quad h(n) = \{1, 2, 3, 4\}$
(d) $x_1(n) = \{1, 2, 0, 1\}$ $\quad x_2(n) = \{2, 2, 1, 1\}$
(e) $x(n) = \{1, 0.5\}$ $\quad h(n) = \{0.5, 1\}$

Solution:

(a) Given $x_1(n) = \{1, 2, -1, -2, 3, 1\}$ and $x_2(n) = \{3, 2, 1\}$. For periodic convolution, both sequences must be of same length. So appending $x_2(n)$ with zeros, we have

$$x_1(n) = \{1, 2, -1, -2, 3, 1\} \quad \text{and} \quad x_2(n) = \{3, 2, 1, 0, 0, 0\}$$

The period $N = 6$. First, we find the linear convolution $y(n)$ using sum-by-column method as follows:

Index n	0	1	2	3	4	5	6	7	8	9	10
$x_1(n)$	1	2	–1	–2	3	1					
$x_2(n)$	3	2	1	0	0	0					
	3	6	–3	–6	9	3					
		2	4	–2	–4	6	2				
			1	2	–1	–2	3	1			
				0	0	0	0	0	0		
					0	0	0	0	0	0	
						0	0	0	0	0	0
$y(n)$	3	8	2	–6	4	7	5	1	0	0	0

Then, we append a zero, wrap-around the last six samples and add.

Index n	0	1	2	3	4	5
First half of $y(n)$	3	8	2	–6	4	7
Wrapped-around half of $y(n)$	5	1	0	0	0	0
Periodic convolution $y_p(n)$	8	9	2	–6	4	7

(b) Given $x_1(n) = \{1, 2, 1, 2\}$ and $x_2(n) = \{4, 3, 2, 1\}$. The period $N = 4$. First, we find the linear convolution $y(n)$ using sum-by-column method as follows:

Index n	0	1	2	3	4	5	6
$x_1(n)$	1	2	1	2			
$x_2(n)$	4	3	2	1			
	4	8	4	8			
		3	6	3	6		
			2	4	2	4	
				1	2	1	2
$y(n)$	4	11	12	16	10	5	2

Then, we append a zero, wrap-around the last four samples and add.

Index n	0	1	2	3
First half of $y(n)$	4	11	12	16
Wrapped-around half of $y(n)$	10	5	2	0
Periodic convolution $y_p(n)$	14	16	14	16

(c) Given $x(n) = \{1, -1, 1, -1\}$ and $h(n) = \{1, 2, 3, 4\}$. The period $N = 4$. First, we find the linear convolution $y(n)$ using sum-by-column method as follows:

Index n	0	1	2	3	4	5	6
$x(n)$	1	–1	1	–1			
$h(n)$	1	2	3	4			
	1	–1	1	–1			
		2	–2	2	–2		
			3	–3	3	–3	
				4	–4	4	–4
$y(n)$	1	1	2	2	–3	1	–4

Then, we append a zero, wrap-around the last four samples and add.

Index n	0	1	2	3
First half of $y(n)$	1	1	2	2
Wrapped-around half of $y(n)$	–3	1	–4	0
Periodic convolution $y_p(n)$	–2	2	–2	2

(d) Given $x_1(n) = \{1, 2, 0, 1\}$ and $x_2(n) = \{2, 2, 1, 1\}$. The period $N = 4$. First, we find the linear convolution $y(n)$ using sum-by-column method as follows.

Index n	0	1	2	3	4	5	6
$x_1(n)$	1	2	0	1			
$x_2(n)$	2	2	1	1			

	2	4	0	2			
		2	4	0	2		
			1	2	0	1	
				1	2	0	1
$y(n)$	2	6	5	5	4	1	1

Then, we append a zero, wrap-around the last four samples and add.

Index n	0	1	2	3
First half of $y(n)$	2	6	5	5
Wrapped-around half of $y(n)$	4	1	1	0
Periodic convolution $y_p(n)$	6	7	6	5

(e) Given $x(n) = \{1, 0.5\}$ and $h(n) = \{0.5, 1\}$. The period $N = 2$. First, we find the linear convolution $y(n)$ using sum-by-column method as follows.

Index n	0	1	2
$x_1(n)$	1	0.5	
$x_2(n)$	0.5	1	
	0.5	0.25	
		1	0.5
$y(n)$	0.5	1.25	0.5

Then, we append a zero, wrap-around the last two samples and add.

Index n	0	1
First half of $y(n)$	0.5	1.25
Wrapped-around half of $y(n)$	0.5	0
Periodic convolution $y_p(n)$	1	1.25

2.13 PERIODIC EXTENSION OF NON-PERIODIC SIGNALS

If we add an absolutely summable signal (or an energy signal) $x(n)$ and its infinitely many replicas shifted by multiples of N, we obtain a periodic signal with period N, which is called the periodic extension of $x(n)$.

$$x_p(n) = \sum_{k=-\infty}^{\infty} x(n + kN)$$

For finite-length sequences, an equivalent way of finding one period of the periodic extension is to wraparound N-sample sections of $x(n)$ and add them all up. If $x(n)$ is shorter than N, we obtain one period of its periodic extension simply by padding $x(n)$ with enough zeros to increase its length to N.

EXAMPLE 2.41 Find the periodic extension of
(a) $x(n) = \{2, 0, 3, 0, 4, 7, 6, 5\}$ with period $N = 3$
(b) $x(n) = a^n u(n)$ with period N

Solution:

(a) Given $x(n) = \{2, 0, 3, 0, 4, 7, 6, 5\}$, its periodic extension with period $N = 3$ is found by wrapping-around blocks of 3 samples and finding the sum to give:

$$\{2, 0, 3, 0, 4, 7, 6, 5\} \rightarrow \text{Wrap-around} \rightarrow \begin{Bmatrix} 2 & 0 & 3 \\ 0 & 4 & 7 \\ 6 & 5 & \end{Bmatrix} \rightarrow \text{sum} \rightarrow \{8, 9, 10\}$$

In other words, if we add $x(n)$ to its shifted versions $x(n + kN)$, where $N = 3$ and $k = 0, \pm 1, \pm 2, \pm 3, \ldots$, we get a periodic signal whose first period is $\{8, 9, 10\}$.

(b) Given $x(n) = a^n u(n)$, its periodic extension with period N is found by

$$x_{pe}(n) = \sum_{k=-\infty}^{\infty} x(n + kN) = \sum_{k=0}^{\infty} a^{n+kN} = a^n \sum_{k=0}^{\infty} (a^N)^k$$

$$= \frac{a^n}{1 - a^N}, \quad 0 \le n \le N - 1$$

2.14 SYSTEM RESPONSE TO PERIODIC INPUTS

The response of a discrete-time system to a periodic input with period N is also periodic with the same period N. To find the response due to periodic input, find the linear convolution of $x(n)$ and $h(n)$ and ignore the first period.

The other methods for finding the response of a discrete-time system to periodic inputs rely on the concepts of periodic extension and wrap-around. One approach is to find the output for one period of the input (using regular convolution) and find one period of the periodic output by superposition (using periodic extension). Another approach is to first find one period of the periodic extension of the impulse response, then find its regular convolution with one period of the input, and finally, wrap-around the regular convolution to generate one period of the periodic output.

EXAMPLE 2.42 Find the response of the system

$$x(n) = \{2, 3, -4, 2, 3, -4, 2, 3, -4, \ldots\} \quad \text{and} \quad h(n) = \{1, 2\}$$

Solution: Given $x(n) = \{2, 3, -4, 2, 3, -4, 2, 3, -4, \ldots\}$ and $h(n) = \{1, 2\}$. Here the period $N = 3$. So $y(n)$ is also periodic with period N. The convolution $y(n) = x(n) * h(n)$, using the sum-by-column method is:

Index n	0	1	2	3	4	5	6	7	8	9	10
$x(n)$	2	3	–4	2	3	–4	2	3	–4	2	…
$h(n)$	1	2									
	2	3	–4	2	3	–4	2	3	–4	2	…
		4	6	–8	4	6	–8	4	6	–8	…
$y(n)$	2	7	2	–6	7	2	–6	7	2	–6	…

The convolution $y(n)$ is periodic with period $N = 3$, except for start up effects (which last for one period). One period of the convolution is $y(n) = \{–6, 7, 2\}$.

EXAMPLE 2.43 Find the response of the system

$$x(n) = \{1, 3, -2, 1, 3, -2, 1, 3, -2, 1 ...\} \quad \text{and} \quad h(n) = \{1, 2, 1, 1\}$$

Solution: Given $x(n) = \{1, 3, -2, 1, 3, -2, 1, 3, -2, 1 ...\}$ and $h(n) = \{1, 2, 1, 1\}$. Here the period $N = 3$. So $y(n)$ is also periodic with period $N = 3$.

The convolution $y(n) = x(n) * h(n)$, using the sum-by-column method is:

Index n	0	1	2	3	4	5	6	7	8	9	10
$x(n)$	1	3	–2	1	3	–2	1	3	–2	1	…
$h(n)$	1	2	1	1							
	1	3	–2	1	3	–2	1	3	–2	1	…
		2	6	–4	2	6	–4	2	6	–4	…
			1	3	–2	1	3	–2	1	3	
				1	3	–2	1	3	–2	1	
$y(n)$	1	5	5	1	6	3	1	6	3	1	…

The convolution $y(n)$ is periodic with period $N = 3$, except for start up effects (which last for one period). One period of the convolution is $y(n) = \{1, 6, 3\}$.

EXAMPLE 2.44 Find the response of the system with $h(n) = \{1, 2\}$ and $N = 3$, for a periodic input $x(n) = \{2, 3, -4, 2, 3, -4, 2, 3, -4, 2, ...\}$.

Solution: $x(n) = \{2, 3, –4\}$ describes one period of an input periodic signal with period $N = 3$. $h(n) = \{1, 2\}$. So $y(n)$ is also periodic with period $N = 3$.

Method 1

To find $y(n)$ for one period, we find the regular convolution $y_1(n)$ of $h(n)$ and one period of $x(n)$ to give

$$y_1(n) = h(n) * x(n) = \{1, 2\} * \{2, 3, -4\} = \{2, 7, 2, -8\}$$

We then wrap-around $y_1(n)$ past three samples to obtain one period of $y(n)$ as:

$$\{2, 7, 2-8\} \rightarrow \text{wrap-around} \rightarrow \left\{\begin{matrix} 2 & 7 & 2 \\ -8 & & \end{matrix}\right\} \rightarrow \text{sum} \rightarrow \{-6, 7, 2\}$$

This is identical to the result obtained in the earlier example.

Method 2

To find $y(n)$ for one period $N = 3$, first find $h_p(n)$, the periodic extension of $h(n)$ for $N = 3$ and then find regular convolution $y(n)$ of $h_p(n)$ and one period of $x(n)$, and finally wrap-around the regular convolution to generate one period of the periodic output $h_p(n)$.

Given $h(n) = \{1, 2\}$ and $N = 3$

$$\therefore \qquad h_p(n) = \{1, 2, 0\}$$

$$y(n) = x_p(n) * h_p(n) = \{2, 3, -4\} * \{1, 2, 0\} = \{2, 7, 2, -8, 0\}$$

$$y_p(n) \text{ is } \{2, 7, 2, -8, 0\} \rightarrow \text{wrap-around} \rightarrow \left\{\begin{matrix} 2 & 7 & 2 \\ -8 & 0 & \end{matrix}\right\} \rightarrow \text{sum} \rightarrow \{-6, 7, 2\}$$

This is same as the result obtained earlier.

EXAMPLE 2.45 Find the response of the system for a periodic input

$$x(n) = \{1, 3, -2, 1, 3, -2, 1, 3, -2, ...\} \text{ with } h(n) = \{2, 1, 2, 1, 1\} \text{ and } N = 3$$

Solution: $x(n) = \{1, 3, -2\}$ describes one period of the input periodic signal with period $N = 3$ and $h(n) = \{2, 1, 2, 1, 1\}$. So $y(n)$ is also periodic with period $N = 3$.

Method 1

To find the response $y(n)$ for one period $N = 3$, first find the output for one period of input by regular convolution of $x(n)$ and $h(n)$, then find one period of the periodic output by superposition (using periodic extension) as shown below.

Regular convolution $y_1(n) = \{1, 3, -2\} * \{2, 1, 2, 1, 1\} = \{2, 7, 1, 5, 0, 1, -2\}$

Wrapping-around $y_1(n)$ past three samples to obtain one period of $y(n)$, we have

$$y(n) = \{2, 7, 1, 5, 0, 1, -2\} \rightarrow \text{wrap-around} \rightarrow \left\{\begin{matrix} 2 & 7 & 1 \\ 5 & 0 & 1 \\ -2 & & \end{matrix}\right\} \rightarrow \text{sum} \rightarrow \{5, 7, 2\}$$

$$\therefore \qquad y_p(n) = \{5, 7, 2\}$$

Method 2

To find $y(n)$ for one period $N = 3$, first find $h_p(n)$, the periodic extension of $h(n)$ for $N = 3$ and then find regular convolution $y(n)$ of $h_p(n)$ and one period of $x(n)$, and finally wrap-around the regular convolution to generate one period of the periodic output $y_p(n)$.

Given $h(n) = \{2, 1, 2, 1, 1\}$ and $N = 3$

$$\therefore \qquad h_p(n) = \begin{Bmatrix} 2 & 1 & 2 \\ 1 & 1 & \end{Bmatrix} = \{3, 2, 2\}$$

$$y(n) = \{h_p(n)\} * \{x(n)\} = \{3, 2, 2\} * \{1, 3, -2\} = \{3, 11, 2, 2, -4\}$$

$$\therefore \quad y_p(n) = \{3, 11, 2, 2, -4\} \rightarrow \text{wrap-around} \rightarrow \begin{Bmatrix} 3 & 11 & 2 \\ 2 & -4 & \end{Bmatrix} \rightarrow \text{sum} \rightarrow \{5, 7, 2\}$$

$$\therefore \quad y_p(n) = \{5, 7, 2\}$$

2.15 DISCRETE CORRELATION

Till now we have discussed about the convolution of two signals. It is used to find the output sequence $y(n)$ if the input sequence $x(n)$ and the impulse response $h(n)$ are known. Here we discuss about correlation which is a mathematical operation similar to convolution. Correlation is used to compare two signals. It is a measure of similarity between signals and is found using a process similar to convolution. It occupies a significant role in signal processing. It is used in radar and sonar systems to find the location of a target by comparing the transmitted and reflected signals. Other applications of correlation are in image processing, control engineering etc. The correlation is of two types: (i) Cross correlation (ii) Auto-correlation.

2.15.1 Cross Correlation

The cross correlation between a pair of sequences $x(n)$ and $y(n)$ is given by

$$R_{xy}(n) = \sum_{k=-\infty}^{\infty} x(k)\, y(k-n)$$

$$= \sum_{k=-\infty}^{\infty} x(k+n)\, y(k) \qquad n = 0, \pm 1, \pm 2, \ldots$$

The index n is the shift (lag) parameter. The order of subscripts xy indicates that $x(n)$ is the reference sequence that remains fixed, i.e. unshifted in time, whereas the sequence $y(n)$ is shifted n units in time w.r.t. $x(n)$.

If we wish to fix the sequence $y(n)$ and to shift the sequence $x(n)$, then the correlation can be written as:

$$R_{yx}(n) = \sum_{k=-\infty}^{\infty} y(k)\, x(k-n)$$

$$= \sum_{k=-\infty}^{\infty} y(k+n)\, x(k) \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$R_{xy}(n) \neq R_{yx}(n)$$

If the time shift $n = 0$, then we get

$$R_{xy}(0) = R_{yx}(0) = \sum_{k=-\infty}^{\infty} x(k)\,y(k)$$

Comparing the above equations for $R_{xy}(n)$ and $R_{yx}(n)$, we observe that

$$R_{xy}(n) = R_{yx}(-n)$$

where $R_{yx}(-n)$ is the folded version of $R_{yx}(n)$ about $n = 0$.

The expression for $R_{xy}(n)$ can be written as:

$$R_{xy}(n) = \sum_{k=-\infty}^{\infty} x(k)y[-(n-k)]$$

$$= x(n) * y(-n)$$

Observing the above equation for $R_{xy}(n)$, we can conclude that the correlation of two sequences is essentially the convolution of two sequences in which one of the sequence has been reversed, i.e., correlation is the convolution of one signal with a flipped version of the other. Therefore, the same algorithm (procedure) can be used to compute the convolution and correlation of two sequences.

2.15.2 Autocorrelation

The autocorrelation of a sequence is correlation of a sequence with itself. It gives a measure of similarity between a sequence and its shifted version. The autocorrelation of a sequence $x(n)$ is defined as:

$$R_{xx}(n) = \sum_{k=-\infty}^{\infty} x(k)\,x(k-n)$$

or equivalently

$$R_{xx}(n) = \sum_{k=-\infty}^{\infty} x(k+n)\,x(k)$$

If the time shift $n = 0$, then we have

$$R_{xx}(0) = \sum_{k=-\infty}^{\infty} x^2(k)$$

$R_{xx}(n)$ is given by $\quad R_{xx}(n) = x(k) * x(-k)$

The autocorrelation has a maximum value at $n = 0$ and satisfies the inequality

$$|R_{xx}(n)| \leq R_{xx}(0) = E_x$$

The autocorrelation is an even symmetric function with

$$R_{xx}(n) = R_{xx}(-n)$$

Correlation is an effective method of detecting signals buried in noise. Noise is essentially uncorrelated with the signal. This means that if we correlate a noisy signal with itself, the correlation will be due only to the signal (if present) and will exhibit a sharp peak at $n = 0$.

The normalized autocorrelation sequence is

$$\rho_{xx}(n) = \frac{R_{xx}(n)}{R_{xx}(0)}$$

Similarly, the normalized cross correlation sequence is:

$$\rho_{xy}(n) = \frac{R_{xy}(n)}{\sqrt{R_{xx}(0)R_{yy}(0)}}$$

$\rho_{xy}(n)$ is also known as the cross correlation coefficient. Its value always lies between –1 and +1. A value 0 for cross correlation means no correlation.

2.15.3 Computation of Correlation

The correlation of two sequences $x(n)$ and $y(n)$ can be obtained by using the procedures for computing the convolution.

We have $$R_{xy}(n) = x(n) * y(-n)$$

So to get the cross correlation $R_{xy}(n)$, first fold the sequence $y(n)$ to obtain $y(-n)$. Then the convolution of $x(n)$ and $y(-n)$ gives the value $R_{xy}(n)$.

Also we have $$R_{xx}(n) = x(n) * x(-n)$$

So to get the autocorrelation $R_{xx}(n)$, first fold the sequence $x(n)$ to obtain $x(-n)$. Then the convolution of $x(n)$ and $x(-n)$ gives the value $R_{xx}(n)$.

EXAMPLE 2.46 Find the cross correlation of two finite length sequences:

$$x(n) = \{2, 3, 1, 4\} \quad \text{and} \quad y(n) = \{1, 3, 2, 1\}$$

Solution: Given $x(n) = \{2, 3, 1, 4\}$ and $y(n) = \{1, 3, 2, 1\}$

$$x(n) = \{2, 3, 1, 4\},\ y(n) = \{1, 3, 2, 1\},\ y(-n) = \{1, 2, 3, 1\}$$

$$R_{xy}(n) = x(n) * y(-n)$$

The cross correlation is computed as given below.

		$y(-n)$			
		1	2	3	1
	2	2	4	6	2
	3	3	6	9	3
$x(n)$	1	1	2	3	1
	4	4	8	12	4

$$R_{xy}(n) = \{2, 3 + 4, 1 + 6 + 6, 4 + 2 + 9 + 2, 8 + 3 + 3, 12 + 1, 4\} = \{2, 7, 13, 17, 14, 13, 4\}$$

EXAMPLE 2.47 Find the autocorrelation of the finite length sequence $x(n) = \{2, 3, 1, 4\}$.

Solution: Given $x(n) = \{2, 3, 1, 4\}$

$$x(n) = \{2, 3, 1, 4\},\ x(-n) = \{4, 1, 3, 2\}$$

$$R_{xx}(n) = x(n) * x(-n)$$

The autocorrelation is computed as given below:

		$x(-n)$			
		4	1	3	2
	2	8	2	6	4
	3	12	3	9	6
$x(n)$	1	4	1	3	2
	4	16	4	12	8

$$R_{xx}(n) = \{8, 12 + 2, 4 + 3 + 6, 16 + 1 + 9 + 4, 4 + 3 + 6, 12 + 2, 8\}$$
$$= \{8, 14, 13, 30, 13, 14, 8\}$$

2.15.4 Correlation of Power and Periodic Signals

Let $x(n)$ and $y(n)$ are power signals. Their cross correlation sequence is defined as:

$$R_{xy}(n) = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{k=-N}^{N} x(k)\, y(k-n)$$

and the autocorrelation of a power signal $x(n)$ is defined as:

$$R_{xy}(n) = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{k=-N}^{N} x(k)\, x(k-n)$$

If $x(n)$ and $y(n)$ are periodic sequences, each with period N, then

$$R_{xy}(n) = \frac{1}{N}\sum_{k=0}^{N-1} x(k)\, y(k-n)$$

and

$$R_{xx}(n) = \frac{1}{N}\sum_{k=0}^{N-1} x(k)\, x(k-n)$$

EXAMPLE 2.48 Find the cross correlation of the sequences

$$x(n) = \left\{\begin{matrix}3, & 5, & 1, & 2\\ & \uparrow & & \end{matrix}\right\} \quad\text{and}\quad h(n) = \left\{\begin{matrix}1, & 4, & 3\\ \uparrow & & \end{matrix}\right\}$$

and show that $R_{xh}(n) \neq R_{hx}(n)$ and $R_{xh}(n) = R_{hx}(-n)$.

Solution: *Computation of* $R_{xh}(n)$

We know that $R_{xh}(n) = x(n) * h(-n)$. So we compute the convolution of $x(n)$ and $h(-n)$.

Here $x(n) = \left\{\begin{matrix}3, & 5, & 1, & 2\\ & \uparrow & & \end{matrix}\right\}$ and $h(-n) = \left\{\begin{matrix}3, & 4, & 1\\ & & \uparrow\end{matrix}\right\}$. So the starting index of $R_{xh}(n)$ is $n = -1 - 2 = -3$. $R_{xh}(n)$ is computed using the sum-by-column method for convolution as follows:

n	–3	–2	–1	0	1	2
$x(n)$	3	5	1	2		
$h(-n)$	3	4	1			
	9	15	3	6		
		12	20	4	8	
			3	5	1	2
$R_{xh}(n)$	9	27	26	15	9	2

So $R_{xh}(n) = \{9, 27, 26, 15, 9, 2\}$

Computation of $R_{hx}(n)$

We know that $R_{hx}(n) = h(n) * x(-n)$. So we compute the convolution of $h(n)$ and $x(-n)$.

Here $h(n) = \left\{\begin{matrix}1, & 4, & 3\\ \uparrow & & \end{matrix}\right\}$ and $x(-n) = \left\{\begin{matrix}2, & 1, & 5, & 3\\ & & \uparrow & \end{matrix}\right\}$. So the starting index of $R_{hx}(n)$ is $n = 0 - 2 = -2$. $R_{hx}(n)$ is computed using the sum-by-column method for convolution as follows:

n	–2	–1	0	1	2	3
$x(-n)$	2	1	5	3		
$h(n)$	1	4	3			
	2	1	5	3		
		8	4	20	12	
			6	3	15	9
$R_{hx}(n)$	2	9	15	26	27	9

So $$R_{hx}(n) = \{2, 9, 15, 26, 27, 9\}$$

Thus, we can observe that $R_{xh}(n) \neq R_{hx}(n)$ and also $R_{xh}(n) = R_{hx}(-n)$.

EXAMPLE 2.49 Find the autocorrelation of $x(n) = \left\{\begin{matrix}2, 5, -4\\ \uparrow \qquad\quad\end{matrix}\right\}$.

Solution: To find $R_{xx}(n)$ we compute the convolution of $x(n) = \left\{\begin{matrix}2, 5, -4\\ \uparrow \qquad\quad\end{matrix}\right\}$ and $x(-n) = \left\{\begin{matrix}-4, 5, 2\\ \qquad\quad \uparrow\end{matrix}\right\}$ as shown below.

The starting index of $R_{xx}(n)$ is $n = -2$. $R_{xx}(n)$ is computed using the sum-by-column method for convolution as follows.

n	–2	–1	0	1	2
$x(n)$	2	5	–4		
$x(-n)$	–4	5	2		
	–8	–20	16		
		10	25	–20	
			4	10	–8
$R_{xx}(n)$	–8	–10	45	–10	–8

So $R_{xx}(n) = \left\{\begin{matrix}-8, -10, 45, -10, -8\\ \uparrow\end{matrix}\right\}$. Note that $R_{xx}(n)$ is even symmetric about the origin $n = 0$ and also that $R_{xx}(0) \geq R_{xx}(n)$ for $n \neq 0$.

EXAMPLE 2.50 Find the cross correlation of $x(n) = (0.6)^n u(n)$ and $h(n) = (0.3)^n u(n)$.

Solution: The cross correlation $R_{xh}(n)$ is computed as follows:

$$R_{xh}(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(k-n) = \sum_{k=-\infty}^{\infty} (0.6)^k u(k)\,(0.3)^{n-k} u(k-n)$$

This summation requires evaluation over two ranges of n. If $n < 0$, the shifted step $u(k - n)$ is non-zero for some $k < 0$. But since $u(k) = 0$, for $k < 0$, the lower limit on the summation reduces to $k = 0$, and we get

For $n < 0$ $$R_{xh}(n) = \sum_{k=0}^{\infty} (0.6)^k (0.3)^{k-n} = (0.3)^{-n} \sum_{k=0}^{\infty} (0.18)^k$$

$$= \frac{(0.3)^{-n}}{1 - 0.18} = 1.22(0.3)^{-n} u(-n-1)$$

If $n \ge 0$, the shifted step $u(k - n)$ is zero for $k < n$, the lower limit on the summation reduces to $k = n$, and we obtain

$$(n \ge 0) \qquad R_{xh}(n) = \sum_{k=n}^{\infty} (0.6)^k (0.3)^{k-n}$$

With the change of variable $m = k - n$, we get

$$(n \ge 0) \qquad R_{xh}(n) = \sum_{m=0}^{\infty} (0.6)^{m+n} (0.3)^m = (0.6)^n \sum_{m=0}^{\infty} (0.18)^m = \frac{(0.6)^n}{1 - 0.18} = 1.22(0.6)^n u(n)$$

So $\qquad R_{xh}(n) = 1.22(0.3)^{-n} u(-n - 1) + 1.22(0.6)^n u(n)$

EXAMPLE 2.51 Find the autocorrelation of $x(n) = a^n u(n)$.

Solution: To compute $R_{xx}(n)$ which is even symmetric, we need to compute the result only for $n \ge 0$ and create its even extension.

$$\text{For } n \ge 0 \qquad R_{xx}(n) = \sum_{k=-\infty}^{\infty} x(k)\, x(k - n) = \sum_{k=n}^{\infty} a^k a^{k-n} = \sum_{m=0}^{\infty} a^{m+n} a^m$$

$$= a^n \sum_{k=-\infty}^{\infty} a^{2m} = \frac{a^n}{1 - a^2}\, u(n)$$

The even extension of this result gives

$$R_{xx}(n) = \frac{a^{|n|}}{1 - a^2}$$

2.16 PERIODIC DISCRETE CORRELATION

For periodic sequences with identical period N, the periodic discrete correlation is defined as:

$$R_{xyp}(n) = x(n) \otimes y(n) = \sum_{k=0}^{N-1} x(k)\, y(k - n)$$

$$R_{yxp}(n) = y(n) \otimes x(n) = \sum_{k=0}^{N-1} y(k)\, x(k - n)$$

As with discrete periodic convolution, an averaging factor of $1/N$ is sometimes included in the summation. We can find one period of the periodic correlation $R_{xyp}(n)$ by first computing the linear correlation of one period segments and then wrapping-around the result. We find that $R_{yxp}(n)$ is a circularly flipped (time reversed) version of $R_{xyp}(n)$ with $R_{yxp}(n) = R_{xyp}(-n)$. We also find that the periodic autocorrelation $R_{xxp}(n)$ or $R_{yyp}(n)$ always display circular even

symmetry. This means that the periodic extension of $R_{xxp}(n)$ or $R_{yyp}(n)$ is even symmetric about the origin $n = 0$. The periodic autocorrelation function also attains a maximum at $n = 0$.

EXAMPLE 2.52 Find the discrete periodic cross correlation $R_{xyp}(n)$ and $R_{yxp}(n)$ of the periodic sequences whose first period is given by

$$x(n) = \left\{\begin{matrix} 1, 3, 0, 4 \\ \uparrow \end{matrix}\right\} \quad \text{and} \quad y(n) = \left\{\begin{matrix} 2, 1, -2, 1 \\ \uparrow \end{matrix}\right\}$$

Solution: To find the periodic cross correlation $R_{xyp}(n)$, we first evaluate the linear cross correlation

$$R_{xy}(n) = x(n) * y(-n) = \left\{\begin{matrix} 1, 3, 0, 4 \\ \uparrow \end{matrix}\right\} * \left\{\begin{matrix} 1, -2, 1, 2 \\ \uparrow \end{matrix}\right\} = \left\{\begin{matrix} 1, 1, -5, 9, -2, 4, 8 \\ \uparrow \end{matrix}\right\}$$

$$= \{1, 1, -5, 9, -2, 4, 8\}$$

Wrapping-around, we get the periodic cross correlation as:

$$R_{xyp}(n) = \left\{\begin{matrix} 1 & 1 & -5 & 9 \\ -2 & 4 & 8 & \end{matrix}\right\} = \left\{\begin{matrix} -1, 5, 3, 9 \\ \uparrow \end{matrix}\right\}$$

Invoking periodicity and describing the result in terms of its first period, we have

$$R_{xyp}(n) = \left\{\begin{matrix} 9, -1, 5, 3 \\ \uparrow \end{matrix}\right\}$$

To find the periodic cross correlation $R_{yxp}(n)$, we first evaluate the linear cross correlation

$$R_{yx}(n) = y(n) * x(-n) = \left\{\begin{matrix} 2, 1, -2, 1 \\ \uparrow \end{matrix}\right\} * \left\{\begin{matrix} 4, 0, 3, 1 \\ \uparrow \end{matrix}\right\}$$

$$= \left\{\begin{matrix} 8, 4, -2, 9, -5, 1, 1 \\ \uparrow \end{matrix}\right\}$$

Wrapping-around, we get the periodic cross correlation as:

$$R_{yxp}(n) = \left\{\begin{matrix} 8 & 4 & -2 & 9 \\ -5 & 1 & 1 & \end{matrix}\right\} = \left\{\begin{matrix} 3, 5, -1, 9 \\ \uparrow \end{matrix}\right\}$$

Invoking periodicity and describing the result in terms of its first period, we have

$$R_{yxp}(n) = \left\{\begin{matrix} 9, 3, 5, -1 \\ \uparrow \end{matrix}\right\}$$

Note the $R_{yxp}(n)$ is a circularly flipped version of $R_{xyp}(n)$ with $R_{yxp}(n) = R_{xyp}(-n)$.

EXAMPLE 2.53 Find the periodic autocorrelation of the sequences

(a) $x(n) = \{1, 3, 0, 4\}$ (b) $y(n) = \{2, 1, -2, 1\}$

Solution:

(a) To find the periodic autocorrelation $R_{xxp}(n)$, we first evaluate the linear autocorrelation

$$R_{xx}(n) = x(n) * x(-n) = \left\{\begin{matrix}1, 3, 0, 4\\ \uparrow\end{matrix}\right\} * \left\{\begin{matrix}4, 0, 3, 1\\ \uparrow\end{matrix}\right\} = \left\{\begin{matrix}4, 12, 3, 26, 3, 12, 4\\ \uparrow\end{matrix}\right\}$$

Wrap-around gives the periodic autocorrelation as:

$$R_{xxp}(n) = \left\{\begin{matrix}4 & 12 & 3 & 26\\ 3 & 12 & 4 & \end{matrix}\right\} = \left\{\begin{matrix}7, 24, 7, 26\\ \uparrow\end{matrix}\right\}$$

Rewriting the result in terms of its first period, we have

$$R_{xxp}(n) = \left\{\begin{matrix}26, 7, 24, 7\\ \uparrow\end{matrix}\right\}$$

This displays circular even symmetry (its periodic extension is even symmetric about the origin $n = 0$).

(b) To find the periodic autocorrelation $R_{yyp}(n)$, we first evaluate the linear autocorrelation

$$R_{yy}(n) = y(n) * y(-n) = \left\{\begin{matrix}2, 1, -2, 1\\ \uparrow\end{matrix}\right\} * \left\{\begin{matrix}1, -2, 1, 2\\ \uparrow\end{matrix}\right\} = \left\{\begin{matrix}2, -3, -2, 10, -2, -3, 2\\ \uparrow\end{matrix}\right\}$$

Wrap-around gives the periodic autocorrelation as:

$$R_{yyp}(n) = \left\{\begin{matrix}2 & -3 & -2 & 10\\ -2 & -3 & 2 & \end{matrix}\right\} = \left\{\begin{matrix}0, -6, 0, 10\\ \uparrow\end{matrix}\right\}$$

Rewriting the result in terms of its first period, we have

$$R_{yyp}(n) = \left\{\begin{matrix}10, 0, -6, 0\\ \uparrow\end{matrix}\right\}$$

This displays the circular even symmetry (its periodic extension is even symmetric about $n = 0$).

SHORT QUESTIONS WITH ANSWERS

1. What is discrete convolution?

Ans. The convolution of discrete-time signals is called discrete convolution. Discrete-time convolution is a method of finding the zero-state response of relaxed linear time invariant systems.

2. Write the expression for discrete convolution.

Ans. The expression for discrete convolution $y(n)$ of two discrete-time signals $x(n)$ and $h(n)$ is:

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \text{ or}$$

$$y(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k)$$

3. If $y(n) = x(n) * h(n)$, how are the parameters of $y(n)$ related to the parameters of $x(n)$ and $h(n)$.

Ans. If $y(n) = x(n) * h(n)$, then

(i) The starting index of $y(n)$ equals the sum of the starting indices of $x(n)$ and $h(n)$.

(ii) The ending index of $y(n)$ equals the sum of the ending indices of $x(n)$ and $h(n)$.

(iii) The length L_y of $y(n)$ is related to the lengths L_x and L_h of $x(n)$ and $h(n)$ by $L_y = L_x + L_h - 1$.

4. What are the methods to compute the convolution sum of two sequences?

Ans. The methods to compute the convolution sum of two sequences are:
(i) Graphical method (ii) Tabular array method (iii) Tabular method (iv) Matrices method (v) Sum-by-column method (vi) The flip, shift, multiply and sum method.

5. Write the expressions for convolution sum of causal and non-causal systems excited by causal and non-causal inputs.

Ans. If $x(n)$ is the input and $h(n)$ is the impulse response of the system, the expression for convolution $y(n)$ is obtained as follows:

For a non-causal system excited by a non-causal input,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k)$$

For a non-causal system excited by a causal input,

$$y(n) = \sum_{k=0}^{\infty} x(k)\,h(n-k) = \sum_{k=-\infty}^{n} h(k)\,x(n-k)$$

For a causal system excited by a non-causal input,

$$y(n) = \sum_{k=-\infty}^{n} x(k)\,h(n-k) = \sum_{k=0}^{\infty} h(k)\,x(n-k)$$

For a causal system excited by a causal input,

$$y(n) = \sum_{k=0}^{n} x(k)\,h(n-k) = \sum_{k=0}^{n} h(k)\,x(n-k)$$

6. Write the properties of discrete convolution.
Ans. The properties of discrete convolution are:

(i) Commutative property: $x(n) * h(n) = h(n) * x(n)$

(ii) Associative property: $[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$

(iii) Distributive property: $x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$

(iv) Shifting property: $x(n-k) * h(n-m) = y(n-k-m)$

(v) Convolution with an impulse $x(n) * \delta(n) = x(n)$

7. What do you mean by flipping a sequence?
Ans. Flipping a sequence means time reversing the sequence.

8. How do you find regular convolution using circular convolution?
Ans. The linear (or regular) convolution of two sequences $x(n)$ (with length N_1) and $h(n)$ (with length N_2) can be found using the periodic convolution by zero padding the sequences $x(n)$ and $h(n)$ to a length $N_1 + N_2 - 1$ and then finding periodic convolution. The regular convolution of original unpadded sequences equals the periodic convolution of the zero padded sequences.

9. What do you mean by deconvolution?
Ans. If the convolution of $x(n)$ and $h(n)$ is $y(n)$, i.e. $y(n) = x(n) * h(n)$, then deconvolution is a process of finding $h(n)$ (or $x(n)$) from $y(n)$ for a given $x(n)$ (or $h(n)$).

10. What are the methods of finding deconvolution?
Ans. Deconvolution may be found using the following methods: (i) Polynomial division method (ii) Recursion method (iii) Tabular method.

11. What is the basic difference between linear and circular convolution?
Ans. The linear convolution and circular convolution basically involve the same four steps, namely folding one sequence, shifting the folded sequence, multiplying the two sequences and finally summing the value of the product sequences. The difference between the two is that in circular convolution, the folding and shifting (rotating) operations are performed in a circular fashion by computing the index of one of the sequences by modulo-N operation. In linear convolution, there is no modulo-N operation.

12. What are the methods of finding circular convolution?
Ans. The methods of finding circular convolution are:
(i) Concentric circle method (Graphical method)
(ii) Tabular array method
(iii) Matrices method
(iv) DFT method

13. How do you obtain periodic extension of a non-periodic signal?
Ans. For finite length sequences, one way of finding one period of the periodic extension is to wrap-around N-sample sections of $x(n)$ and add them all up. If $x(n)$ is shorter than N, one period of its periodic extension is obtained simply by padding $x(n)$ with enough zeros to increase its length to N.

14. How do you obtain periodic convolution from linear convolution?
Ans. To find periodic convolution from linear convolution, first make the sequences to be of equal length N, find the linear convolution of one period of the sequences which will have $(2N - 1)$ samples, then extend its length to $2N$ (by appending a zero), slice it into two halves (of length N each), line up the second half with the first, and add the two halves to get the periodic convolution.

15. How do you find the system response to periodic inputs?
Ans. The system response to periodic inputs with period N is obtained by finding the linear convolution of input $x(n)$ and impulse response $h(n)$ and ignoring the first period.

Another approach is to find the output for one period of the input (using regular convolution) and find one period of the periodic output by superposition (using periodic extension).

One more approach is to first find one period of the periodic extension of the impulse response, then find its regular convolution with one period of the input, and finally, wrap-around the regular convolution to generate one period of the periodic output.

16. What is correlation?
Ans. Correlation is a measure of similarity between two signals and is found using a process similar to convolution. The correlation of two signals is equal to the convolution of one signal with the flipped version of second signal.

17. What is cross correlation?
Ans. The correlation of two different signals is called cross correlation.

18. What is autocorrelation?
Ans. The correlation of a signal with itself is called autocorrelation. It gives a measure of similarity between a sequence and its shifted version.

19. What is the use of correlation?
Ans. Correlation is an effective method of detecting signals buried in noise.

20. Where does the autocorrelation function attain its maximum value?
Ans. The autocorrelation function attains its maximum value at the origin, i.e. at $n = 0$.

REVIEW QUESTIONS

1. Write the properties of discrete convolution.
2. Discuss the methods of computing the linear convolution.
3. What is deconvolution? Discuss the methods of computing deconvolution.
4. Discuss the methods of performing circular convolution.
5. Explain how linear convolution can be obtained from circular convolution, and how circular convolution can be obtained from linear convolution.

FILL IN THE BLANKS

1. Convolution is a mathematical operation equivalent to _______ filtering.
2. Convolution is a method of finding the _______ response of relaxed LTI systems.
3. Convolving two sequences in time domain is equivalent to _______ the sequences in frequency domain.
4. _______ is a measure of similarity between two signals.
5. If the input to the system is a unit impulse then the output of the system is known as _______.
6. The expression for the convolution $y(n)$ of two sequences $x(n)$ and $h(n)$ is _______.
7. For a non-causal system $h(n)$ excited by a non-causal input $x(n)$, the output $y(n)$ is given by _______.
8. For a non-causal system $h(n)$ excited by a causal input $x(n)$, the output $y(n)$ is given by _______.
9. For a causal system $h(n)$ excited by a non-causal input $x(n)$, the output $y(n)$ is given by _______.
10. For a causal system $h(n)$ excited by a causal input $x(n)$, the output $y(n)$ is given by _______.
11. The commutative property of convolution says that _______.
12. The associate property of convolution says that _______.
13. The distributive property of convolution says that _______.
14. The shifting property of convolution says that _______.
15. The convolution with an impulse property of convolution says that _______.
16. The step response is the _______ of the impulse response.
17. If N_1 is the length of $x(n)$ and N_2 is the length of $h(n)$, the length of $y(n)$, the convolution of $x(n)$ and $h(n)$ is _______.
18. If n_1 is the starting index of $x(n)$ and n_2 is the starting index of $h(n)$, the starting index of $y(n)$ is _______.
19. If n_3 is the ending index of $x(n)$ and n_4 is the ending index of $h(n)$, the ending index of $y(n)$ is _______.
20. Leading zeros appended to a sequence will appear as _______ zeros in the convolution result.
21. Trailing zeros appended to a sequence will show up as _______ zeros in the convolution.
22. _______ is the process of finding the impulse response $h(n)$ from $y(n)$.
23. The regular convolution of two signals, both of which are _______ does not exist.

24. The circular convolution of two sequences, each of length N yields a sequence of length ________.

25. ________ is a measure of similarity between two signals.

26. ________ is used to compare two signals.

27. The correlation between $x(n)$ and $y(n)$ is given by ________.

28. The periodic autocorrelation attains a maximum at ________.

OBJECTIVE TYPE QUESTIONS

1. The commutative property of convolution states that
(a) $x(n) * h(n) = h(n) * x(n)$
(b) $[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$
(c) $x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$
(d) none of these

2. The associate property of convolution states that
(a) $x(n) * h(n) = h(n) * x(n)$
(b) $[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$
(c) $x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$
(d) none of these

3. The distributive property of convolution states that
(a) $x(n) * h(n) = h(n) * x(n)$
(b) $[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$
(c) $x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$
(d) none of these

4. For a non-causal system $h(n)$ excited by a non-causal input $x(n)$, the output $y(n)$ is given by

(a) $y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$ (b) $y(n) = \sum_{k=0}^{\infty} x(k)\, h(n-k)$

(c) $y(n) = \sum_{k=-\infty}^{n} x(k)\, h(n-k)$ (d) $y(n) = \sum_{k=0}^{n} x(k)\, h(n-k)$

5. For a non-causal system $h(n)$ excited by a causal input $x(n)$, the output $y(n)$ is given by

(a) $y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$ (b) $y(n) = \sum_{k=0}^{\infty} x(k) h(n-k)$

(c) $y(n) = \sum_{k=-\infty}^{n} x(k) h(n-k)$ (d) $y(n) = \sum_{k=0}^{n} x(k) h(n-k)$

6. For a causal system $h(n)$ excited by a non-causal input $x(n)$, the output $y(n)$ is given by

(a) $y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$ (b) $y(n) = \sum_{k=0}^{\infty} x(k)h(n-k)$

(c) $y(n) = \sum_{k=-\infty}^{n} x(k)h(n-k)$ (d) $y(n) = \sum_{k=0}^{n} x(k)h(n-k)$

7. For a causal system $h(n)$ excited by a causal input $x(n)$, the output $y(n)$ is given by

(a) $y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$ (b) $y(n) = \sum_{k=0}^{\infty} x(k)h(n-k)$

(c) $y(n) = \sum_{k=-\infty}^{n} x(k)h(n-k)$ (d) $y(n) = \sum_{k=0}^{n} x(k)h(n-k)$

8. If $x(n) = \{1, 2, 3, 0\}$ and $h(n) = \{3, 1, 0, 0, 0\}$, the length of $y(n) = x(n) * h(n)$ is

(a) 8 (b) 7

(c) 9 (d) none of these

9. $\{1, 2, 3\} * \{3, 2, 1\} =$

(a) $\{3, 8, 1, 4, 8, 3\}$ (b) $\{3, 8, 8, 3\}$

(c) $\{3, 8, 12, 8, 3\}$ (d) $\{2, 3, 8, 14, 8, 3\}$

10. $\left\{1, \underset{\uparrow}{0}, 2\right\} * \left\{2, \underset{\uparrow}{0}, 1\right\} =$

(a) $\left\{2, \underset{\uparrow}{0}, 5, 0, 2\right\}$ (b) $\left\{2, 0, \underset{\uparrow}{5}, 0, 2\right\}$

(c) $\left\{\underset{\uparrow}{2}, 0, 5, 0, 2\right\}$ (d) $\left\{2, 0, 5, \underset{\uparrow}{0}, 2\right\}$

11. $[x(n) = \{\delta(n+2) - \delta(n) + \delta(n-2)\}] * [h(n) = \{\delta(n) + \delta(n-1)\}]$ is

(a) $\left\{1, 1, -1, \underset{\uparrow}{-1}, 1, 1\right\}$ (b) $\left\{1, 1, \underset{\uparrow}{-1}, 1, -1\right\}$

(c) $\left\{1, 1, -1, \underset{\uparrow}{-2}, 1, 1\right\}$ (d) none of these

12. The convolution of $x(n) = \{1, 2, 0, 0, 0\}$ and $h(n) = \{2, 1, 0\}$ is

(a) $\{2, 5, 2, 0, 0, 0\}$ (b) $\{2, 5, 2, 0, 0, 0, 0\}$

(c) $\{2, 5, 0, 0, 0, 0, 0\}$ (d) $\{2, 5, 1, 0, 0, 0, 0\}$

13. If $x(n) = \{1, 2, 3, 0, 4, 0, 6\}$, then circularly shifted signal $x(n-2)$ =

(a) $\{0, 6, 1, 2, 3, 0, 4\}$ (b) $\{0, 0, 1, 2, 3, 0, 4\}$

(c) $\{0, 0, 1, 2, 3, 0, 4, 0, 6\}$ (d) $\{-1, 0, 1, 0, 2, 0, 4\}$

14. If $x(n) = \{1, 2, 3, 0, 4, 0, 6\}$, then circularly shifted signal $x(n+2)$ =

(a) $\{1, 2, 3, 0, 4, 0, 6, 0, 0\}$ (b) $\{3, 0, 4, 0, 6, 1, 2\}$

(c) $\{3, 4, 5, 0, 6, 0, 8\}$ (d) $\{0, 0, 1, 2, 3, 0, 4, 0, 6\}$

15. If $x(n) = \{1, 2, 3, 0, 4, 0, 6\}$, then circularly flipped signal $x(-n)$ =

(a) $\{1, 6, 0, 4, 0, 3, 2, 1\}$ (b) $\{6, 0, 4, 0, 3, 2, 1\}$

(c) $\{-1, -2, -3, 0, -4, 0, -6\}$ (d) none of these

16. The circular convolution of $x(n) = \{1, 2, 1\}$ and $h(n) = \{2, 1, 2\}$ is

(a) $\{7, 7, 6\}$ (b) $\{6, 7, 6\}$

(c) $\{6, 7, 6, 0\}$ (d) $\{0, 7, 7, 6\}$

17. What is the periodic extension of $x(n) = \{1, 0, 2, 0, 3, 0, 4\}$ for period $N = 3$?

(a) $\{2, 3, 5\}$ (b) $\{5, 3, 2\}$

(c) $\{1, 2, 3\}$ (d) $\{1, 0, 2\}$

18. What is the periodic extension of $x(n) = \{2, 2\}$ for $N = 3$?

(a) $\{2, 2, 0\}$ (b) $\{0, 2, 2\}$

(c) $\{2, 2, 2\}$ (d) $\{0, 2, 2, 0\}$

19. The cross correlation of $x(n) = \{1, 2, 1\}$ and $h(n) = \{1, 2\}$ is

(a) $\{1, 4, 5, 2\}$ (b) $\{2, 5, 4, 1\}$

(c) $\{1, 2, 1, 1, 2\}$ (d) $\{1, 3, 5, 2\}$

20. The autocorrelation of $x(n) = \{2, 1\}$ is

(a) $\{2, 5, 2\}$ (b) $\{4, 4, 1\}$

(c) $\{2, 1\}$ (d) $\{2, 1, 2, 1\}$

PROBLEMS

1. Determine the response of the system characterized by the impulse response

$$h(n) = \left(\frac{1}{2}\right)^n u(n) \text{ to the input signal } x(n) = 2^n u(n)$$

2. Evaluate the step response for the LTI system represented by the following impulse response:

$$h(n) = \left(\frac{1}{5}\right)^n u(n)$$

3. Compute the linear convolution $y(n)$ of the following signals by all the methods.

(a) $x(n) = \{4, \ -2, 1\}; \quad h(n) = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$

(b) $x(n) = \{2, 1, \underset{\uparrow}{0}, 5\}; \quad h(n) = \{2, 0, \underset{\uparrow}{1}, 1\}$

(c) $x(n) = \{1, \underset{\uparrow}{2}, 2, 3\}; \quad h(n) = \{2, \ -1, \underset{\uparrow}{3}\}$

(d) $x(n) = \{1, \underset{\uparrow}{2}, 3\}; \quad h(n) = \{2, \underset{\uparrow}{0}, 1\}$

(e) $x(n) = \{1, 0, 2, 0, 3\}; \quad h(n) = \{2, 0, 0, 0, 1\}$

(f) $x(n) = \{1, 2, 3, 0, 0\}; \quad h(n) = \{2, 0, 1, 0, 0\}$

4. Find the periodic extension of the following for $N = 3$.

(a) $x(n) = \{1, 3, 2, 0, 5, 6, 4, 7\}$

(b) $x(n) = \{1, 0, 0, 5, 6, 4, 1\}$

5. Find the response $y(n)$ of the system for $N = 3$.

(a) Input $x(n) = \{2, 3, 4\}$ with $N = 3$ and impulse response $h(n) = \{2, 2\}$

(b) Input $x(n) = \{1, 0, 3\}$ with $N = 3$ and $h(n) = \{1, 2, 3, 4, 1\}$

6. What is the input signal $x(n)$ that will generate the output sequence $y(n) = \{8, 22, 11, 31, 4, 12\}$ for a system with impulse response $h(n) = \{2, 5, 0, 4\}$?

7. The input $x(n) = \{1, 2\}$ to an LTI system produces an output $y(n) = \{2, 3, 1, 6\}$. Use deconvolution to find the impulse response $h(n)$.

8. Find the circular convolution of the following signals by all the methods:

(a) $x(n) = \{1, 0, 2, 0\}; \quad h(n) = \{2, \ -1, \ -1, 2\}$

(b) $x(n) = \{0, 2, \ -4, 6\}; \quad h(n) = \{2, \ -2, 1, \ -1\}$

(c) $x(n) = \{0, 2, 4, 6, 8\}; \quad h(n) = \{1, 3, 5, 7, 9\}$

9. Find the circular convolution of the following sequences and compare it with linear convolution:

(a) $x(n) = \{1, 0, \ -2, \ -3\}, \quad h(n) = \{2, 0, \ -1, \ -4\}$

(b) $x(n) = \{1, 2, 3\}, \quad h(n) = \{3, 2, 1\}$

10. The input $x(n)$ and the impulse response $h(n)$ of a LTI system are given by

$$x(n) = \{1, \underset{\uparrow}{0}, 2, 0\}; \quad h(n) = \{\underset{\uparrow}{3}, 0, 0, 2\}$$

Determine the response of the system using (a) Linear convolution (b) Circular convolution.

11. Find the regular convolution of the following sequences using circular convolution.

$$x(n) = \{2,\ -3,\ 1,\ -2,\ 0,\ 3\};\quad h(n) = \{1,\ 3,\ 2\}$$

12. Let $x(n) = \{2, 1, 0, -5, 2\}$ and $h(n) = \{1, 3\}$

(a) How many zeros must be appended to $x(n)$ and $h(n)$ in order to generate their regular convolution from the zero-padded sequences?

(b) What is the regular convolution of the original sequences?

(c) What is the circular convolution of the zero-padded sequences?

(d) What is the regular convolution of the zero-padded sequences?

13. Find the periodic convolution of the following sequences using linear convolution:

(a) $x_1(n) = \{3,\ -1,\ 2,\ -4,\ 5,\ 7\};\ \ x_2(n) = \{1,\ 0,\ 3\}$

(b) $x_1(n) = \{1,\ 0,\ -2,\ 0,\ 3\};\ \ x_2(n) = \{3,\ 2\}$

14. Find the cross correlation of $x(n) = \{2, 3, 1, 4\}$ and $h(n) = \{1, 2, 1, 2\}$

15. Given $x(n) = \{1, 2, 3, 4\}$ and $h(n) = \{1, 1, 2, 2\}$, show that $R_{xh}(n) \neq R_{hx}(n)$ and $R_{xh}(n) = R_{hx}(-n)$.

16. Find the autocorrelation of $x(n) = \{4, -1, 3, -2\}$.

17. Find the discrete periodic cross correlation $R_{xyp}(n)$ and $R_{yxp}(n)$ of the periodic sequences whose first period is given by

(a) $x(n) = \{2, 4, 0, -3\}$ and $y(n) = \{1, -2, 3, -1\}$

(b) $x(n) = \{3, -1, 0, -2\}$ and $y(n) = \{2, 3\}$

18. Find the periodic autocorrelation of the sequences:

(a) $x(n) = \{4,\ -3,\ 1,\ -2\}$ (b) $x(n) = \{2,\ -1,\ 3\}$

MATLAB PROGRAMS

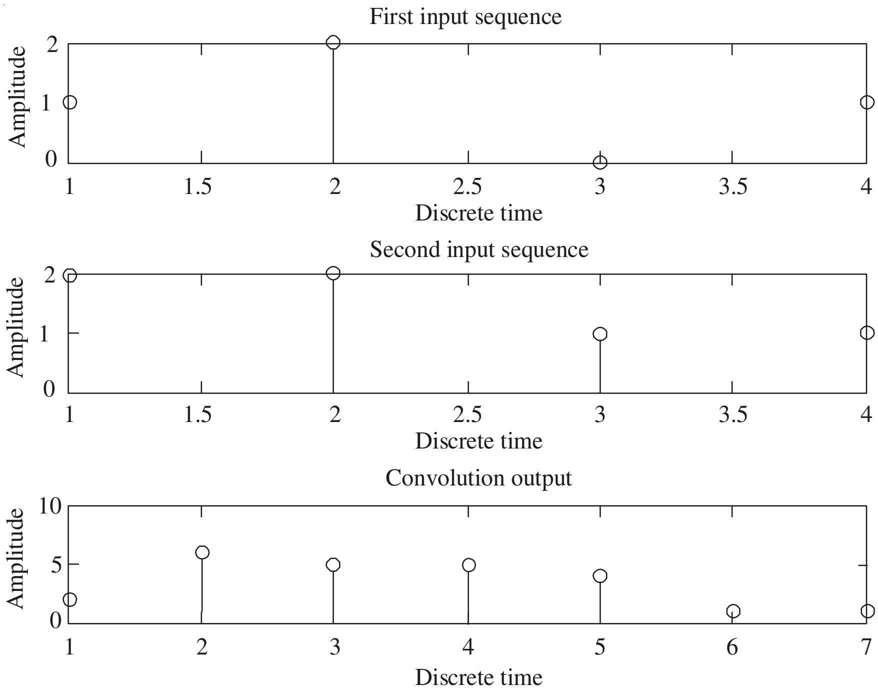
Program 2.1

% Convolution of two sequences

```
clc; clear all; close all;
x1=[1 2 0 1];
x2=[2 2 1 1];
y=conv(x1,x2);
disp('the convolution output is')
disp(y)
subplot(3,1,1),stem(x1);
xlabel('Discrete time')
ylabel('Amplitude')
title('first input sequence')
subplot(3,1,2),stem(x2);
xlabel('Discrete time')
ylabel('Amplitude')
title('Second input sequence')
subplot(3,1,3),stem(y);
xlabel('Discrete time')
ylabel('Amplitude')
title('convolution output')
```

```
the convolution output is

    [2   6   5   5   4   1   1]
```

Output:



Program 2.2

%Linear convolution via circular convolution

```
clc; clear all; close all;
x1=[1 2 3 4 5];
x2=[2 2 0 1 1];
x1e=[x1 zeros(1,length(x2)–1)];
x2e=[x2 zeros(1,length(x1)–1)];
ylin=cconv(x1e,x2e,length(x1e));
disp(‘linear convolution via circular convolution’)
disp(‘ylin’)
y=conv(x1,x2);
disp(‘Direct convolution’)
disp(y)
```

```
linear convolution via circular convolution

  Columns 1 through 3

   1.999999999999999    5.999999999999998   10.000000000000000

  Columns 4 through 6

  15.000000000000000   21.000000000000000   15.000000000000000

  Columns 7 through 9

   7.000000000000000    9.000000000000000    5.000000000000000

Direct convolution

     2     6    10    15    21    15     7     9     5
```

Program 2.3

% Linear convolution using DFT

```
clc; clear all; close all;
x=[1 2];
h=[2 1];
x1=[x zeros(1,length(h)–1)];
h1=[h zeros(1,length(x)–1)];
X=fft(x1);
H=fft(h1);
y=X.*H;
y1=ifft(y);
disp('the linear convolution of the given sequence')
disp(y1)

the linear convolution of the given sequence

                  [ 2     5     2]
```

Program 2.4

% Circular convolution using DFT based approach

```
clc; clear all; close all;
x1=[1 2 0 1];
x2=[2 2 1 1];
d4=[1 1 1 1;1 –j –1 j;1 –1 1 –1; 1 j –1 –j];
x11=d4*x1';
x21=d4*x2';
X=x11.*x21;
x=conj((d4)*X/4);
disp('circular convolution by using DFT method')
disp(x)
x3=cconv(x1,x2,4);
disp('circular convolution by using time domain method')
disp(x3)
```

circular convolution by using DFT method

[6 5 6 7]

circular convolution by using time domain method

[6 7 6 5]

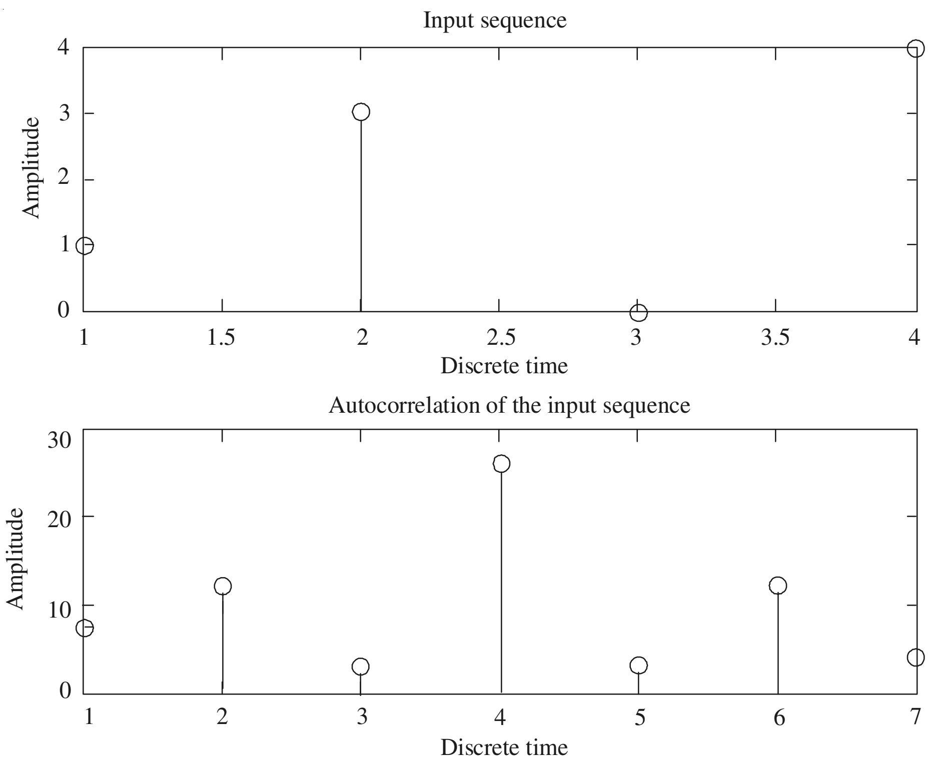
Program 2.5

% Computation of correlation

```
x1=[1 3 0 4];
y=xcorr(x1,x1);
subplot(2,1,1);stem(x1);
xlabel('Discrete time')
ylabel('Amplitude')
title('input sequence')
subplot(2,1,2);stem(y);
```

```
title('Autocorelation of the input sequence')
xlabel('Discrete time')
ylabel('Amplitude')
```

Output



3
Z-Transforms

3.1 INTRODUCTION

A linear time-invariant discrete-time system is represented by difference equations. The direct solution of higher order difference equations is quite tedious and time consuming. So usually they are solved by indirect methods. The Z-transform plays the same role for discrete-time systems as that played by Laplace transform for continuous-time systems. The Z-transform is the discrete-time counterpart of the Laplace transform. It is the Laplace transform of the discretized version of the continuous-time signal $x(t)$. To solve the difference equations which are in time domain, they are converted first into algebraic equations in z-domain using Z-transform, the algebraic equations are manipulated in z-domain and the result obtained is converted back into time domain using inverse Z-transform. The Z-transform has the advantage that it is a simple and systematic method and the complete solution can be obtained in one step and the initial conditions can be introduced in the beginning of the process itself. The Z-transform plays an important role in the analysis and representation of discrete-time Linear Shift Invariant (LSI) systems. It is the generalization of the Discrete-Time Fourier Transform (DTFT). The Z-transform may be one-sided (unilateral) or two-sided (bilateral). It is the one-sided or unilateral Z-transform that is more useful, because we mostly deal with causal sequences. Further, it is eminently suited for solving difference equations with initial conditions.

The *bilateral* or *two-sided* Z-transform of a discrete-time signal or a sequence $x(n)$ is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

where z is a complex variable.

The *one-sided* or *unilateral* Z-transform is defined as:

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

If $x(n) = 0$, for $n < 0$, the one-sided and two-sided Z-transforms are equivalent.

In the z-domain, the convolution of two time domain signals is equivalent to multiplication of their corresponding Z-transforms. This property simplifies the analysis of the response of an LTI system to various signals.

Region of convergence (ROC)

For any given sequence, the Z-transform may or may not converge.

The set of values of z or equivalently the set of points in z-plane, for which $X(z)$ converges is called the region of convergence (ROC) of $X(z)$. In general ROC can be $R_{x^-} < |z| < R_{x^+}$ where R_{x^-} can be as small as zero and R_{x^+} can be as large as infinity.

If there is no value of z (i.e. no point in the z-plane) for which $X(z)$ converges, then the sequence $x(n)$ is said to be having no Z-transform.

3.1.1 Advantages of Z-transform

1. The Z-transform converts the difference equations of a discrete-time system into linear algebraic equations so that the analysis becomes easy and simple.
2. Convolution in time domain is converted into multiplication in z-domain.
3. Z-transform exists for most of the signals for which Discrete-Time Fourier Transform (DTFT) does not exist.
4. Also since the Fourier transform is nothing but the Z-transform evaluated along the unit circle in the z-plane, the frequency response can be determined.

3.2 RELATION BETWEEN DISCRETE-TIME FOURIER TRANSFORM (DTFT) AND Z-TRANSFORM

The Discrete-Time Fourier Transform (DTFT) of a sequence $x(n)$ is given by

$$X(e^{j\omega}) \quad \text{or} \quad X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

For the existence of DTFT, the above summation should converge, i.e. $x(n)$ must be absolutely summable. The Z-transform of the sequence $x(n)$ is given by

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

where z is a complex variable and is given by

$$z = re^{j\omega}$$

where r is the radius of a circle.

$$\therefore \qquad X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\,(re^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [x(n)\,r^{-n}]\,e^{-j\omega n}$$

For the existence of Z-transform, the above summation should converge, i.e. $x(n)\,r^{-n}$ must be absolutely summable, i.e.

$$\sum_{n=-\infty}^{\infty} |x(n)\,r^{-n}| < \infty$$

The above equation represents the Discrete-Time Fourier Transform of a signal $x(n)\,r^{-n}$. Hence, we can say that the Z-transform of $x(n)$ is same as the Discrete-Time Fourier Transform of $x(n)\,r^{-n}$.

For the DTFT to exist, the discrete sequence $x(n)$ must be absolutely summable, i.e.

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

So for many sequences, the DTFT may not exist but the Z-transform may exist. When $r = 1$, the DTFT is same as the Z-transform, i.e. the DTFT is nothing but the Z-transform evaluated along the unit circle centred at the origin of the z-plane.

EXAMPLE 3.1 Prove that, for causal sequences, the ROC is the exterior of a circle of radius r.

Solution: Causal sequences are the sequences defined for only positive integer values of n and do not exist for negative times, i.e.

$$x(n) = 0 \quad \text{for } n < 0$$

Consider a causal sequence,

$$x(n) = r^n\,u(n)$$

From the definition of Z-transform of $x(n)$, we have

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\,z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} r^n\,u(n)\,z^{-n}$$

$$= \sum_{n=0}^{\infty} r^n\,z^{-n}$$

$$= \sum_{n=0}^{\infty} (rz^{-1})^n$$

The above summation converges for

$$|rz^{-1}| < 1, \text{ i.e. for } |z| > r$$

Hence, for the causal sequences, the ROC is the exterior of a circle of radius r.

EXAMPLE 3.2 Prove that the sequences

(a) $x(n) = a^n u(n)$ and (b) $x(n) = -a^n u(-n-1)$

have the same $X(z)$ and differ only in ROC. Also plot their ROCs.

Solution:

(a) The given sequence $a^n u(n)$ is a causal infinite duration sequence, i.e.

$$x(n) = \begin{cases} a^n, & n \ge 0 \\ 0, & n < 0 \end{cases} \quad \text{because } u(n) = \begin{cases} 1 & \text{for } n \ge 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$\therefore \qquad Z[x(n)] = Z[a^n u(n)] = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} [az^{-1}]^n = 1 + az^{-1} + (az^{-1})^2 + (az^{-1})^3 + \cdots$$

This is a geometric series of infinite length, and converges if $|az^{-1}| < 1$, i.e. if $|z| > |a|$.

$$\therefore \qquad X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}; \text{ ROC; } |z| > |a|$$

which implies that the ROC is exterior to the circle of radius a as shown in Figure 3.1(a)

$$\boxed{a^n u(n) \xleftrightarrow{\text{ZT}} \frac{1}{1 - az^{-1}} = \frac{z}{z-a}; \text{ ROC; } |z| > a}$$

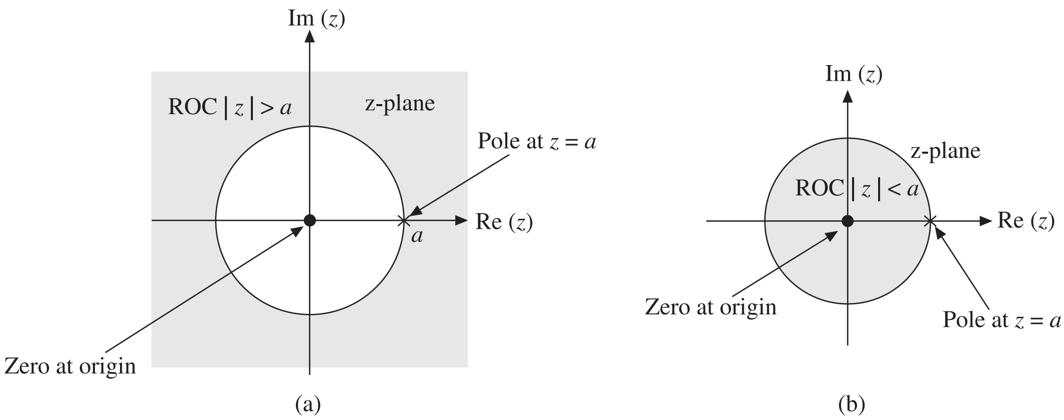


Figure 3.1 (a) ROC of $a^n u(n)$ (b) ROC of $-a^n u(-n-1)$.

(b) The given signal $x(n) = -a^n u(-n-1)$ is a non-causal infinite duration sequence, i.e.

$$x(n) = \begin{cases} -a^n, & n \le -1 \\ 0, & n \ge 0 \end{cases} \quad \text{because} \quad u(-n-1) = \begin{cases} 1 & \text{for } n \le -1 \\ 0 & \text{for } n \ge 0 \end{cases}$$

$$\therefore \quad X(z) = Z[-a^n u(-n-1)] = \sum_{n=-\infty}^{\infty} -a^n u(-n-1)\, z^{-n} = \sum_{n=-\infty}^{-1} -a^n z^{-n}$$

$$= \sum_{n=1}^{\infty} -a^{-n} z^n = -\sum_{n=1}^{\infty} (a^{-1} z)^n$$

This is a geometric series of infinite length and converges if $|a^{-1}z| < 1$ or $|z| < |a|$. Hence

$$X(z) = -\left[\sum_{n=0}^{\infty} (a^{-1}z)^n - 1\right] = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

$$= 1 - \frac{1}{1-a^{-1}z} = -\frac{a^{-1}z}{1-a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}; \text{ ROC; } |z| < |a|$$

That is, the ROC is the interior of the circle of radius a as shown in Figure 3.1(b).

From this example, we can observe that the Z-transform of the sequences $a^n u(n)$ and $-a^n u(-n-1)$ are same, even though the sequences are different. Only ROC differentiates them. Therefore, to find the correct inverse Z-transform, it is essential to know the ROC. The ROCs are shown in Figure 3.1[(a) and (b)].

In general, the ROC of a causal signal is $|z| > a$ and the ROC of a non-causal signal is $|z| < a$, where a is some constant.

EXAMPLE 3.3 Find the Z-transform of the sequence

$$x(n) = \left(\frac{1}{4}\right)^n \cos\left(\frac{\pi}{3} n\right) u(n)$$

Also sketch the ROC and pole-zero location.

Solution: Given $x(n) = \left(\frac{1}{4}\right)^n \cos\left(\frac{\pi}{3} n\right) u(n) = \left(\frac{1}{4}\right)^n \left[\frac{e^{j(\pi/3)n} + e^{-j(\pi/3)n}}{2}\right] u(n)$

$$= \frac{1}{2}\left[\left(\frac{1}{4}e^{j(\pi/3)}\right)^n + \left(\frac{1}{4}e^{-j(\pi/3)}\right)^n\right] u(n)$$

We have

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2}\left[\left(\frac{1}{4}e^{j(\pi/3)}\right)^n + \left(\frac{1}{4}e^{-j(\pi/3)}\right)^n\right] u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1}{4}e^{j(\pi/3)}\right)^n z^{-n} + \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1}{4}e^{-j(\pi/3)}\right)^n z^{-n}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left[\frac{1}{4}e^{j(\pi/3)} z^{-1}\right]^n + \frac{1}{2}\sum_{n=0}^{\infty}\left[\frac{1}{4}e^{-j(\pi/3)} z^{-1}\right]^n$$

$$= \frac{1}{2}\left(\frac{1}{1-(1/4)e^{j(\pi/3)} z^{-1}}\right) + \frac{1}{2}\left(\frac{1}{1-(1/4)\, e^{-j(\pi/3)} z^{-1}}\right)$$

The first series converges if $\left|\frac{1}{4} z^{-1} e^{j(\pi/3)}\right| < 1$, i.e. $|z| > \frac{1}{4}$ and the second series converges if $\left|\frac{1}{4} z^{-1} e^{-j(\pi/3)}\right| < 1$, i.e. $|z| > \frac{1}{4}$. So the ROC is $|z| > \frac{1}{4}$.

$$\therefore \quad X(z) = \frac{1}{2}\left(\frac{z}{z-(1/4)\, e^{j(\pi/3)}}\right) + \frac{1}{2}\left(\frac{z}{z-(1/4)\, e^{-j(\pi/3)}}\right)$$

$$= \frac{z\,[z-(1/4)\cos(\pi/3)]}{[z-(1/4)\, e^{j(\pi/3)}]\,[z-(1/4)\, e^{-j(\pi/3)}]} = \frac{z[z-(1/8)]}{[z-(1/8)-j(\sqrt{3}/8)]\,[z-(1/8)+j(\sqrt{3}/8)]}$$

The poles are at $z = (1/4)\, e^{j(\pi/3)}$ and $z = (1/4)\, e^{-j(\pi/3)}$ and the zeros are at $z = 0$ and $z = (1/8)$. The pole-zero location and the ROC are sketched in Figure 3.2.

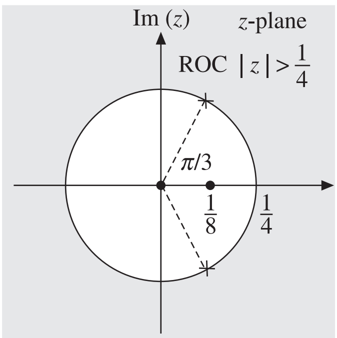


Figure 3.2 Pole-zero location and ROC for Example 3.3.

EXAMPLE 3.4 Find the Z-transform and ROC of

$$x(n) = 2\left(\frac{5}{6}\right)^n u(-n-1) + 3\left(\frac{1}{2}\right)^{2n} u(n)$$

Sketch the ROC and pole-zero location.

Solution: Given $x(n) = 2\left(\frac{5}{6}\right)^n u(-n-1) + 3\left(\frac{1}{2}\right)^{2n} u(n) = 2\left(\frac{5}{6}\right)^n u(-n-1) + 3\left(\frac{1}{4}\right)^n u(n)$

We have

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=-\infty}^{\infty} \left[2\left(\frac{5}{6}\right)^n u(-n-1) + 3\left(\frac{1}{4}\right)^n u(n)\right] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} 2\left(\frac{5}{6}\right)^n u(-n-1)\, z^{-n} + \sum_{n=-\infty}^{\infty} 3\left(\frac{1}{4}\right)^n u(n)\, z^{-n} = \sum_{n=-\infty}^{-1} 2\left(\frac{5}{6}\right)^n z^{-n} + \sum_{n=0}^{\infty} 3\left(\frac{1}{4}\right)^n z^{-n}$$

$$= \sum_{n=1}^{\infty} 2\left[\left(\frac{5}{6}\right)^{-1} z\right]^n + \sum_{n=0}^{\infty} 3\left(\frac{1}{4}z^{-1}\right)^n$$

The first series converges if $|(5/6)^{-1}z| < 1$ or $|z| < (5/6)$ and the second series converges if $|(1/4)\, z^{-1}| < 1$ or $|z| > (1/4)$.

So the region of convergence for $X(z)$ is $(1/4) < |z| < (5/6)$, i.e. it is a ring with $(1/4) < |z| < (5/6)$.

$$\therefore \qquad X(z) = 2\left\{-1 + \sum_{n=0}^{\infty} \left[\left(\frac{5}{6}\right)^{-1} z\right]^n\right\} + \sum_{n=0}^{\infty} 3\left(\frac{1}{4}z^{-1}\right)^n$$

$$= \left\{2\left[-1 + \frac{1}{1-(5/6)^{-1}z}\right] + 3\frac{1}{1-(1/4)\,z^{-1}}\right\} = -2\frac{z}{z-(5/6)} + 3\frac{z}{z-(1/4)}$$

$$= \frac{-2z^2 + (z/2) + 3z^2 - (5z/2)}{[z-(5/6)]\,[z-(1/4)]} = \frac{z(z-2)}{[z-(5/6)]\,[z-(1/4)]}$$

The ROC and the pole-zero plot are shown in Figure 3.3.

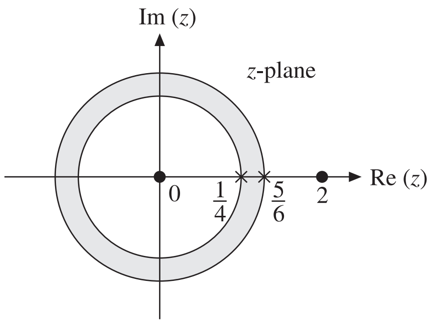


Figure 3.3 Pole-zero plot and ROC for Example 3.4.

EXAMPLE 3.5 Consider the sequence

$$x(n) = \begin{cases} a^n & 0 \le n \le N-1,\ a < 0 \\ 0 & \text{otherwise} \end{cases}$$

Find $X(z)$.

Solution: $$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n$$

$$= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1-(a/z)^N}{1-(a/z)} = \frac{(z^N - a^N)/z^N}{(z-a)/z} = \frac{z^N - a^N}{z^{N-1}(z-a)}$$

$\therefore$ $$X(z) = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$

3.3 Z-TRANSFORM AND ROC OF FINITE DURATION SEQUENCES

Finite duration sequences are sequences having a finite number of samples. Finite duration sequences may be right-sided sequences or left-sided sequences or two-sided sequences.

3.3.1 Right-sided Sequence

A right-sided sequence is one for which $x(n) = 0$ for $n < n_0$, where n_0 is positive or negative but finite. The Z-transform of such a sequence is $X(z) = \sum_{n=n_0}^{\infty} x(n)\, z^{-n}$. The ROC of the above series is the exterior of a cirle. If $n_0 \ge 0$, the resulting sequence is a causal or a positive time sequence. For a causal or a positive finite time sequence, the ROC is entire z-plane except at $z = 0$.

EXAMPLE 3.6 Find the ROC and Z-transform of the causal sequence

$$x(n) = \{\underset{\uparrow}{1}, 0, -2, 3, 5, 4\}$$

Solution: The given sequence values are:

$$x(0) = 1,\ x(1) = 0,\ x(2) = -2,\ x(3) = 3,\ x(4) = 5 \text{ and } x(5) = 4.$$

We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

For the given sample values,

$$X(z) = x(0) + x(1)z^{-1} + x(2)\, z^{-2} + x(3)\, z^{-3} + x(4)\, z^{-4} + x(5)\, z^{-5}$$

$\therefore$ $$Z[x(n)] = X(z) = 1 - 2z^{-2} + 3z^{-3} + 5z^{-4} + 4z^{-5}$$

The $X(z)$ converges for all values of z except at $z = 0$.

EXAMPLE 3.7 A finite sequence $x(n)$ is defined as $x(n) = \{5, 3, -2, 0, 4, -3\}$. Find $X(z)$ and its ROC.

Solution: Given $\quad x(n) = \{5, 3, -2, 0, 4, -3\}$

$$\therefore \quad x(n) = 5\delta(n) + 3\delta(n-1) - 2\delta(n-2) + 4\delta(n-4) - 3\delta(n-5)$$

The given sequence is a right-sided sequence. So the ROC is entire z-plane except at $z = 0$. Taking Z-transform on both sides of the above equation, we have

$$X(z) = 5 + 3z^{-1} - 2z^{-2} + 4z^{-4} - 3z^{-5}$$

ROC: Entire z-plane except at $z = 0$.

3.3.2 Left-sided Sequence

A left-sided sequence is one for which $x(n) = 0$ for $n \geq n_0$ where n_0 is positive or negative, but finite. The Z-transform of such a sequence is $X(z) = \sum_{n=-\infty}^{n_0} x(n)\, z^{-n}$. The ROC of the above series is the interior of a cirle. If $n_0 \leq 0$, the resulting sequence is anticausal sequence. For an anticausal finite duration sequence, the ROC is entire z-plane except at $z = \infty$.

EXAMPLE 3.8 Find the Z-transform and ROC of the anticausal sequence.

$$x(n) = \{4, 2, 3, -1, -2, \underset{\uparrow}{1}\}$$

Solution: The given sequence values are:

$$x(-5) = 4,\; x(-4) = 2,\; x(-3) = 3,\; x(-2) = -1,\; x(-1) = -2,\; x(0) = 1$$

We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

For the given sample values, $X(z)$ is:

$$X(z) = x(-5)\, z^5 + x(-4)\, z^4 + x(-3)\, z^3 + x(-2)z^2 + x(-1)\, z + x(0)$$

$$\therefore \quad Z[x(n)] = X(z) = 4z^5 + 2z^4 + 3z^3 - z^2 - 2z + 1$$

The $X(z)$ converges for all values of z except at $z = \infty$.

3.3.3 Two-sided Sequence

A sequence that has finite duration on both the left and right sides is known as a two-sided sequence. A two-sided sequence is one that extends from $n = -\infty$ to $n = +\infty$. In general, we can write $X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=-\infty}^{-1} x(n)\, z^{-n} + \sum_{n=0}^{\infty} x(n)\, z^{-n}$. The first series converges for

$|z| < R_{x^-}$ and the second series converges for $|z| > R_{x^+}$. So the ROC of such a sequence $R_{x^-} < |z| < R_{x^+}$ is a ring. For a two-sided finite duration sequence, the ROC is entire z-plane except at $z = 0$ and $z = \infty$.

EXAMPLE 3.9 Find the Z-transform and ROC of the sequence

$$x(n) = \{2, 1, -3, 0, \underset{\uparrow}{4}, 3, 2, 1, 5\}$$

Solution: The given sequence values are:

$$x(-4) = 2,\ x(-3) = 1,\ x(-2) = -3,\ x(-1) = 0,\ x(0) = 4,\ x(1) = 3,\ x(2) = 2,\ x(3) = 1,\ x(4) = 5$$

We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

For the given sample values,

$$X(z) = x(-4)z^4 + x(-3)z^3 + x(-2)z^2 + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4}$$

$$= 2z^4 + z^3 - 3z^2 + 4 + 3z^{-1} + 2z^{-2} + z^{-3} + 5z^{-4}$$

The ROC is entire z-plane except at $z = 0$ and $z = \infty$.

EXAMPLE 3.10 Find the Z-transform of the following sequences:

(a) $u(n) - u(n-4)$ (b) $u(-n) - u(-n-3)$ (c) $u(2-n) - u(-2-n)$

Solution:

(a) The given sequence is:

$$x(n) = u(n) - u(n-4)$$

From Figure 3.4, we notice that the sequence values are:

$$x(n) = 1, \quad \text{for } 0 \le n \le 3$$
$$= 0, \quad \text{otherwise}$$

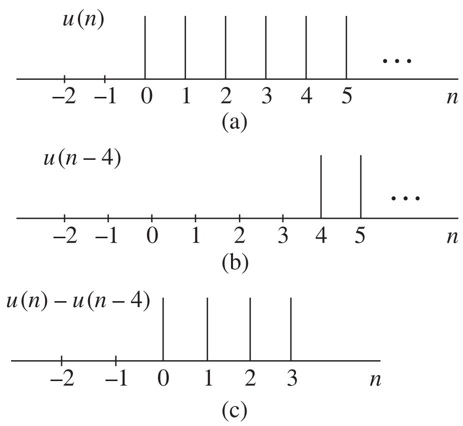


Figure 3.4 Sequences (a) $u(n)$, (b) $u(n-4)$ and (c) $u(n) - u(n-4)$.

We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

Substituting the sequence values, we get

$$X(z) = 1 + z^{-1} + z^{-2} + z^{-3}$$

The ROC is entire z-plane except at $z = 0$.

(b) The given sequence is:

$$x(n) = u(-n) - u(-n-3)$$

From Figure 3.5, we notice that the sequence values are:

$$\begin{aligned} x(n) &= 1, \quad \text{for } -2 \le n \le 0 \\ &= 0, \quad \text{otherwise} \end{aligned}$$

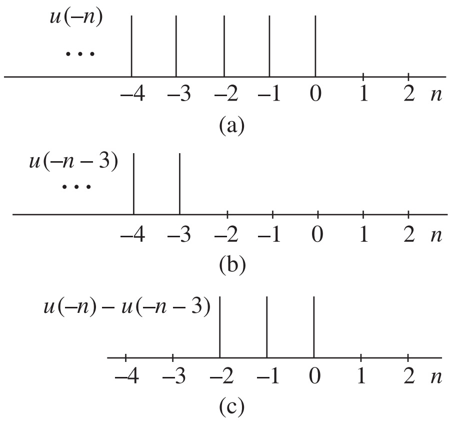


Figure 3.5 Sequences (a) $u(-n)$, (b) $u(-n-3)$ and (c) $u(-n) - u(-n-3)$.

We know that $$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Substituting the sequence values, we get

$$X(z) = 1 + z + z^2$$

The ROC is entire z-plane except at $z = \infty$.

(c) The given sequence is:

$$x(n) = u(2-n) - u(-2-n)$$

From Figure 3.6, we notice that the sequence values are:

$$\begin{aligned} x(n) &= 1, \quad \text{for } -1 \le n \le 2 \\ &= 0, \quad \text{otherwise} \end{aligned}$$

Substituting the sequence values, we get

$$X(z) = z + 1 + z^{-1} + z^{-2}$$

The ROC is entire z-plane except at $z = 0$ and $z = \infty$.

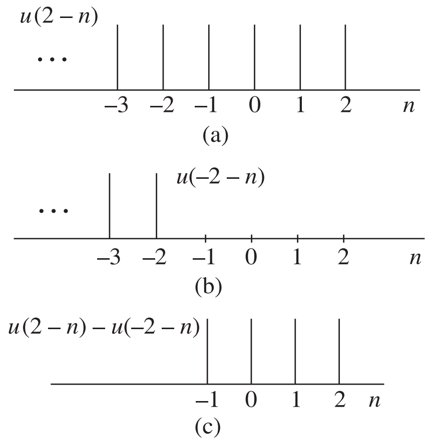


Figure 3.6 Sequences (a) $u(2 - n)$, (b) $u(-2 - n)$ and (c) $u(2 - n) - u(-2 - n)$.

3.4 PROPERTIES OF ROC

1. The ROC is a ring or disk in the z-plane centred at the origin.
2. The ROC cannot contain any poles.
3. If $x(n)$ is an infinite duration causal sequence, the ROC is $|z| > \alpha$, i.e. it is the exterior of a circle of radius α.
   If $x(n)$ is a finite duration causal sequence (right-sided sequence), the ROC is entire z-plane except at $z = 0$.
4. If $x(n)$ is an infinite duration anticausal sequence, the ROC is $|z| < \beta$, i.e. it is the interior of a circle of radius β.
   If $x(n)$ is a finite duration anticausal sequence (left-sided sequence), the ROC is entire z-plane except at $z = \infty$.
5. If $x(n)$ is a finite duration two-sided sequence, the ROC is entire z-plane except at $z = 0$ and $z = \infty$.
6. If $x(n)$ is an infinite duration, two-sided sequence, the ROC consists of a ring in the z-plane (ROC; $\alpha < |z| < \beta$) bounded on the interior and exterior by a pole, not containing any poles.
7. The ROC of an LTI stable system contains the unit circle.
8. The ROC must be a connected region. If $X(z)$ is rational, then its ROC is bounded by poles or extends up to infinity.
9. $x(n) = \delta(n)$ is the only signal whose ROC is entire z-plane.

3.5 PROPERTIES OF Z-TRANSFORM

The Z-transform has several properties that can be used in the study of discrete-time signals and systems. They can be used to find the closed form expression for the Z-transform of a given sequence. Many of the properties are analogous to those of the DTFT. They make the Z-transform a powerful tool for the analysis and design of discrete-time LTI systems. In general, both one-sided and two-sided Z-transforms have almost same properties.

3.5.1 Linearity Property

The linearity property of Z-transform states that, the Z-transform of a weighted sum of two signals is equal to the weighted sum of individual Z-transforms. That is, the linearity property states that

If $$x_1(n) \xleftrightarrow{\text{ZT}} X_1(z), \text{ with ROC} = R_1$$

and $$x_2(n) \xleftrightarrow{\text{ZT}} X_2(z), \text{ with ROC} = R_2$$

Then $$ax_1(n) + bx_2(n) \xleftrightarrow{\text{ZT}} aX_1(z) + bX_2(z), \text{ with ROC} = R_1 \cap R_2$$

Proof: We know that $$\text{Z}[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$\therefore$
$$\text{Z}[ax_1(n) + bx_2(n)] = \sum_{n=-\infty}^{\infty} [ax_1(n) + bx_2(n)]\, z^{-n}$$
$$= \sum_{n=-\infty}^{\infty} ax_1(n)\, z^{-n} + \sum_{n=-\infty}^{\infty} bx_2(n)\, z^{-n}$$
$$= a\sum_{n=-\infty}^{\infty} x_1(n)\, z^{-n} + b\sum_{n=-\infty}^{\infty} x_2(n)\, z^{-n}$$
$$= aX_1(z) + bX_2(z);\ \text{ROC};\ R_1 \cap R_2$$

$$\boxed{ax_1(n) + bx_2(n) \xleftrightarrow{\text{ZT}} aX_1(z) + bX_2(z)}$$

The ROC for the Z-transform of a sum of sequences is equal to the intersection of the ROCs of the individual transforms.

3.5.2 Time Shifting Property

The time shifting property of Z-transform states that

If $$x(n) \xleftrightarrow{\text{ZT}} X(z), \text{ with zero initial conditions with ROC} = R$$

Then $$x(n-m) \xleftrightarrow{\text{ZT}} z^{-m}\, X(z)$$

with ROC = R except for the possible addition or deletion of the origin or infinity.

Proof: We know that

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$\therefore \qquad Z[x(n-m)] = \sum_{n=-\infty}^{\infty} x(n-m)\, z^{-n}$$

Let $n - m = p$ in the summation, then $n = m + p$.

$$\therefore \qquad Z[x(n-m)] = \sum_{p=-\infty}^{\infty} x(p)\, z^{-(m+p)}$$

$$= z^{-m} \sum_{p=-\infty}^{\infty} x(p)\, z^{-p}$$

$$= z^{-m} X(z)$$

$$\boxed{x(n-m) \xleftrightarrow{\text{ZT}} z^{-m} X(z)}$$

Similarly, $$\boxed{x(n+m) \xleftrightarrow{\text{ZT}} z^{m} X(z)}$$

If the initial conditions are not neglected, we have

(a) Time delay $\quad Z[x(n-m)] = z^{-m} X(z) + z^{-m} \sum_{k=1}^{m} x(-k)\, z^{k}$

i.e. $\quad Z[x(n-m)] = z^{-m} X(z) + z^{-(m-1)} x(-1) + z^{-(m-2)} x(-2) + z^{-(m-3)} x(-3) + \cdots$

(b) Time advance $\quad Z[x(n+m)] = z^{m} X(z) - z^{m} \sum_{k=0}^{m-1} x(k)\, z^{-k}$

i.e. $\quad Z[x(n+m)] = z^{m} X(z) - z^{m} x(0) - z^{m-1} x(1) - z^{m-2} x(2)$

This time shifting property is very useful in finding the output $y(n)$ of a system described in difference equation for an input $x(n)$.

$$Z[x(n+1)] = zX(z) - zx(0)$$

$$Z[x(n+2)] = z^2 X(z) - z^2 x(0) - zx(1)$$

$$Z[x(n+3)] = z^3 X(z) - z^3 x(0) - z^2 x(1) - zx(2)$$

$$Z[x(n-1)] = z^{-1} X(z) + x(-1)$$

$$Z[x(n-2)] = z^{-2} X(z) + z^{-1} x(-1) + x(-2)$$

$$Z[x(n-3)] = z^{-3} X(z) + z^{-2} x(-1) + z^{-1} x(-2) + x(-3)$$

3.5.3 Multiplication by an Exponential Sequence Property

The multiplication by an exponential sequence property of Z-transform states that

If $$x(n) \xleftrightarrow{\text{ZT}} X(z) \text{ with ROC} = R$$

Then $$a^n x(n) \xleftrightarrow{\text{ZT}} X\left(\frac{z}{a}\right) \text{ with ROC} = |a|\,R$$

where a is a complex number.

Proof: We know that

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$\therefore \quad Z[a^n x(n)] = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left(\frac{z}{a}\right)^{-n}$$

$$= X\left(\frac{z}{a}\right)$$

$$\boxed{a^n x(n) \xleftrightarrow{\text{ZT}} X\left(\frac{z}{a}\right)}$$

If $X(z)$ has a pole at $z = z_1$, then $X(z/a)$ will have a pole at $z = az_1$. In general, all the pole-zero locations are scaled by a factor a.

Note:

$$e^{j\omega n} x(n) \xleftrightarrow{\text{ZT}} X\left(\frac{z}{e^{j\omega}}\right) = X(e^{-j\omega} z)$$

$$e^{-j\omega n} x(n) \xleftrightarrow{\text{ZT}} X\left(\frac{z}{e^{-j\omega}}\right) = X(e^{j\omega} z)$$

3.5.4 Time Reversal Property

The time reversal property of Z-transform states that

If $$x(n) \xleftrightarrow{\text{ZT}} X(z), \text{ with ROC} = R$$

Then $$x(-n) \xleftrightarrow{\text{ZT}} X\left(\frac{1}{z}\right), \text{ with ROC} = \frac{1}{R}$$

Proof: We know that

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$\therefore \qquad Z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n)\, z^{-n}$$

Let $p = -n$ in the above summation, then

$$Z\,[x(-n)] = \sum_{p=-\infty}^{\infty} x(p)\, z^{p}$$

$$= \sum_{p=-\infty}^{\infty} x(p)\, (z^{-1})^{-p}$$

$$= X(z^{-1}) = X\left(\frac{1}{z}\right)$$

$$\boxed{x(-n) \xleftrightarrow{\text{ZT}} X(z^{-1})}$$

3.5.5 Time Expansion Property

The time expansion property of Z-transform states that

If $\qquad x(n) \xleftrightarrow{\text{ZT}} X(z)$, with ROC $= R$

Then $\qquad x_k(n) \xleftrightarrow{\text{ZT}} X(z^k)$, with ROC $= R^{1/k}$

where $\qquad x_k(n) = x\left(\frac{n}{k}\right)$, if n is an integer multiple of k

$\qquad = 0$, otherwise

$x_k(n)$ has $k - 1$ zeros inserted between successive values of the original signal.

Proof: We know that

$$Z\,[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$\therefore \qquad Z[x_k(n)] = \sum_{n=-\infty}^{\infty} x_k(n)\, z^{-n} = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{k}\right) z^{-n}$$

Let $\qquad \frac{n}{k} = p$

$$\therefore \qquad Z[x_k(n)] = \sum_{p=-\infty}^{\infty} x(p)\, z^{-pk}$$

$$= \sum_{p=-\infty}^{\infty} x(p)\, (z^k)^{-p}$$

$$= X(z^k)$$

$$\boxed{x\left(\frac{n}{k}\right) \xleftrightarrow{\text{ZT}} X(z^k)}$$

3.5.6 Multiplication by *n* or Differentiation in *z*-domain Property

The multiplication by n or differentiation in z-domain property of Z-transform states that

If $\qquad x(n) \xleftrightarrow{\text{ZT}} X(z)$, with ROC = R

Then $\qquad nx(n) \xleftrightarrow{\text{ZT}} -z\dfrac{d}{dz}X(z)$, with ROC = R

Proof: We know that

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

Differentiating both sides with respect to z, we get

$$\frac{d}{dz}X(z) = \frac{d}{dz}\left[\sum_{n=-\infty}^{\infty} x(n)\, z^{-n}\right] = \sum_{n=-\infty}^{\infty} x(n)\, \frac{d}{dz}(z^{-n})$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, (-n)\, z^{-n-1}$$

$$= -z^{-1} \sum_{n=-\infty}^{\infty} [nx(n)] z^{-n}$$

$$= -z^{-1} Z[nx(n)]$$

$$\therefore \qquad Z[nx(n)] = -z\frac{d}{dz}X(z)$$

$$\boxed{nx(n) \xleftrightarrow{\text{ZT}} -z\frac{d}{dz}X(z)}$$

In the same way, $\qquad Z[n^k x(n)] = (-1)^k z^k \dfrac{d^k X(z)}{dz^k}$

3.5.7 Convolution Property

The convolution property of Z-transform states that the Z-transform of the convolution of two signals is equal to the multiplication of their Z-transforms, i.e.

If $$x_1(n) \xleftrightarrow{\text{ZT}} X_1(z), \text{ with ROC} = R_1$$

and $$x_2(n) \xleftrightarrow{\text{ZT}} X_2(z) \text{ with ROC} = R_2$$

Then $$x_1(n) * x_2(n) \xleftrightarrow{\text{ZT}} X_1(z)\, X_2(z), \text{ with ROC} = R_1 \cap R_2$$

Proof: We know that

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

Let $$x(n) = x_1(n) * x_2(n)$$

We have

$$\text{Z}[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$\therefore$ $$\text{Z}[x_1(n) * x_2(n)] = \sum_{n=-\infty}^{\infty} [x_1(n) * x_2(n)]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k) \right] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)\, z^{-(n-k)}\, z^{-k}$$

Interchanging the order of summations,

$$X(z) = \sum_{k=-\infty}^{\infty} x_1(k)\, z^{-k} \sum_{n=-\infty}^{\infty} x_2(n-k)\, z^{-(n-k)}$$

Replacing $(n - k)$ by p in the second summation, we get

$$X(z) = \sum_{k=-\infty}^{\infty} x_1(k)\, z^{-k} \sum_{p=-\infty}^{\infty} x_2(p)\, z^{-p}$$

$$= X_1(z)\, X_2(z)$$

$$\boxed{x_1(n) * x_2(n) \xleftrightarrow{\text{ZT}} X_1(z)\, X_2(z);\ \text{ROC};\ R_1 \cap R_2}$$

3.5.8 The Multiplication Property or Complex Convolution Property

The multiplication property or complex convolution property of Z-transform states that

If $\qquad x_1(n) \xleftrightarrow{\text{ZT}} X_1(z) \quad \text{and} \quad x_2(n) \xleftrightarrow{\text{ZT}} X_2(z)$

Then $\qquad x_1(n)\, x_2(n) \xleftrightarrow{\text{ZT}} \dfrac{1}{2\pi j} \oint_c X_1(v)\, X_2\left(\dfrac{z}{v}\right) v^{-1}\, dv$

Proof: From the definition of Z-transform, we have

$$\text{Z}\,[x_1(n)\, x_2(n)] = \sum_{n=-\infty}^{\infty} [x_1(n)\, x_2(n)]\, z^{-n}$$

But from the inverse Z-transform, we have

$$x(n) = \frac{1}{2\pi j} \oint X(z)\, z^{n-1}\, dz$$

Changing the complex variable z to v and substituting back, we get

$$\text{Z}\,[x_1(n)\, x_2(n)] = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi j} \oint X_1(v)\, v^{n-1}\, dv\right] x_2(n)\, z^{-n}$$

$$= \frac{1}{2\pi j} \oint X_1(v) \left[\sum_{n=-\infty}^{\infty} x_2(n)\, (v^{-1} z)^{-n}\right] v^{-1}\, dv$$

$$= \frac{1}{2\pi j} \oint X_1(v)\, X_2\left(\frac{z}{v}\right) v^{-1}\, dv$$

$$\boxed{x_1(n)\, x_2(n) \xleftrightarrow{\text{ZT}} \frac{1}{2\pi j} \oint_c X_1(v)\, X_2\left(\frac{z}{v}\right) v^{-1}\, dv}$$

3.5.9 Correlation Property

The correlation property of Z-transform states that

If $\qquad x_1(n) \xleftrightarrow{\text{ZT}} X_1(z) \quad \text{and} \quad x_2(n) \xleftrightarrow{\text{ZT}} X_2(z)$

Then $\qquad R_{12}(n) = x_1(n) \otimes x_2(n) \xleftrightarrow{\text{ZT}} X_1(z)\, X_2(z^{-1})$

Proof: From the definition of the Z-transform, we have

$$\text{Z}\,[x_1(n) \otimes x_2(n)] = \sum_{n=-\infty}^{\infty} [x_1(n) \otimes x_2(n)]\, z^{-n}$$

But from the correlation sum, we have

$$x_1(n) \otimes x_2(n) = \sum_{l=-\infty}^{\infty} x_1(l)\, x_2(l-n) \quad \text{or} \quad x_1(n) \otimes x_2(n) = \sum_{l=-\infty}^{\infty} x_1(l-n)\, x_2(l)$$

Substituting back, we get

$$Z[x_1(n) \otimes x_2(n)] = \sum_{n=-\infty}^{\infty} \left[\sum_{l=-\infty}^{\infty} x_1(l)\, x_2(l-n) \right] z^{-n}$$

Interchanging the order of summation, we have

$$Z[x_1(n) \otimes x_2(n)] = \sum_{l=-\infty}^{\infty} x_1(l) \left[\sum_{n=-\infty}^{\infty} x_2(l-n)\, z^{-n} \right]$$

Letting $l - n = m$ in the second summation, we have $n = l - m$.

$$\therefore \quad Z[x_1(n) \otimes x_2(n)] = \sum_{l=-\infty}^{\infty} x_1(l) \left[\sum_{m=\infty}^{-\infty} x_2(m)\, z^{-(l-m)} \right]$$

$$= \left[\sum_{l=-\infty}^{\infty} x_1(l)\, z^{-l} \right] \left[\sum_{m=-\infty}^{\infty} x_2(m)\, z^{m} \right]$$

$$= \left[\sum_{l=-\infty}^{\infty} x_1(l)\, z^{-l} \right] \left[\sum_{m=-\infty}^{\infty} x_2(m)\, (z^{-1})^{-m} \right]$$

$$= X_1(z)\, X_2(z^{-1})$$

$$\boxed{R_{12}(n) = x_1(n) \otimes x_2(n) \xleftrightarrow{\text{ZT}} X_1(z)\, X_2(z^{-1})}$$

3.5.10 Parseval's Theorem or Relation or Property

The Parseval's relation or theorem or property of Z-transform states that

If $\quad x_1(n) \xleftrightarrow{\text{ZT}} X_1(z)$ and $x_2(n) \xleftrightarrow{\text{ZT}} X_2(z)$

Then $\quad \displaystyle\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j} \oint X_1(z)\, X_2^*[(z^*)^{-1}]\, z^{-1}\, dz$ for complex $x_1(n)$ and $x_2(n)$

Proof: Substituting the relation of the inverse Z-transform at LHS, we get

$$\text{LHS} = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi j} \oint X_1(z)\, z^{n-1}\, dz \right] x_2^*(n)$$

$$= \frac{1}{2\pi j} \oint X_1(z) \left[\sum_{n=-\infty}^{\infty} x_2^*(n)\,(z^{-1})^{-n} \right] z^{-1}\, dz$$

$$= \frac{1}{2\pi j} \oint X_1(z) \left\{ \sum_{n=-\infty}^{\infty} x_2(n)\,[(z^*)^{-1}]^{-n} \right\}^* z^{-1}\, dz$$

$$= \frac{1}{2\pi j} \oint X_1(z) \left\{ X_2\,[(z^*)^{-1}] \right\}^* z^{-1}\, dz$$

$$= \frac{1}{2\pi j} \oint X_1(z)\, X_2^*\,[(z^*)^{-1}]\, z^{-1}\, dz = \text{RHS}$$

$$\boxed{\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j} \oint X_1(z)\, X_2^*\,[(z^*)^{-1}]\, z^{-1}\, dz}$$

3.5.11 Initial Value Theorem

The initial value theorem of Z-transform states that, for a causal signal $x(n)$

If $$x(n) \xleftrightarrow{\text{ZT}} X(z)$$

Then $$\underset{n \to 0}{\text{Lt}}\, x(n) = x(0) = \underset{z \to \infty}{\text{Lt}}\, X(z)$$

Proof: We know that for a causal signal

$$Z\,[x(n)] = X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} = x(0) + x(1)\, z^{-1} + x(2)\, z^{-2} + \cdots$$

$$= x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \cdots$$

Taking the limit $z \to \infty$ on both sides, we have

$$\underset{z \to \infty}{\text{Lt}}\, X(z) = \underset{z \to \infty}{\text{Lt}} \left[x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \frac{x(3)}{z^3} + \cdots \right] = x(0) + 0 + 0 + \cdots = x(0)$$

$\therefore$ $$\underset{n \to 0}{\text{Lt}}\, x(n) = x(0) = \underset{z \to \infty}{\text{Lt}}\, X(z)$$

$$\boxed{x(0) = \underset{z \to \infty}{\text{Lt}}\, X(z)}$$

This theorem helps us to find the initial value of $x(n)$ from $X(z)$ without taking its inverse Z-transform.

3.5.12 Final Value Theorem

The final value theorem of Z-transform states that, for a causal signal

If $$x(n) \xleftrightarrow{\text{ZT}} X(z)$$

and if $X(z)$ has no poles outside the unit circle, and it has no double or higher order poles on the unit circle centred at the origin of the z-plane, then

$$\underset{n\to\infty}{\text{Lt}}\ x(n) = x(\infty) = \underset{z\to 1}{\text{Lt}}\ (z-1)\,X(z)$$

Proof: We know that for a causal signal

$$\text{Z}\,[x(n)] = X(z) = \sum_{n=0}^{\infty} x(n)\,z^{-n}$$

$$\text{Z}\,[x(n+1)] = zX(z) - zx(0) = \sum_{n=0}^{\infty} x(n+1)\,z^{-n}$$

$\therefore$ $$\text{Z}\,[x\,(n+1)] - \text{Z}\,[x(n)] = zX(z) - zx(0) - X(z) = \sum_{n=0}^{\infty} x(n+1)\,z^{-n} - \sum_{n=0}^{\infty} x(n)\,z^{-n}$$

i.e. $$(z-1)\,X(z) - z\,x(0) = \sum_{n=0}^{\infty} [x(n+1) - x(n)]\,z^{-n}$$

i.e. $$(z-1)\,X(z) - z\,x(0) = [x(1) - x(0)]\,z^{-0} + [x(2) - x(1)]\,z^{-1} + [x(3) - x(2)]\,z^{-2} + \cdots$$

Taking limit $z \to 1$ on both sides, we have

$$\underset{z\to 1}{\text{Lt}}\ (z-1)\,X(z) - x(0) = [x(1) - x(0) + x(2) - x(1) + x(3) - x(2) + \cdots + x(\infty) - x(\infty - 1)$$

$$= x(\infty) - x(0)$$

$\therefore$ $$x(\infty) = \underset{z\to 1}{\text{Lt}}\ (z-1)\,X(z)$$

or $$x(\infty) = \underset{z\to 1}{\text{Lt}}\ (1 - z^{-1})\,X(z)$$

$$\boxed{x(\infty) = \underset{z\to 1}{\text{Lt}}\ (z-1)\,X(z)}$$

This theorem enables us to find the steady-state value of $x(n)$, i.e. $x(\infty)$ without taking the inverse Z-transform of $X(z)$.

Some common Z-transform pairs are given in Table 3.1. The properties of Z-transform are given in Table 3.2.

TABLE 3.1 Some Common Z-transform Pairs

	Sequence x(n)	*Z-transform X(z)*	ROC
1.	$\delta(n)$	1	All z
2.	$u(n)$	$z/(z-1) = 1/(1-z^{-1})$	$\lvert z \rvert > 1$
3.	$u(-n)$	$\dfrac{1}{1-z} = -\dfrac{1}{z-1} = -\dfrac{z^{-1}}{1-z^{-1}}$	$\lvert z \rvert < 1$
4.	$u(-n-1)$	$z/(z-1) = 1/(1-z^{-1})$	$\lvert z \rvert < 1$
5.	$u(-n-2)$	$z^2/(z-1)$	$\lvert z \rvert < 1$
6.	$u(-n-k)$	$z^k/(z-1)$	$\lvert z \rvert < 1$
7.	$\delta(n-k)$	z^{-k}	All z except at $z = 0$ (if $k > 0$) All z except at $z = \infty$ (if $k < 0$)
8.	$a^n u(n)$	$(z/(z-a)) = 1/(1-az^{-1})$	$\lvert z \rvert > \lvert a \rvert$
9.	$-a^n u(-n)$	$a/(z-a)$	$\lvert z \rvert < \lvert a \rvert$
10.	$-a^n u(-n-1)$	$z/(z-a) = 1/(1-az^{-1})$	$\lvert z \rvert < \lvert a \rvert$
11.	$nu(n)$	$-z/(z-1)^2 = -[z^{-1}/(1-z^{-1})^2]$	$\lvert z \rvert > 1$
12.	$na^n u(n)$	$az/(z-a)^2 = az^{-1}/(1-az^{-1})^2$	$\lvert z \rvert > \lvert a \rvert$
13.	$-na^n u(-n-1)$	$az/(z-a)^2 = az^{-1}/(1-az^{-1})^2$	$\lvert z \rvert < \lvert a \rvert$
14.	$e^{-j\omega n} u(n)$	$z/(z-e^{-j\omega}) = 1/(1-z^{-1}e^{-j\omega})$	$\lvert z \rvert > \lvert 1 \rvert$
15.	$\cos \omega n\, u(n)$	$\dfrac{z(z-\cos\omega)}{z^2-2z\cos\omega+1} = \dfrac{1-z^{-1}\cos\omega}{1-2z^{-1}\cos\omega+z^{-2}}$	$\lvert z \rvert > 1$
16.	$\sin \omega n\, u(n)$	$\dfrac{z\sin\omega}{z^2-2z\cos\omega+1} = \dfrac{z^{-1}\sin\omega}{1-2z^{-1}\cos\omega+z^{-2}}$	$\lvert z \rvert > 1$
17.	$a^n \cos \omega n\, u(n)$	$\dfrac{z(z-a\cos\omega)}{z^2-2az\cos\omega+a^2} = \dfrac{1-z^{-1}a\cos\omega}{1-2az^{-1}\cos\omega+a^2 z}$	$\lvert z \rvert > \lvert a \rvert$
18.	$a^n \sin \omega n\, u(n)$	$\dfrac{az\sin\omega}{z^2-2az\cos\omega+a^2} = \dfrac{az^{-1}\sin\omega}{1-2az^{-1}\cos\omega+a^2 z^{-2}}$	$\lvert z \rvert > \lvert a \rvert$
19.	$(n+1)\, a^n u(n)$	$z^2/(z-a)^2 = 1/(1-az^{-1})^2$	$\lvert z \rvert > \lvert a \rvert$
20.	$-nu(-n-1)$	$z/(z-1)^2 = z^{-1}/(1-z^{-1})^2$	$\lvert z \rvert < \lvert 1 \rvert$
21.	$na^n u(n)$	$z/(z-a)^2$	$\lvert z \rvert > \lvert a \rvert$
22.	$[n(n-1)\, a^{n-2} u(n)]/2!$	$z/(z-a)^3$	$\lvert z \rvert > \lvert a \rvert$
23.	$\dfrac{n(n-1)\ldots[n-(k-2)]\, a^{n-k+1}}{(k-1)!} u(n)$	$z/(z-a)^k$	$\lvert z \rvert > \lvert a \rvert$
24.	$1/n, \quad n > 0$	$-\ln(1-z^{-1})$	$\lvert z \rvert > \lvert 1 \rvert$
25.	$n^k a^n, \quad k < 0$	$-\left(-z\dfrac{d}{dz}\right)^n \dfrac{1}{1-az^{-1}}$	$\lvert z \rvert < \lvert a \rvert$
26.	$a^{\lvert n \rvert}$ for all n	$(1-a^2)/[(1-az)(1-az^{-1})]$	$\lvert a \rvert < \lvert z \rvert < \lvert 1/a \rvert$

TABLE 3.2 Properties of Z-transforms

Property	*Response*	ROC
	$x_1(n) \xleftrightarrow{\text{ZT}} X_1(z)$	R_1
	$x_2(n) \xleftrightarrow{\text{ZT}} X_2(z)$	R_2
Linearity	$ax_1(n) + bx_2(n) \xleftrightarrow{\text{ZT}} aX_1(z) + bX_2(z)$	$R_1 \cap R_2$
Time shifting	$x(n-m) \xleftrightarrow{\text{ZT}} z^{-m} X(z)$	Same as $X(z)$ except $z = 0$
	$x(n+m) \xleftrightarrow{\text{ZT}} z^{m} X(z)$	Same as $X(z)$ except $z = \infty$
Multiplication by exponential sequence or scaling in z-domain	$a^n u(n) \xleftrightarrow{\text{ZT}} X\left(\frac{z}{a}\right)$	$\lvert a \rvert R_1 < \lvert z \rvert < \lvert a \rvert R_2$
Time reversal	$x(-n) \xleftrightarrow{\text{ZT}} X(z^{-1})$	$\frac{1}{R_2} < \lvert z \rvert < \frac{1}{R_1}$
Time expansion	$x\left(\frac{n}{k}\right) \xleftrightarrow{\text{ZT}} X(z^k)$	
Differentiation in z-domain	$nx(n) \xleftrightarrow{\text{ZT}} -z\frac{d}{dz}X(z)$	$R_1 < \lvert z \rvert < R_2$
Conjugation	$x^*(n) \xleftrightarrow{\text{ZT}} X^*(z)^*$	$R_1 < \lvert z \rvert < R_2$
Accumulation	$\sum_{k=-\infty}^{n} x(k) \xleftrightarrow{\text{ZT}} \frac{1}{1-z^{-1}} X(z)$	
Convolution	$x_1(n) * x_2(n) \xleftrightarrow{\text{ZT}} X_1(z)\, X_2(z)$	At least the intersection of R_1 and R_2
Correlation	$R_{x_1x_2}(n) = x_1(n) \otimes x_2(n) \xleftrightarrow{\text{ZT}} X_1(z)\, X_2(z^{-1})$	At least the intersection of the ROC of $X_1(z)$ and $X_2(z^{-1})$
Multiplication	$x_1(n)\, x_2(n) \xleftrightarrow{\text{ZT}} \frac{1}{2\pi j} \oint_c X_1(v)\, X_2\left(\frac{z}{v}\right) v^{-1}\, dv$	At least $R_{1l}R_{2l} < \lvert z \rvert < R_{1u}R_{2u}$
Parseval's theorem	$\sum_{n=-\infty}^{\infty} x_1(n)\, x_2{}^*(n) = \frac{1}{2\pi j} \oint_c X_1(v)\, X_2{}^*\left(\frac{1}{v^*}\right) v^{-1}\, dv$	
Initial value theorem	$x(0) = \underset{n\to 0}{\text{Lt}}\ x(n) = \underset{z\to\infty}{\text{Lt}}\ X(z)$	
Final value theorem	$x(\infty) = \underset{n\to\infty}{\text{Lt}}\ x(n) = \underset{z\to 1}{\text{Lt}}\ (z-1)\, X(z)$, If $(1 - z^{-1})$ has no pole on or outside the unit circle	

Note: The initial value theorem and the final value theorem hold true only for causal signals.

EXAMPLE 3.11 Using properties of Z-transform, find the Z-transform of the following signals:

(a) $x(n) = u(-n)$ (b) $x(n) = u(-n+1)$
(c) $x(n) = u(-n-2)$ (d) $x(n) = 2^n u(n-2)$

Solution:

(a) Given $x(n) = u(-n)$

We know that $Z[u(n)] = \dfrac{z}{z-1} = \dfrac{1}{1-z^{-1}}$; ROC; $|z| > 1$

Using the time reversal property,

$$Z[u(-n)] = \left.\frac{z}{z-1}\right|_{z=(1/z)} = \frac{1/z}{(1/z)-1} = \frac{1}{1-z} = -\frac{1}{z-1}; \text{ ROC; } |z| < 1$$

(b) Given $x(n) = u(-n+1) = u[-(n-1)]$

$$\therefore \quad Z[x(n)] = X(z) = Z[u(-n+1)] = Z\{u[-(n-1)]\}$$

$$= z^{-1}Z[u(-n)] = z^{-1}\frac{1}{1-z} = -\frac{1}{z(z-1)}$$

(c) Given $x(n) = u(-n-2) = u[-(n+2)]$

$$\therefore \quad Z[x(n)] = X(z) = Z[u(-n-2)] = Z\{u[-(n+2)]\}$$

$$= z^2 Z[u(-n)] = \frac{z^2}{1-z} = -\frac{z^2}{z-1}$$

(d) Given $x(n) = 2^n u(n-2)$

$$Z[u(n-2)] = z^{-2}Z[u(n)] = z^{-2}\frac{z}{z-1} = \frac{z^{-1}}{z-1} = \frac{1}{z(z-1)}$$

$$Z[2^n u(n-2)] = Z[u(n-2)]\big|_{z=(z/2)} = \left.\frac{1}{z(z-1)}\right|_{z=(z/2)} = \frac{1}{(z/2)[(z/2)-1]} = \frac{4}{z(z-2)}$$

EXAMPLE 3.12 Using properties of Z-transform, find the Z-transform of the sequence

(a) $x(n) = \alpha^{n-2}u(n-2)$ (b) $x(n) = \begin{cases} 1, & \text{for } 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases}$

Solution:

(a) The Z-transform of the sequence $x(n) = \alpha^n u(n)$ is given by

$$X(z) = \frac{z}{z-\alpha}; \text{ ROC; } |z| > |\alpha|$$

Using the time shifting property of Z-transform, we have

$$Z[x(n-m)] = z^{-m}X(z)$$

In the same way,

$$Z[\alpha^{n-2}u(n-2)] = z^{-2}Z[\alpha^n u(n)] = z^{-2}\frac{z}{z-\alpha} = \frac{1}{z(z-\alpha)}; \text{ ROC; } |z| > |\alpha|$$

(b) Given $$x(n) = \begin{cases} 1, & \text{for } 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases}$$

implies that $x(n) = u(n) - u(n-N)$.

We know that $$Z[u(n)] = \frac{z}{z-1}$$

Using the time shifting property, we have

$$Z[u(n-N)] = z^{-N}Z[u(n)] = z^{-N}\frac{z}{z-1}$$

Using the linearity property, we have

$$Z[u(n) - u(n-N)] = Z[u(n)] - Z[u(n-N)] = \frac{z}{z-1} - z^{-N}\frac{z}{z-1} = \frac{z}{z-1}[1 - z^{-N}]$$

EXAMPLE 3.13 Using appropriate properties of Z-transform, find the Z-transform of the signal.

$$x(n) = n2^n \sin\left(\frac{\pi}{2}n\right)u(n)$$

Solution: Given $$x(n) = n2^n \sin\left(\frac{\pi}{2}n\right)u(n)$$

We know that

$$Z\left[\sin\left(\frac{\pi}{2}n\right)u(n)\right] = \frac{z\sin(\pi/2)}{z^2 - 2z\cos(\pi/2) + 1} = \frac{z}{z^2+1}$$

Using the multiplication by an exponential property, we have

$$Z\left[2^n\sin\left(\frac{\pi}{2}n\right)u(n)\right] = Z\left[\sin\left(\frac{\pi}{2}n\right)u(n)\right]\Bigg|_{z\to(z/2)}$$

$$= \frac{z}{z^2+1}\Bigg|_{z\to(z/2)} = \frac{z/2}{(z/2)^2+1}$$

$$= \frac{2z}{z^2+4}$$

Using differentiation in z-domain property, we have

$$Z\left[n2^n \sin\left(\frac{\pi}{2}n\right)u(n)\right] = -z\frac{d}{dz}\left\{Z\left[2^n \sin\left(\frac{\pi}{2}n\right)u(n)\right]\right\}$$

$$= -z\frac{d}{dz}\left(\frac{2z}{z^2+4}\right) = -z\left[\frac{(z^2+4)(2)-2z(2z)}{(z^2+4)^2}\right]$$

$$= -z\left[\frac{-2z^2+8}{(z^2+4)^2}\right] = \frac{2z(z^2-4)}{(z^2+4)^2}$$

EXAMPLE 3.14 Find the Z-transform of the following signal using convolution property of Z-transforms.

$$x(n) = \left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{4}\right)^n u(n)$$

Solution: Let $$x_1(n) = \left(\frac{1}{2}\right)^n u(n)$$

$\therefore$ $$X_1(z) = \frac{z}{z-(1/2)};\ \text{ROC};\ |z| > \frac{1}{2}$$

and $$x_2(n) = \left(\frac{1}{4}\right)^n u(n)$$

$\therefore$ $$X_2(z) = \frac{z}{z-(1/4)};\ \text{ROC};\ |z| > \frac{1}{4}$$

Now, $$x(n) = x_1(n) * x_2(n)$$

$\therefore$ $$Z[x(n)] = X(z) = Z[x_1(n) * x_2(n)] = X_1(z)\,X_2(z);\ \text{ROC};\ |z| > \frac{1}{2}$$

$$= \frac{z}{z-(1/2)}\,\frac{z}{z-(1/4)};\ \text{ROC};\ |z| > \frac{1}{2}$$

EXAMPLE 3.15 Find the Z-transform of the signal

$$x(n) = n\left[\left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{3}\right)^n u(n)\right]$$

Solution: Let $$x_1(n) = \left(\frac{1}{2}\right)^n u(n)$$

$\therefore$ $$X_1(z) = \frac{1}{1-(1/2)z^{-1}} = \frac{z}{z-(1/2)}; \text{ ROC; } |z| > \frac{1}{2}$$

and $$x_2(n) = \left(\frac{1}{3}\right)^n u(n)$$

$\therefore$ $$X_2(z) = \frac{1}{1-(1/3)z^{-1}} = \frac{z}{z-(1/3)}; \text{ ROC; } |z| > \frac{1}{3}$$

Using convolution in the time domain property, we have

$$Z[x_1(n) * x_2(n)] = X_1(z)\, X_2(z)$$

$\therefore$ $$Z\left[\left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{3}\right)^n u(n)\right] = \frac{z}{z-(1/2)} \frac{z}{z-(1/3)}$$

Using differentiation in z-domain property, we have

$$Z\left\{n\left[\left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{3}\right)^n u(n)\right]\right\} = -z\frac{d}{dz}\left\{Z\left[\left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{3}\right)^n u(n)\right]\right\}$$

$$= -z\frac{d}{dz}\left\{\frac{z^2}{[z-(1/2)]\,[z-(1/3)]}\right\}$$

$$= -z\left[\frac{[z^2-(5/6)z+(1/6)]\,2z - z^2\,[2z-(5/6)]}{[z^2-(5/6)z+(1/6)]^2}\right]$$

$$= -z\left[\frac{2z^3-(5/3)z^2+(1/3)z-2z^3+(5/6)z^2}{[z^2-(5/6)z+(1/6)]^2}\right]$$

$$= \frac{(5/6)\,z^2\,[z-(2/5)]}{[z-(1/2)]^2\,[z-(1/3)]^2}$$

EXAMPLE 3.16 Using Z-transform, find the convolution of the sequences

$$x_1(n) = \{2, 1, 0, -1, 3\};\ x_2(n) = \{1, -3, 2\}$$

Solution: From the convolution property of Z-transforms, we have

$$Z\{x_1(n) * x_2(n)\} = X_1(z)\, X_2(z) \text{ which implies that}$$

$$x_1(n) * x_2(n) = Z^{-1}[X_1(z)\, X_2(z)]$$

Given $$x_1(n) = \{2, 1, 0, -1, 3\}$$

$\therefore \qquad X_1(z) = 2 + z^{-1} - z^{-3} + 3z^{-4}$

and $\qquad x_2(n) = \{1, -3, 2\}$

$\therefore \qquad X_2(z) = 1 - 3z^{-1} + 2z^{-2}$

$$\therefore \qquad X_1(z)\, X_2(z) = (2 + z^{-1} - z^{-3} + 3z^{-4})\,(1 - 3z^{-1} + 2z^{-2})$$

$$= 2 - 5z^{-1} + z^{-2} + z^{-3} + 6z^{-4} - 11z^{-5} + 6z^{-6}$$

Taking inverse Z-transform on both sides,

$$x(n) = \{2, -5, 1, 1, 6, -11, 6\}$$

EXAMPLE 3.17 Find the convolution of the sequences

$$x_1(n) = \left(\frac{1}{2}\right)^n u(n) \text{ and } x_2(n) = \left(\frac{1}{3}\right)^{n-2} u(n-2)$$

using (a) Convolution property of Z-transforms and (b) Time domain method.

Solution:

(a) Given $\qquad x_1(n) = \left(\frac{1}{2}\right)^n u(n) \text{ and } x_2(n) = \left(\frac{1}{3}\right)^{n-2} u(n-2)$

$$\therefore \quad X_1(z) = \mathrm{Z}\left[\left(\frac{1}{2}\right)^n u(n)\right] = \frac{1}{1-(1/2)\,z^{-1}} = \frac{z}{z-(1/2)};\ \text{ROC; } |z| > \frac{1}{2}$$

and $$X_2(z) = \mathrm{Z}\left[\left(\frac{1}{3}\right)^{n-2} u(n-2)\right] = z^{-2}\,\mathrm{Z}\left[\left(\frac{1}{3}\right)^n u(n)\right]$$

$$= z^{-2}\,\frac{1}{1-(1/3)\,z^{-1}} = \frac{z^{-1}}{z-(1/3)};\ \text{ROC; } |z| > \frac{1}{3}$$

We know that

$$x(n) = x_1(n) * x_2(n)$$

$$\therefore \qquad \mathrm{Z}[x(n)] = X(z) = \mathrm{Z}\,[x_1(n) * x_2(n)] = X_1(z)\, X_2(z)$$

$$\therefore \qquad \mathrm{Z}\,[x_1(n) * x_2(n)] = \frac{z}{z-(1/2)}\,\frac{z^{-1}}{z-(1/3)} = \frac{1}{[z-(1/2)]\,[z-(1/3)]}$$

$$\therefore \quad x(n) = \mathrm{Z}^{-1}\left\{\frac{1}{[z-(1/2)]\,[z-(1/3)]}\right\} = \mathrm{Z}^{-1}\left[\frac{1}{z-(1/2)} - \frac{1}{z-(1/3)}\right]6$$

$$= 6\left[\left(\frac{1}{2}\right)^{n-1} u(n-1) - \left(\frac{1}{3}\right)^{n-1} u(n-1)\right]$$

(b) $$x_1(n) * x_2(n) = \sum_{k=0}^{n} x_1(k)\, x_2(n-k)$$

$$= \sum_{k=0}^{n-2} \left(\frac{1}{2}\right)^k u(k) \left(\frac{1}{3}\right)^{n-2-k} u(n-2-k)$$

$$= \sum_{k=0}^{n-2} \left(\frac{1}{2}\right)^k \left(\frac{1}{3}\right)^{n-2-k} = \sum_{k=0}^{n-2} \left(\frac{1}{2}\right)^k \left(\frac{1}{3}\right)^n \left(\frac{1}{3}\right)^{-2} \left(\frac{1}{3}\right)^{-k}$$

$$= 9\left(\frac{1}{3}\right)^n \sum_{k=0}^{n-2} \left(\frac{3}{2}\right)^k = 9\left(\frac{1}{3}\right)^n \left[\frac{1-(3/2)^{n-1}}{1-(3/2)}\right] = -18\left(\frac{1}{3}\right)^n \left[1-\left(\frac{3}{2}\right)^{n-1}\right]$$

$$= -6\left[\left(\frac{1}{3}\right)^{n-1} u(n-1) - \left(\frac{1}{3}\right)^{n-1} \left(\frac{3}{2}\right)^{n-1} u(n-1)\right]$$

$$= -6\left[\left(\frac{1}{3}\right)^{n-1} u(n-1) - \left(\frac{1}{2}\right)^{n-1} u(n-1)\right]$$

EXAMPLE 3.18 Determine the cross correlation sequence $r_{x_1x_2}(l)$ of the sequences

$$x_1(n) = (1, 2, 3, 4)$$
$$x_2(n) = (4, 3, 2, 1)$$

Solution: The cross correlation sequence can be obtained using the correlation property of Z-transforms.

Therefore, for the given $x_1(n)$ and $x_2(n)$, we have

$$X_1(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

and $$X_2(z) = 4 + 3z^{-1} + 2z^{-2} + z^{-3}$$

Thus, we have $$X_2(z^{-1}) = 4 + 3z + 2z^2 + z^3$$

Now $$R_{x_1x_2}(z) = X_1(z)X_2(z^{-1}) = (1 + 2z^{-1} + 3z^{-2} + 4z^{-3})(4 + 3z + 2z^2 + z^3)$$

or $$R_{x_1x_2}(z) = z^3 + 4z^2 + 10z + 20 + 25z^{-1} + 24z^{-2} + 16z^{-3}$$

Therefore, $$r_{x_1x_2}(l) = Z^{-1}[R_{x_1x_2}(z)] = \{1, 4, 10, \underset{\uparrow}{20}, 25, 24, 16\}$$

Another method

The cross correlation can be written as

$$r_{x_1x_2}(l) = \sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n-l)$$

or equivalently, we have

$$r_{x_1x_2}(l) = \sum_{n=-\infty}^{\infty} x_1(n+l)\, x_2(n)$$

where l is the time shift index.

For $l = 0$, we have $$r_{x_1x_2}(0) = \sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n)$$

The product sequence $P_0(n) = x_1(n)x_2(n)$ is

$$P_0(n) = \{4, 6, 6, 4\}$$

and therefore, the sum of all values of $P_0(n)$ is

$$r_{x_1x_2}(0) = 20$$

For $l = 1$, we have $$r_{x_1x_2}(1) = \sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n-1)$$

The product sequence $P_1(n) = x_1(n)x_2(n-1)$ is

$$P_1(n) = \{8, 9, 8\}$$

and therefore, the sum of all values of $P_1(n)$ is

$$r_{x_1x_2}(1) = 25$$

Similarly, $$r_{x_1x_2}(2) = 24,\ r_{x_1x_2}(3) = 16$$

For $l < 0$, $$r_{x_1x_2}(-1) = 10,\ r_{x_1x_2}(-2) = 4,\ r_{x_1x_2}(-3) = 1$$

Therefore, the cross correlation of sequences $x_1(n)$ and $x_2(n)$ is

$$r_{x_1x_2}(l) = \{1, 4, 10, \underset{\uparrow}{20}, 25, 24, 16\}$$

EXAMPLE 3.19 Using final value theorem, find $x(\infty)$, if $X(z)$ is given by

(a) $\dfrac{z+1}{(z-0.6)^2}$ (b) $\dfrac{z+2}{4(z-1)(z+0.7)}$ (c) $\dfrac{2z+3}{(z+1)(z+3)(z-1)}$

Solution:

(a) Given $$X(z) = \frac{z+1}{(z-0.6)^2}$$

Looking at $X(z)$, we notice that the ROC of $X(z)$ is $|z| > 0.6$ and $(z-1)\,X(z)$ has no poles on or outside the unit circle. Therefore,

$$x(\infty) = \underset{z\to 1}{\text{Lt}}\,(z-1)\,X(z) = \underset{z\to 1}{\text{Lt}}\,(z-1)\,\frac{z+1}{(z-0.6)^2} = 0$$

(b) Given $$X(z) = \frac{z+2}{4(z-1)\,(z+0.7)}$$

$$(z-1)\,X(z) = \frac{z+2}{4\,(z+0.7)}$$

$(z-1)\,X(z)$ has no poles on or outside the unit circle.

$$\therefore \qquad x(\infty) = \underset{z\to 1}{\text{Lt}}\,(z-1)\,X(z) = \underset{z\to 1}{\text{Lt}}\left[\frac{z+2}{4(z+0.7)}\right] = \frac{3}{6.8} = 0.44$$

(c) Given $$X(z) = \frac{2z+3}{(z+1)(z+3)(z-1)}$$

$$(z-1)\,X(z) = \frac{(z-1)\,(2z+3)}{(z+1)(z+3)(z-1)} = \frac{2z+3}{(z+1)\,(z+3)}$$

$(z-1)\,X(z)$ has one pole on the unit circle and one pole outside the unit circle. So $x(\infty)$ tends to infinity as $n \to \infty$.

EXAMPLE 3.20 Find $x(0)$ if $X(z)$ is given by

(a) $\dfrac{z^2+2z+2}{(z+1)\,(z+0.5)}$ (b) $\dfrac{z+3}{(z+1)(z+2)}$

Solution:

(a) Given $$X(z) = \frac{z^2+2z+2}{(z+1)(z+0.5)} = \frac{1+(2/z)+(2/z^2)}{[1+(1/z)]\,[1+(0.5/z)]}$$

$$x(0) = \underset{z\to\infty}{\text{Lt}}\,X(z) = \underset{z\to\infty}{\text{Lt}}\,\frac{[1+(2/z)+(2/z^2)]}{[1+(1/z)]\,[1+(0.5/z)]} = 1$$

(b) Given $$X(z) = \frac{z+3}{(z+1)\,(z+2)} = \frac{z[1+(3/z)]}{z^2\,[1+(1/z)]\,[1+(2/z)]} = \frac{1}{z}\,\frac{1+(3/z)}{[1+(1/z)]\,[1+(2/z)]}$$

$$x(0) = \underset{z\to\infty}{\text{Lt}}\,X(z) = \underset{z\to\infty}{\text{Lt}}\,\frac{1}{z}\,\frac{1+(3/z)}{[1+(1/z)]\,[1+(2/z)]} = 0$$

EXAMPLE 3.21 Prove that the final value of $x(n)$ for $X(z) = z^2/(z-1)(z-0.2)$ is 1.25 and its initial value is unity.

Solution: Given $$X(z) = \frac{z^2}{(z-1)(z-0.2)}$$

The final value theorem states that

$$\underset{n\to\infty}{\text{Lt}}\; x(n) = x(\infty) = \underset{z\to 1}{\text{Lt}}\,(z-1)\,X(z)$$

$$\therefore \qquad x(\infty) = \underset{z\to 1}{\text{Lt}}\,(z-1)\frac{z^2}{(z-1)(z-0.2)} = \underset{z\to 1}{\text{Lt}}\,\frac{z^2}{z-0.2} = \frac{1}{1-0.2} = 1.25$$

The initial value theorem states that

$$\underset{n\to 0}{\text{Lt}}\; x(n) = x(0) = \underset{z\to\infty}{\text{Lt}}\; X(z)$$

$$\therefore \qquad x(0) = \underset{z\to\infty}{\text{Lt}}\; \frac{1}{[1-(1/z)][1-(0.2/z)]} = 1$$

3.6 INVERSE Z-TRANSFORM

The process of finding the time domain signal $x(n)$ from its Z-transform $X(z)$ is called the inverse Z-transform which is denoted as:

$$x(n) = Z^{-1}[X(z)]$$

We have

$$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} [x(n)\,r^{-n}]\,e^{-j\omega n}$$

This is the DTFT of the signal $x(n)\,r^{-n}$. Hence the Inverse Discrete-Time Fourier Transform (IDTFT) of $X(re^{j\omega})$ must be $x(n)\,r^{-n}$. Therefore, we can write

$$x(n)r^{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(re^{j\omega})\,e^{j\omega n}\,d\omega$$

i.e.
$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(re^{j\omega})(re^{j\omega})^n\,d\omega$$

We have

$$z = re^{j\omega}$$

$$\therefore \qquad \frac{dz}{d\omega} = jre^{j\omega}, \text{ i.e. } d\omega = \frac{dz}{jre^{j\omega}}$$

$$\therefore \qquad x(n) = \frac{1}{2\pi j}\oint_c X(z)\,z^{n-1}\,dz$$

where the symbol $\oint_c$ denotes integration around the circle of radius $|z| = r$ in a counter clockwise direction.

This is the direct method of finding the inverse Z-transform of $X(z)$. It is quite tedious. So inverse Z-transform is normally found using indirect methods. The Z-transform $X(z)$ is a ratio of two polynomials in z given by

$$X(z) = \frac{b_0 z^M + b_1 z^{M-1} + b_2 z^{M-2} + \cdots + b_M}{z^N + a_1 z^{N-1} + a_2 z^{N-2} + \cdots + a_N}$$

The roots of the numerator polynomial are those values of z for which $X(z) = 0$ and are referred to as the zeros of $X(z)$. The roots of the denominator polynomial are those values of z for which $X(z) = \infty$ and are referred to as poles of $X(z)$. In z-plane, zero locations are denoted by • (a small circle) symbol and the pole locations with × (cross) symbol.

Basically, there are four methods that are often used to find the inverse Z-transform. They are:

(a) Power series method or long division method
(b) Partial fraction expansion method
(c) Complex inversion integral method (also known as the residue method)
(d) Convolution integral method

The long division method is simple, but does not give a closed form expression for the time signal. Further, it can be used only if the ROC of the given $X(z)$ is either of the form $|z| > \alpha$ or of the form $|z| < \alpha$, i.e. it is useful only if the sequence $x(n)$ is either purely right-sided or purely left-sided. The partial fraction expansion method enables us to determine the time signal $x(n)$ making use of our knowledge of some basic Z-transform pairs and Z-transform theorems. The inversion integral method requires a knowledge of the theory of complex variables, but is quite powerful and useful. The convolution integral method uses convolution property of Z-transforms and can be used when given $X(z)$ can be written as the product of two functions.

3.6.1 Long Division Method

The Z-transform of a two-sided sequence $x(n)$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

The $X(z)$ has both positive powers of z as well as negative powers of z. We cannot obtain a two-sided sequence by long division. If the sequence $x(n)$ is causal, then

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} = x(0)\, z^0 + x(1)\, z^{-1} + x(2)\, z^{-2} + \cdots$$

has only negative powers of z, with ROC; $|z| > \alpha$.

If the sequence $x(n)$ is anticausal, then

$$X(z)=\sum_{n=-\infty}^{0} x(n)\,z^{-n}=\cdots+x(-3)\,z^{3}+x(-2)\,z^{2}+x(-1)\,z^{1}+x(0)\,z^{0}$$

has only positive powers of z, with ROC; $|z|<\alpha$.

Since determination of the inverse Z-transform of $X(z)$ is only the determination of $x(n)$, i.e. $x(0)$, $x(1)$, $x(2)$, ... if it is a causal signal or $x(0)$, $x(-1)$, $x(-2)$, ... if it is an anticausal signal, to determine the inverse Z-transform, if $X(z)$ is a ratio of the polynomials

$$X(z)=\frac{N(z)}{D(z)}=\frac{b_0+b_1z^{-1}+b_2z^{-2}+\cdots+b_Mz^{-M}}{1+a_1z^{-1}+a_2z^{-2}+\cdots+a_Nz^{-N}}$$

We can generate a series in z by dividing the numerator of $X(z)$ by its denominator.

If $X(z)$ converges for $|z|>\alpha$, we obtain the series

$$X(z)=x(0)+x(1)\,z^{-1}+x(2)\,z^{-2}+\cdots$$

We can identify the coefficients of z^{-n} as $x(n)$ of a causal sequence.

If $X(z)$ converges for $|z|<\alpha$, we obtain the series

$$X(z)=x(0)+x(-1)\,z^{1}+x(-2)\,z^{2}+\cdots$$

We can identify the coefficients of z^{-n} as $x(n)$ of a non-causal sequence.

For getting a causal sequence, first put $N(z)$ and $D(z)$ either in descending powers of z or in ascending powers of z^{-1} before long division.

For getting a non-causal sequence, first put $N(z)$ and $D(z)$ either in ascending powers of z or in descending powers of z^{-1} before long division. This method is best illustrated by the following examples.

EXAMPLE 3.22 Find the inverse Z-transform of

$$X(z)=z^{3}+2z^{2}+z+1-2z^{-1}-3z^{-2}+4z^{-3}$$

Solution: We know that

$$X(z)=\sum_{n=-\infty}^{\infty} x(n)\,z^{-n}=\ldots x(-3)\,z^{3}+x(-2)\,z^{2}+x(-1)\,z^{1}+x(0)+x(1)\,z^{-1}$$

$$+\,x(2)\,z^{-2}+x(3)\,z^{-3}+\cdots$$

Comparing this $X(z)$ with the given $X(z)$, we have

$$x(n)=\{1,2,1,\underset{\uparrow}{1},-2,-3,4\}$$

Alternatively, taking inverse Z-transform of $X(z)$, we have

$$x(n)=\delta(n+3)+2\delta(n+2)+\delta(n+1)+\delta(n)-2\delta(n-1)-3\delta(n-2)+4\delta(n-3)$$

EXAMPLE 3.23 Determine the inverse Z-transform of

(a) $X(z)=\dfrac{1}{z-a}$; ROC; $|z|>a$ (b) $X(z)=\dfrac{1}{1-az^{-1}}$; ROC; $|z|>a$

(c) $X(z)=\dfrac{1}{1-z^{-4}}$; ROC; $|z|>1$

Solution:

(a) Given $X(z)=\dfrac{1}{z-a}$; ROC; $|z|>a$

$$=\frac{1}{z(1-az^{-1})}=z^{-1}(1-az^{-1})^{-1}=z^{-1}(1+az^{-1}+a^2z^{-2}+a^3z^{-3}+\cdots)$$

$$=z^{-1}+az^{-2}+a^2z^{-3}+\cdots=\sum_{n=1}^{\infty}a^{n-1}z^{-n}=\sum_{n=0}^{\infty}a^{n-1}u(n-1)z^{-n}$$

$\therefore$ $$x(n)=a^{n-1}u(n-1)$$

(b) Given $$X(z)=\frac{1}{1-az^{-1}};\ \text{ROC};\ |z|>a$$

By Taylor's series expansion, we have

$$X(z)=\frac{1}{1-az^{-1}}=1+az^{-1}+a^2z^{-2}+a^3z^{-3}+\cdots=\sum_{n=0}^{\infty}a^nz^{-n}=\sum_{n=0}^{\infty}a^n u(n)z^{-n}$$

Therefore, $$x(n)=a^n u(n)$$

(c) From infinite sum formula, we have

$$\sum_{k=0}^{\infty}\alpha^k=\frac{1}{1-\alpha};\ |\alpha|<1$$

Given $X(z)=\dfrac{1}{1-z^{-4}}=\displaystyle\sum_{k=0}^{\infty}(z^{-4})^k=\sum_{k=0}^{\infty}z^{-4k}$ $\left[|z^{-4}|<1,\ \text{i.e.}\ |z|>1\right]$

Taking inverse Z-transform on both sides, we get

$$x(n)=\sum_{k=0}^{\infty}\delta(n-4k)$$

$\therefore$ $x(n)=1$, when $n=4k$, i.e. when n is an integer multiple of 4

$=0$, otherwise

EXAMPLE 3.24 Determine the inverse Z-transform of the sequences:

(a) $X(z)=\cos(3z)$; ROC; $|z|<\infty$ (b) $X(z)=\sin(z)$; ROC; $|z|<\infty$

Solution:

(a) Given $$X(z) = \cos(3z)$$

The corresponding $x(n)$ must be a left-sided sequence because ROC is $|z| < \infty$. From the trigonometric series or Taylor series, we have

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k}}{(2k)!}$$

$$\therefore \qquad X(z) = \cos(3z) = \sum_{k=0}^{\infty} (-1)^k \frac{(3z)^{2k}}{(2k)!} = 1 - \frac{(3z)^2}{2!} + \frac{(3z)^4}{4!} - \frac{(3z)^6}{6!} + \cdots$$

$$= \cdots - \frac{81}{80}z^6 + \frac{27}{8}z^4 - \frac{9}{2}z^2 + 1$$

Therefore, the inverse Z-transform is:

$$x(n) = \left[\begin{array}{c} \cdots, -\frac{81}{80}, 0, \frac{27}{8}, 0, -\frac{9}{2}, 0, \underset{\uparrow}{1} \end{array}\right]$$

(b) Given $X(z) = \sin(2z)$. The corresponding $x(n)$ must be a left-sided sequence because ROC is $|z| < \infty$. From the trigonometric series or Taylor series, we have

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k+1}}{(2k+1)!}$$

$$\therefore \qquad X(z) = \sin(2z) = \sum_{k=0}^{\infty} (-1)^k \frac{(2z)^{2k+1}}{(2k+1)!} = (2z) - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} - \frac{(2z)^7}{7!} + \cdots$$

$$= \cdots - \frac{8}{315}z^7 + \frac{4}{15}z^5 - \frac{4}{3}z^3 + 2z$$

Therefore, the inverse Z-transform is:

$$x(n) = \left[\begin{array}{c} \cdots, -\frac{8}{315}, 0, \frac{4}{15}, 0, -\frac{4}{3}, 0, 2, \underset{\uparrow}{0} \end{array}\right]$$

EXAMPLE 3.25 Determine the inverse Z-transform of the following transformed signals:

$$X(z) = \log_{10}(1 + az^{-1}); \text{ ROC; } |z| > a$$

Solution: Given

$$X(z) = \log_{10}(1 + az^{-1})$$

Since the given signal is of base 10, manipulating, we get

$$X(z) = \log_{10}(1 + az^{-1}) = \frac{\log_e(1 + az^{-1})}{\log_e(10)}$$

$$= \frac{1}{\log_e(10)}\left[az^{-1} - \frac{(az^{-1})^2}{2} + \frac{(az^{-1})^3}{3} - \frac{(az^{-1})^4}{4} + \frac{(az^{-1})^5}{5} - \cdots\right]$$

$$= -\sum_{n=1}^{\infty} \frac{(-az^{-1})^n}{n\log_e(10)} = -\sum_{n=1}^{\infty} \frac{(-a)^n}{n\log_e(10)} z^{-n}$$

Therefore, the inverse Z-transform is:

$$x(n) = \left[\underset{\uparrow}{0}, \frac{a}{\log_e(10)}, -\frac{a^2}{2\log_e(10)}, \frac{a^3}{3\log_e(10)}, -\frac{a^4}{4\log_e(10)}, \cdots\right]$$

$$= -\frac{(-a)^n}{n\log_e(10)} u(n-1)$$

EXAMPLE 3.26 Using power series expansion method, determine the inverse Z-transform of

$$X(z) = \ln(1 + z^{-1}); \text{ ROC; } |z| > 0$$

Solution: Given $\qquad X(z) = \ln(1 + z^{-1})$

The corresponding $x(n)$ must be right-sided sequence because ROC is $|z| > 0$. We know that

$$\ln(1 + \theta) = \theta - \frac{\theta^2}{2} + \frac{\theta^3}{3} - \frac{\theta^4}{4} + \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\theta^k}{k}; \text{ if } |\theta| < 1$$

$$\therefore \qquad X(z) = \ln(1 + z^{-1}) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(z^{-1})^k}{k} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{z^{-k}}{k}$$

Taking inverse Z-transform on both sides, we get

$$x(n) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\delta(n-k)}{k}$$

EXAMPLE 3.27 Determine the inverse Z-transform of

(a) $X(z) = \log_e\left(\dfrac{1}{1 - a^{-1}z}\right)$; ROC; $|z| < |a|$ (b) $X(z) = \log_e\left(\dfrac{1}{1 - az^{-1}}\right)$; ROC; $|z| > |a|$

Solution:

(a) Given $$X(z) = \log_e\left(\frac{1}{1-a^{-1}z}\right);\ \text{ROC};\ |z|<|a|$$

$$X(z) = \log_e\left(\frac{1}{1-a^{-1}z}\right) = -\log_e(1-a^{-1}z)$$

$$= -\left[-a^{-1}z - \frac{(a^{-1}z)^2}{2} - \frac{(a^{-1}z)^3}{3} - \frac{(a^{-1}z)^4}{4} - \cdots\right]$$

$\therefore$ $$X(z) = \cdots + \frac{1}{4a^4}z^4 + \frac{1}{3a^3}z^3 + \frac{1}{2a^2}z^2 + \frac{1}{a}z = \sum_{n=-\infty}^{-1}\left(\frac{a^n}{-n}\right)z^{-n}$$

Hence $$x(n) = -\frac{a^n}{n} \text{ for } n<0$$

that is, $$x(n) = \left(-\frac{a^n}{n}\right)u(-n-1)$$

(b) Given $$X(z) = \log_e\left(\frac{1}{1-az^{-1}}\right);\ \text{ROC};\ |z|>|a|$$

$$X(z) = \log_e\left(\frac{1}{1-az^{-1}}\right) = -\log_e(1-az^{-1})$$

$$= -\left[-az^{-1} - \frac{(az^{-1})^2}{2} - \frac{(az^{-1})^3}{3} - \frac{(az^{-1})^4}{4} - \cdots\right]$$

$\therefore$ $$X(z) = az^{-1} + \frac{(az^{-1})^2}{2} + \frac{(az^{-1})^3}{3} + \frac{(az^{-1})^4}{4} + \cdots = \sum_{k=1}^{\infty}\frac{(az^{-1})^k}{k} = \sum_{k=1}^{\infty}\frac{a^k}{k}z^{-k}$$

Taking inverse Z-transform, we have

$$x(n) = \sum_{k=1}^{\infty}\frac{a^k}{k}\delta(n-k) = \sum_{n=1}^{\infty}\frac{a^n}{n} = \frac{a^n}{n}u(n-1)$$

EXAMPLE 3.28 Determine the inverse *Z*-transform of

$$X(z) = \log_e(1+az^{-1});\ \text{ROC};\ |z|>a$$

Solution: Given $$X(z) = \log_e(1+az^{-1})$$

$$X(z) = \log_e(1+az^{-1}) = az^{-1} - \frac{(az^{-1})^2}{2} + \frac{(az^{-1})^3}{3} - \frac{(az^{-1})^4}{4} + \cdots$$

$$= -\sum_{n=1}^{\infty}\frac{(-az^{-1})^n}{n} = -\sum_{n=1}^{\infty}\frac{(-a)^n}{n}z^{-n} = -\frac{(-a)^n}{n}u(n-1)$$

$$\therefore \qquad x(n) = -\frac{(-a)^n}{n}u(n-1)$$

EXAMPLE 3.29 Using long division, determine the inverse Z-transform of

$$X(z) = \frac{z^2+2z}{z^3-3z^2+4z+1};\ \text{ROC};\ |z|>1$$

Solution: Since ROC is $|z| > 1$, $x(n)$ must be a causal sequence. For getting a causal sequence, the $N(z)$ and $D(z)$ of $X(z)$ must be put either in descending powers of z or in ascending powers of z^{-1} before performing long division.

In the given $X(z)$ both $N(z)$ and $D(z)$ are already in descending powers of z.

$$\begin{array}{r|l}
 & z^{-1} + 5z^{-2} + 11z^{-3} + 12z^{-4} - 13z^{-5} \\ \hline
z^3 - 3z^2 + 4z + 1 & z^2 + 2z \\
 & \underline{z^2 - 3z + 4 + z^{-1}} \\
 & \quad 5z - 4 - z^{-1} \\
 & \quad \underline{5z - 15 + 20z^{-1} + 5z^{-2}} \\
 & \qquad 11 - 21z^{-1} - 5z^{-2} \\
 & \qquad \underline{11 - 33z^{-1} + 44z^{-2} + 11z^{-3}} \\
 & \qquad\quad 12z^{-1} - 49z^{-2} - 11z^{-3} \\
 & \qquad\quad \underline{12z^{-1} - 36z^{-2} + 48z^{-3} + 12z^{-4}} \\
 & \qquad\qquad -13z^{-2} - 59z^{-3} - 12z^{-4}
\end{array}$$

$$\therefore \qquad X(z) = z^{-1} + 5z^{-2} + 11z^{-3} + 12z^{-4} - 13z^{-5} \cdots$$

$$\therefore \qquad x(n) = \{0, 1, 5, 11, 12, -13, \ldots\}$$

Writing $N(z)$ and $D(z)$ of $X(z)$ in ascending powers of z^{-1}, we have

$$X(z) = \frac{N(z)}{D(z)} = \frac{z^2+2z}{z^3-3z^2+4z+1} = \frac{z^{-1}+2z^{-2}}{1-3z^{-1}+4z^{-2}+z^{-3}}$$

$$\begin{array}{r|l}
 & z^{-1} + 5z^{-2} + 11z^{-3} + 12z^{-4} - 13z^{-5} \\ \hline
1 - 3z^{-1} + 4z^{-2} + z^{-3} & z^{-1} + 2z^{-2} \\
 & \underline{z^{-1} - 3z^{-2} + 4z^{-3} + z^{-4}} \\
 & \quad 5z^{-2} - 4z^{-3} - z^{-4} \\
 & \quad \underline{5z^{-2} - 15z^{-3} + 20z^{-4} + 5z^{-5}} \\
 & \qquad 11z^{-3} - 21z^{-4} - 5z^{-5} \\
 & \qquad \underline{11z^{-3} - 33z^{-4} + 44z^{-5} + 11z^{-6}} \\
 & \qquad\quad 12z^{-4} - 49z^{-5} - 11z^{-6} \\
 & \qquad\quad \underline{12z^{-4} - 36z^{-5} + 48z^{-6} + 12z^{-7}} \\
 & \qquad\qquad -13z^{-5} - 59z^{-6} - 12z^{-7}
\end{array}$$

$$\therefore \qquad X(z) = z^{-1} + 5z^{-2} + 11z^{-3} + 12z^{-4} - 13z^{-5} \cdots$$

$$\therefore \qquad x(n) = \{0, 1, 5, 11, 12, -13, \ldots\}$$

Observe that both the methods give the same sequence $x(n)$.

EXAMPLE 3.30 Using long division, determine the inverse Z-transform of

$$X(z) = \frac{z^2 + z + 2}{z^3 - 2z^2 + 3z + 4}; \text{ ROC; } |z| < 1$$

Solution: Since ROC is $|z| < 1$, $x(n)$ must be a non-causal sequence. For getting a non-causal sequence, the $N(z)$ and $D(z)$ must be put either in ascending powers of z or in descending powers of z^{-1} before performing long division.

$$X(z) = \frac{z^2 + z + 2}{z^3 - 2z^2 + 3z + 4} = \frac{2 + z + z^2}{4 + 3z - 2z^2 + z^3}$$

$$\begin{array}{r|l}
 & \frac{1}{2} - \frac{1}{8}z + \frac{19}{32}z^2 - \frac{81}{128}z^3 + \frac{411}{512}z^4 \\
\hline
4 + 3z - 2z^2 + z^3 & 2 + z + z^2 \\
 & 2 + \frac{3}{2}z - z^2 + \frac{1}{2}z^3 \\
\cline{2-2}
 & -\frac{1}{2}z + 2z^2 - \frac{1}{2}z^3 \\
 & -\frac{1}{2}z - \frac{3}{8}z^2 + \frac{1}{4}z^3 - \frac{1}{8}z^4 \\
\cline{2-2}
 & \frac{19}{8}z^2 - \frac{3}{4}z^3 + \frac{1}{8}z^4 \\
 & \frac{19}{8}z^2 + \frac{57}{32}z^3 - \frac{19}{16}z^4 + \frac{19}{32}z^5 \\
\cline{2-2}
 & -\frac{81}{32}z^3 + \frac{21}{16}z^4 - \frac{19}{32}z^5 \\
 & -\frac{81}{32}z^3 - \frac{243}{128}z^4 + \frac{81}{64}z^5 - \frac{81}{128}z^6 \\
\cline{2-2}
 & \frac{411}{128}z^4 - 129z^5 + \frac{81}{128}z^6
\end{array}$$

$$\therefore \qquad X(z) = \frac{1}{2} - \frac{1}{8}z + \frac{19}{32}z^2 - \frac{81}{128}z^3 + \frac{411}{512}z^4 \ldots$$

$$\therefore \qquad x(n) = \left\{ \ldots, \frac{411}{512}, -\frac{81}{128}, \frac{19}{32}, -\frac{1}{8}, \underset{\uparrow}{\frac{1}{2}} \right\}$$

Also
$$X(z) = \frac{2 + z + z^2}{4 + 3z - 2z^2 + z^3} = \frac{2z^{-3} + z^{-2} + z^{-1}}{4z^{-3} + 3z^{-2} - 2z^{-1} + 1}$$

$$\begin{array}{r|l}
 & \frac{1}{2} - \frac{1}{8}z + \frac{19}{32}z^2 - \frac{81}{128}z^3 + \frac{411}{512}z^4 + \cdots \\
\hline
4z^{-3} + 3z^{-2} - 2z^{-1} + 1 & 2z^{-3} + z^{-2} + z^{-1} \\
 & 2z^{-3} + \frac{3}{2}z^{-2} - z^{-1} + \frac{1}{2} \\
\hline
 & -\frac{1}{2}z^{-2} + 2z^{-1} - \frac{1}{2} \\
 & -\frac{1}{2}z^{-2} - \frac{3}{8}z^{-1} + \frac{1}{4} - \frac{1}{8}z \\
\hline
 & \frac{19}{8}z^{-1} - \frac{3}{4} + \frac{1}{8}z \\
 & \frac{19}{8}z^{-1} + \frac{57}{32} - \frac{19}{16}z + \frac{19}{32}z^2 \\
\hline
 & -\frac{81}{32} + \frac{21}{16}z - \frac{19}{32}z^2 \\
 & -\frac{18}{32} - \frac{243}{128}z + \frac{81}{64}z^2 - \frac{81}{128}z^3 \\
\hline
 & \frac{411}{128}z - \frac{119}{69}z^2 + \frac{81}{128}z^3
\end{array}$$

$$\therefore \qquad X(z) = \frac{1}{2} - \frac{1}{8}z + \frac{19}{32}z^2 - \frac{81}{128}z^3 + \frac{411}{512}z^4 + \cdots$$

$$\therefore \qquad x(n) = \left\{ \ldots, \frac{411}{512}, -\frac{81}{128}, \frac{19}{32}, -\frac{1}{8}, \underset{\uparrow}{\frac{1}{2}} \right\}$$

Observe that both the methods give the same sequence $x(n)$.

We can say from the above examples that, this method does not give $x(n)$ in a closed form expression in terms of n, and hence, is useful only if one is interested in determining the first few terms of the sequence $x(n)$.

3.6.2 Partial Fraction Expansion Method

To find the inverse Z-transform of $X(z)$ using partial fraction expansion method, its denominator must be in factored form. It is similar to the partial fraction expansion method used earlier for the inversion of Laplace transforms. However, in this case, we try to obtain the partial fraction expansion of $X(z)/z$ instead of $X(z)$. This is because, the Z-transform of time domain signals have z in their numerators. This method can be applied only if $X(z)/z$ is a proper rational function (i.e. the order of its denominator is greater than the order of its numerator). If $X(z)/z$ is not proper, then it should be written as the sum of a polynomial and a proper function before applying this method. The disadvantage of this method is that, the denominator must be factored. Using known Z-transform pairs and the properties of Z-transform, the inverse Z-transform of each partial fraction can be found.

Consider a rational function $X(z)/z$ given by

$$\frac{X(z)}{z} = \frac{b_0 z^M + b_1 z^{M-1} + b_2 z^{M-2} + \cdots + b_M}{z^N + a_1 z^{N-1} + a_2 z^{N-2} + \cdots + a_N}$$

When $M < N$, it is a proper function.

When $M \geq N$, it is not a proper function, so write it as:

$$\frac{X(z)}{z} = \underbrace{c_0 z^{N-M} + c_1 z^{N-M-1} + \cdots + c_{N-M}}_{\text{polynomial}} + \underbrace{\frac{N_1(z)}{D(z)}}_{\text{Proper rational function}}$$

There are two cases for the proper rational function $X(z)/z$.

Case 1 $X(z)/z$ has all distinct poles.
When all the poles of $X(z)/z$ are distinct, then $X(z)/z$ can be expanded in the form

$$\frac{X(z)}{z} = \frac{C_1}{z - P_1} + \frac{C_2}{z - P_2} + \cdots + \frac{C_N}{z - P_N}$$

The coefficients $C_1, C_2, \ldots, C_N$ can be determined using the formula

$$C_k = (z - P_k)\left.\frac{X(z)}{z}\right|_{z=P_k}, \quad k = 1, 2, \ldots, N$$

Case 2 $X(z)/z$ has l-repeated poles and the remaining N-l poles are simple. Let us say the kth pole is repeated l times. Then, $X(z)/z$ can be written as:

$$\frac{X(z)}{z} = \underbrace{\frac{C_1}{z - P_1} + \frac{C_2}{z - P_2} + \cdots +}_{(N\text{-}l)\text{ terms}} \frac{C_{k1}}{z - P_k} + \frac{C_{k2}}{(z - P_k)^2} + \cdots + \frac{C_{kl}}{(z - P_k)^l}$$

where $$C_{kl} = (z - P_k)^l \left.\frac{X(z)}{z}\right|_{z=P_k}$$

In general, $$C_{ki} = \frac{1}{(l-i)!} \frac{d^{l-i}}{dz^{l-i}} \left[(z-P_k)^l \frac{X(z)}{z} \right] \Bigg|_{z=P_k}$$

If $X(z)$ has a complex pole, then the partial fraction can be expressed as:

$$\frac{X(z)}{z} = \frac{C_1}{z-P_1} + \frac{C_1^*}{z-P_1^*}$$

where C_1^* is complex conjugate of C_1 and P_1^* is complex conjugate of P_1.

In other words, complex conjugate poles result in complex conjugate coefficients in the partial fraction expansion.

EXAMPLE 3.31 Find the inverse Z-transform of

$$X(z) = \frac{z^{-1}}{3-4z^{-1}+z^{-2}}; \text{ ROC; } |z| > 1$$

Solution: Given $$X(z) = \frac{z^{-1}}{3-4z^{-1}+z^{-2}} = \frac{z}{3z^2-4z+1}$$

$$= \frac{z}{3[z^2-(4z/3)+(1/3)]} = \frac{1}{3} \frac{z}{(z-1)[z-(1/3)]}$$

$$\therefore \qquad \frac{X(z)}{z} = \frac{1}{3} \frac{1}{(z-1)[z-(1/3)]} = \frac{A}{z-1} + \frac{B}{z-(1/3)}$$

where A and B can be evaluated as follows:

$$A = (z-1)\frac{X(z)}{z}\bigg|_{z=1} = (z-1)\frac{1}{3}\frac{1}{(z-1)[z-(1/3)]}\bigg|_{z=1} = \frac{1}{3}\frac{1}{1-(1/3)} = \frac{1}{2}$$

$$B = \left(z-\frac{1}{3}\right)\frac{X(z)}{z}\bigg|_{z=1/3} = \left(z-\frac{1}{3}\right)\frac{1}{3}\frac{1}{(z-1)[z-(1/3)]}\bigg|_{z=1/3} = \frac{1}{3}\frac{1}{(1/3)-1} = -\frac{1}{2}$$

$$\therefore \qquad \frac{X(z)}{z} = \frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{z-(1/3)}$$

or $$X(z) = \frac{1}{2}\left[\frac{z}{z-1} - \frac{z}{z-(1/3)}\right]; \text{ ROC; } |z| > 1$$

Since ROC is $|z| > 1$, both the sequences must be causal. Therefore, taking inverse Z-transform, we have

$$x(n) = \frac{1}{2}\left[u(n) - \left(\frac{1}{3}\right)^n u(n)\right]; \text{ ROC; } |z| > 1$$

EXAMPLE 3.32 Find the inverse Z-transform of

$$X(z) = \frac{z(z-1)}{(z+1)^3\,(z+2)};\ \text{ROC; } |z| > 2$$

Solution: Given $$X(z) = \frac{z(z-1)}{(z+1)^3\,(z+2)};\ \text{ROC; } |z| > 2$$

$$\therefore \qquad \frac{X(z)}{z} = \frac{z-1}{(z+1)^3\,(z+2)} = \frac{C_1}{z+1} + \frac{C_2}{(z+1)^2} + \frac{C_3}{(z+1)^3} + \frac{C_4}{z+2}$$

where the constants C_1, C_2, C_3 and C_4 can be obtained as follows:

$$C_4 = (z+2)\frac{X(z)}{z}\bigg|_{z=-2} = \frac{z-1}{(z+1)^3}\bigg|_{z=-2} = \frac{-2-1}{(-2+1)^3} = 3$$

$$C_3 = (z+1)^3\,\frac{X(z)}{z}\bigg|_{z=-1} = \frac{z-1}{(z+2)}\bigg|_{z=-1} = \frac{-1-1}{-1+2} = -2$$

$$C_2 = \frac{1}{1!}\frac{d}{dz}\left[(z+1)^3\,\frac{X(z)}{z}\right]\bigg|_{z=-1} = \frac{d}{dz}\left(\frac{z-1}{z+2}\right)\bigg|_{z=-1} = \frac{(z+2)\,(1)-(z-1)\,(1)}{(z+2)^2}\bigg|_{z=-1} = 3$$

$$C_1 = \frac{1}{2!}\frac{d^2}{dz^2}\left[(z+1)^3\,\frac{X(z)}{z}\right]\bigg|_{z=-1} = \frac{1}{2!}\frac{d^2}{dz^2}\left(\frac{z-1}{z+2}\right)\bigg|_{z=-1}$$

$$= \frac{1}{2!}\frac{d}{dz}\left[\frac{3}{(z+2)^2}\right]\bigg|_{z=-1} = \frac{1}{2}\,\frac{-3\times 2(z+2)}{(z+2)^4}\bigg|_{z=-1} = \frac{-3(-1+2)}{(-1+2)^3} = -3$$

$$\therefore \qquad \frac{X(z)}{z} = \frac{-3}{z+1} + \frac{3}{(z+1)^2} - \frac{2}{(z+1)^3} + \frac{3}{z+2}$$

$$\therefore \qquad X(z) = \frac{-3z}{z+1} + \frac{3z}{(z+1)^2} - \frac{2z}{(z+1)^3} + \frac{3z}{z+2};\ \text{ROC; } |z| > 2$$

Since ROC is $|z| > 2$, all the above sequences must be causal. Taking inverse Z-transform on both sides, we have

$$x(n) = -3(-1)^n\,u(n) + 3n(-1)^n\,u(n) - 2(n)\,(n-1)\,(-1)^n\,u(n) + 3(-2)^n\,u(n)$$

$$= [-3 + 3n - 2n(n-1)]\,(-1)^n\,u(n) + 3(-2)^n\,u(n)$$

EXAMPLE 3.33 Determine all possible signals $x(n)$ associated with Z-transform.

$$X(z) = \frac{(1/4)z^{-1}}{[1-(1/2)z^{-1}][1-(1/4)z^{-1}]}$$

Solution: Given
$$X(z) = \frac{(1/4)z^{-1}}{[1-(1/2)z^{-1}][1-(1/4)z^{-1}]}$$

Multiplying the numerator and denominator with z^2, we obtain

$$X(z) = \frac{(1/4)z}{[z-(1/2)][z-(1/4)]}$$

Now, $X(z)$ has two poles, one at $z = (1/2)$ and the other at $z = 1/4$ as shown in Figure 3.7. The possible ROCs are:

(a) ROC; $|z| > \frac{1}{2}$ (b) ROC; $|z| < \frac{1}{4}$ (c) ROC; $\frac{1}{4} < |z| < \frac{1}{2}$

Hence there are three possible signals $x(n)$ corresponding to these ROCs.

Now, $$\frac{X(z)}{z} = \frac{1/4}{[z-(1/2)][z-(1/4)]} = \frac{C_1}{z-(1/2)} + \frac{C_2}{z-(1/4)} = \frac{1}{z-(1/2)} - \frac{1}{z-(1/4)}$$

or $$X(z) = \frac{z}{z-(1/2)} - \frac{z}{z-(1/4)}$$

(a) ROC; $|z| > \frac{1}{2}$

Here both the poles, i.e. $z = (1/2)$ and $z = (1/4)$ correspond to causal terms.

$\therefore$ $$x(n) = \left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{4}\right)^n u(n)$$

(b) ROC; $|z| < \frac{1}{4}$

Here both the poles must correspond to anticausal terms.

$\therefore$ $$x(n) = -\left(\frac{1}{2}\right)^n u(-n-1) + \left(\frac{1}{4}\right)^n u(-n-1)$$

(c) ROC; $\frac{1}{4} < |z| < \frac{1}{2}$

Here the pole at $z = (1/4)$ must correspond to causal term and the pole at $z = (1/2)$ must correspond to anticausal term.

$\therefore$ $$x(n) = -\left(\frac{1}{2}\right)^n u(-n-1) - \left(\frac{1}{4}\right)^n u(n)$$

The ROCs are shown in Figure 3.7.

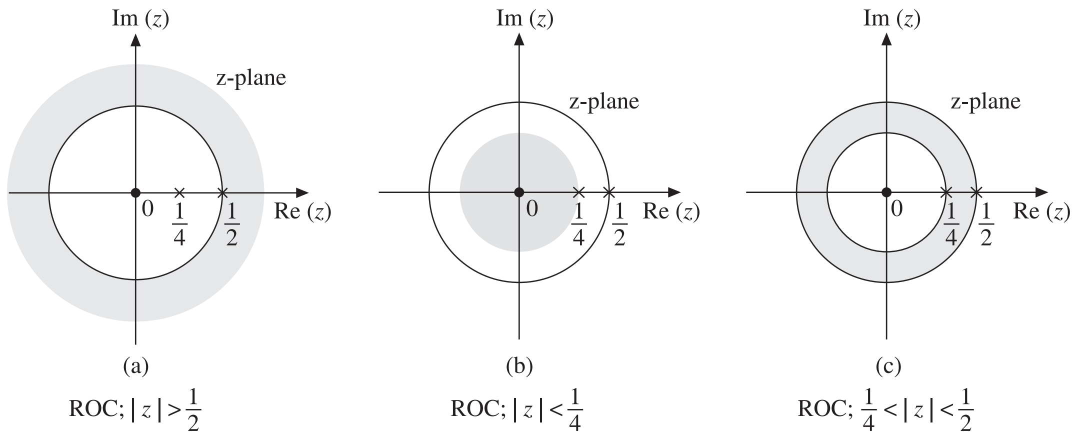


Figure 3.7 ROCs for Example 3.33.

EXAMPLE 3.34 Find all possible inverse Z-transforms of the following function:

$$X(z)=\frac{z(z^2-4z+5)}{z^3-6z^2+11z-6}$$

Solution: Given $X(z)=\dfrac{z(z^2-4z+5)}{z^3-6z^2+11z-6}=\dfrac{z(z^2-4z+5)}{(z-1)(z-2)(z-3)}$

The poles of $X(z)$ are at $z = 1$, $z = 2$ and $z = 3$. So the possible ROCs are:

(a) $|z| > 3$ (b) $|z| < 1$ (c) $2 < |z| < 3$ and (d) $1 < |z| < 2$

Using partial fraction method, we have

$$\frac{X(z)}{z}=\frac{z^2-4z+5}{(z-1)(z-2)(z-3)}=\frac{A}{z-1}+\frac{B}{z-2}+\frac{C}{z-3}=\frac{1}{z-1}-\frac{1}{z-2}+\frac{1}{z-3}$$

or $$X(z)=\frac{z}{z-1}-\frac{z}{z-2}+\frac{z}{z-3}$$

Therefore, the possible inverse Z-transforms are:

(a) $x(n)=u(n)-2^n u(n)+3^n u(n)$; ROC; $|z|>3$

(b) $x(n)=-u(-n-1)+2^n u(-n-1)-3^n u(-n-1)$; ROC; $|z|<1$

(c) $x(n)=u(n)-2^n u(n)-3^n u(-n-1)$; ROC; $2<|z|<3$

(d) $x(n)=u(n)+2^n u(-n-1)-3^n u(-n-1)$; ROC; $1<|z|<2$

EXAMPLE 3.35 Determine the causal signal $x(n)$ having Z-transform

$$X(z) = \frac{z^2 + z}{[z - (1/2)]^2 \, [z - (1/4)]}$$

Solution: Given $X(z) = \dfrac{z^2 + z}{[z - (1/2)]^2 \, [z - (1/4)]} = \dfrac{z(z+1)}{[z - (1/2)]^2 \, [z - (1/4)]}$

Taking partial fractions of $\dfrac{X(z)}{z}$, we have

$$\frac{X(z)}{z} = \frac{(z+1)}{[z - (1/2)]^2 \, [z - (1/4)]} = \frac{A}{[z - (1/2)]^2} + \frac{B}{[z - (1/2)]} + \frac{C}{[z - (1/4)]}$$

$$= \frac{6}{[z - (1/2)]^2} - \frac{20}{[z - (1/2)]} + \frac{20}{[z - (1/4)]}$$

$$\therefore \qquad X(z) = 6\frac{z}{[z - (1/2)]^2} - 20\frac{z}{[z - (1/2)]} + 20\frac{z}{[z - (1/4)]}$$

Taking inverse Z-transform on both sides, we have the causal signal

$$x(n) = 6n\left(\frac{1}{2}\right)^{n-1} u(n) - 20\left(\frac{1}{2}\right)^{n} u(n) + 20\left(\frac{1}{4}\right)^{n} u(n)$$

3.6.3 Residue Method

The inverse Z-transform of $X(z)$ can be obtained using the equation:

$$x(n) = \frac{1}{2\pi j} \oint_c X(z)\, z^{n-1}\, dz$$

where c is a circle in the z-plane in the ROC of $X(z)$. The above equation can be evaluated by finding the sum of all residues of the poles that are inside the circle c. Therefore,

$$x(n) = \sum \text{Residues of } X(z) z^{n-1} \text{ at the poles inside } c$$

$$= \sum_i (z - z_i)\, X(z)\, z^{n-1} \Big|_{z = z_i}$$

If $X(z)\, z^{n-1}$ has no poles inside the contour c for one or more values of n, then $x(n) = 0$ for these values.

EXAMPLE 3.36 Using residue method, find the inverse Z-transform of

$$X(z) = \frac{1+2z^{-1}}{1+4z^{-1}+3z^{-2}}; \text{ ROC; } |z| > 3$$

Solution: Given $X(z) = \dfrac{1+2z^{-1}}{1+4z^{-1}+3z^{-2}} = \dfrac{z(z+2)}{z^2+4z+3} = \dfrac{z(z+2)}{(z+1)(z+3)}$

$$\therefore \quad x(n) = \sum \text{Residues of } X(z)\,z^{n-1} \text{ at the poles of } X(z)\,z^{n-1} \text{ within } c$$

$$= \sum \text{Residues of } \frac{z(z+2)\,z^{n-1}}{(z+1)\,(z+3)} = \frac{z^n(z+2)}{(z+1)(z+3)} \text{ at the poles of same within } c$$

$$\therefore \quad x(n) = \sum \text{Residues of } \frac{z^n(z+2)}{(z+1)\,(z+3)} \text{ at poles } z = -1 \text{ and } z = -3$$

$$= (z+1)\frac{z^n(z+2)}{(z+1)\,(z+3)}\bigg|_{z=-1} + \frac{(z+3)\,z^n(z+2)}{(z+1)\,(z+3)}\bigg|_{z=-3}$$

$$= \frac{1}{2}(-1)^n\,u(n) + \frac{1}{2}(-3)^n\,u(n)$$

EXAMPLE 3.37 Determine the inverse Z-transform using the complex integral

$$X(z) = \frac{3z^{-1}}{[1-(1/2)\,z^{-1}]^2}; \text{ ROC; } |z| > \frac{1}{4}$$

Solution: We know that the inverse Z-transform of $X(z)$ can be obtained using the equation:

$$x(n) = \frac{1}{2\pi j}\oint_c X(z)\,z^{n-1}\,dz$$

at the poles inside c where c is a circle in the z-plane in the ROC of $X(z)$.

This can be evaluated by finding the sum of all residues of the poles that are inside the circle c. Therefore, the above equation can be written as:

$$x(n) = \sum \text{Residues of } X(z)\,z^{n-1} \text{ at the poles inside } c$$

$$= \sum_i (z-z_i)\,X(z)\,z^{n-1}\Big|_{z=z_i}$$

If there is a pole of multiplicity k, then the residue at that pole is:

$$\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}[(z-z_i)^k\;X(z)\,z^{n-1}] \text{ at the pole } z = z_i$$

Given $$X(z)=\frac{3z^{-1}}{[1-(1/2)z^{-1}]^2}=\frac{3z}{[z-(1/2)]^2}$$

The given $X(z)$ has a pole of order 2 at z = 1/2.

$$x(n)=\sum \text{Residues of } X(z)z^{n-1} \text{ at its poles}$$

$$=\sum \text{Residue of } 3z^n/[z-(1/2)]^2 \text{ at the pole } z=(1/2) \text{ of multiplicity 2}$$

$$\therefore \qquad x(n)=\frac{1}{1!}\frac{d}{dz}\left[\left(z-\frac{1}{2}\right)^2\frac{3z^n}{[z-(1/2)]^2}\right]\Bigg|_{z=1/2}=3nz^{n-1}\Big|_{z=1/2}=3n\left(\frac{1}{2}\right)^{n-1}u(n)$$

EXAMPLE 3.38 Find all possible inverse Z-transforms of $X(z)=\frac{z}{(z+1)^2(z+2)^3}$ using contour integration (residue) method.

Solution: Given $$X(z)=\frac{z}{(z+1)^2(z+2)^3}$$

$X(z)$ has two poles, one of order 2 at z = –1 and the second one of order 3 at z = –2. So there are three possible inverse Z-transforms:

(a) with ROC; $|z| > 2$
(b) with ROC; $|z| < 1$
(c) with ROC; $1 < |z| < 2$

We know that $x(n)$ is given by the sum of the residues of $X(z)\, z^{n-1}$ at the poles of $X(z)$.

Residue of $X(z)z^{n-1}=\frac{z^n}{(z+1)^2(z+2)^3}$ at the pole z = –1 of order 2 is:

$$\frac{1}{1!}\frac{d}{dz}\left[(z+1)^2\frac{z^n}{(z+1)^2(z+2)^3}\right]\Bigg|_{z=-1}=\frac{d}{dz}\left[\frac{z^n}{(z+2)^3}\right]\Bigg|_{z=-1}$$

$$=\frac{(z+2)^3(nz^{n-1})-z^n3(z+2)^2}{(z+2)^6}\Bigg|_{z=-1}$$

$$=-(n+3)(-1)^n$$

Residue of $X(z)z^{n-1}=\frac{z^n}{(z+1)^2(z+2)^3}$ at the pole z = –2 of order 3 is:

$$\frac{1}{2!}\frac{d^2}{dz^2}\left[(z+2)^3\frac{z^n}{(z+1)^2(z+2)^3}\right]\Bigg|_{z=-2}=\frac{1}{2}\frac{d^2}{dz^2}\left[\frac{z^n}{(z+1)^2}\right]\Bigg|_{z=-2}$$

$$=(-2)^n[0.125n^2-1.125n+3]$$

(a) When ROC; $|z| > 2$, both the residues must be positive.

$$\therefore \quad x(n) = -(n+3)(-1)^n u(n) + (0.125n^2 - 1.125n + 3)(-2)^n u(n)$$

(b) When ROC; $|z| < 1$, both the residues must be negative.

$$\therefore \; x(n) = (n+3)(-1)^n u(-n-1) - (0.125n^2 - 1.125n + 3)(-2)^n u(-n-1)$$

(c) When ROC; $1 \le |z| \le 2$, residue at pole $z = -1$ is positive and the residue at pole at $z = -2$ is negative.

$$\therefore \quad x(n) = -(n+3)(-1)^n u(n) - (0.125n^2 - 1.125n + 3)(-2)^n u(-n-1)$$

3.6.4 Convolution Method

The inverse Z-transform can also be determined using convolution method. In this method, the given $X(z)$ is splitted into $X_1(z)$ and $X_2(z)$ such that $X(z) = X_1(z)\, X_2(z)$. Then, $x_1(n)$ and $x_2(n)$ are obtained by taking the inverse Z-transform of $X_1(z)$ and $X_2(z)$ respectively. Then, $x(n)$ is obtained by performing convolution of $x_1(n)$ and $x_2(n)$ in time domain.

$$\mathrm{Z}\,[x_1(n) * x_2(n)] = X_1(z)\, X_2(z) = X(z)$$

$$\therefore \qquad x(n) = \mathrm{Z}^{-1}[X(z)] = \mathrm{Z}^{-1}[\mathrm{Z}\{x_1(n) * x_2(n)\}] = x_1(n) * x_2(n) = \sum_{k=0}^{n} x_1(k)\, x_2(n-k)$$

EXAMPLE 3.39 Explain how the analysis of discrete time-invariant system can be obtained using convolution properties of Z-transform.

Solution: One of the most important properties of Z-transform used in the analysis of discrete-time systems is convolution property. According to this property, the Z-transform of the convolution of two signals is equal to the multiplication of their Z-transforms. The analysis of a discrete-time system means the input to the system $x(n)$ and its impulse response $h(n)$ are known and we have to determine the output $y(n)$ of the system. If the input sequence to the LTI system is $x(n)$ and the impulse response is $h(n)$, then first determine the Z-transforms of $x(n)$ and $h(n)$.

Let Z $[x(n)] = X(z)$ and Z $[h(n)] = H(z)$, then obtain the product of these Z-transforms, i.e. $Y(z) = H(z)\, X(z)$. We know that the output $y(n) = x(n) * h(n)$.

As per the linear convolution property,

$$Y(z) = \mathrm{Z}[x(n) * h(n)] = \mathrm{Z}[x(n)]\, \mathrm{Z}[h(n)] = X(z)\, H(z)$$

So having obtained $Y(z) = X(z)\, H(z)$, take the inverse Z-transform of $X(z)\, H(z)$. This gives $y(n)$ which is the response of the system for the input $x(n)$.

EXAMPLE 3.40 Find the inverse Z-transform of $X(z) = \dfrac{z^2}{(z-2)(z-3)}$ using convolution property of Z-transforms.

Solution: Given
$$X(z) = \frac{z^2}{(z-2)(z-3)}$$

Let
$$X(z) = X_1(z) X_2(z) = \frac{z}{z-2}\,\frac{z}{z-3}$$

$\therefore$
$$x_1(n) = Z^{-1}[X_1(z)] = Z^{-1}\left(\frac{z}{z-2}\right) = 2^n u(n)$$

$$x_2(n) = Z^{-1}[X_2(z)] = Z^{-1}\left(\frac{z}{z-3}\right) = 3^n u(n)$$

$\therefore$
$$x_1(n) * x_2(n) = \sum_{k=0}^{n} x_1(k)\, x_2(n-k)$$
$$= \sum_{k=0}^{n} 2^k u(k)\, 3^{n-k} u(n-k)$$
$$= 3^n \sum_{k=0}^{n} \left(\frac{2}{3}\right)^k = 3^n \left[\frac{1-(2/3)^{n+1}}{1-(2/3)}\right]$$
$$= 3^{n+1}\left[1-\left(\frac{2}{3}\right)^{n+1}\right] = 3^{n+1} u(n) - 2^{n+1} u(n)$$

EXAMPLE 3.41 Find the inverse Z-transform of $X(z) = \dfrac{z}{(z-1)\,[z-(1/2)]}$ using convolution property of Z-transforms.

Solution: Given
$$X(z) = \frac{z}{(z-1)\,[z-(1/2)]}$$

Let
$$X(z) = X_1(z)\, X_2(z) = \frac{z}{(z-1)}\,\frac{1}{[z-(1/2)]}$$

$\therefore$
$$x_1(n) = Z^{-1}[X_1(z)] = Z^{-1}\left(\frac{z}{z-1}\right) = u(n)$$

and
$$x_2(n) = Z^{-1}[X_2(z)] = Z^{-1}\left[\frac{1}{z-(1/2)}\right] = \left(\frac{1}{2}\right)^{n-1} u(n-1)$$

$\therefore$
$$x_1(n) * x_2(n) = \sum_{k=0}^{n} x_1(k) x_2(n-k)$$

$$= \sum_{k=0}^{n-1} u(k)\left(\frac{1}{2}\right)^{n-1-k} u(n-1-k)$$

$$= \left(\frac{1}{2}\right)^{n-1}\left[\sum_{k=0}^{n-1}\left(\frac{1}{2}\right)^{-k}\right] = \left(\frac{1}{2}\right)^{n-1}\sum_{k=0}^{n-1}\left[\left(\frac{1}{2}\right)^{-1}\right]^{k}$$

$$= \left(\frac{1}{2}\right)^{n-1}\left\{\frac{1-[(1/2)^{-1}]^{n}}{1-(1/2)^{-1}}\right\} = \left(\frac{1}{2}\right)^{n-1}\left[\frac{1-(1/2)^{-n}}{-1}\right]$$

$$= \left(\frac{1}{2}\right)^{-1} - \left(\frac{1}{2}\right)^{n}\left(\frac{1}{2}\right)^{-1} = 2u(n) - 2\left(\frac{1}{2}\right)^{n} u(n)$$

3.7 TRANSFORM ANALYSIS OF LTI SYSTEMS

The Z-transform plays an important role in the analysis and design of discrete-time LTI systems.

3.7.1 System Function and Impulse Response

Consider a discrete-time LTI system having an impulse response $h(n)$ as shown in Figure 3.8.

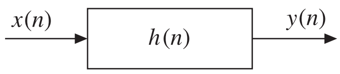


Figure 3.8 Discrete-time LTI system.

Let us say it gives an output $y(n)$ for an input $x(n)$. Then, we have

$$y(n) = x(n) * h(n)$$

Taking Z-transform on both sides, we get

$$Y(z) = X(z)\,H(z)$$

where

$Y(z)$ = Z-transform of the output $y(n)$
$X(z)$ = Z-transform of the input $x(n)$
$H(z)$ = Z-transform of the impulse response $h(n)$

$$\therefore \qquad H(z) = \frac{Y(z)}{X(z)}$$

$H(z)$ is called the *system function* or the *transfer function* of the LTI discrete system and is defined as:

The ratio of the Z-transform of the output sequence $y(n)$ to the Z-transform of the input sequence $x(n)$ when the initial conditions are neglected.

If the input $x(n)$ is an impulse sequence, then $X(z) = 1$. So $Y(z) = H(z)$. So the transfer function is also defined as the Z-transform of the impulse response of the system.

The poles and zeros of the system function offer an insight into the system characteristics. The poles of the system are defined as the values of z for which the system function $H(z)$ is infinity and the zeros of the system are the values of z for which the system function $H(z)$ is zero.

3.7.2 Relationship between Transfer Function and Difference Equation

In terms of a difference equation, an nth order discrete-time LTI system is specified as:

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

Expanding it, we have

$$a_0 y(n) + a_1 y(n-1) + a_2 y(n-2) + \cdots + a_N y(n-N) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \cdots + b_M x(n-M)$$

Taking Z-transform on both sides and neglecting the initial conditions, we obtain

$$a_0 Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + \cdots + a_N z^{-N} Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + \cdots + b_M z^{-M} X(z)$$

i.e. $$Y(z)\,[a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}] = X(z)[b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}]$$

$\therefore$ $$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Now, $Y(z)/X(z) = H(z)$ is called the transfer function of the system or the system function. The frequency response of a system is obtained by substituting $z = e^{j\omega}$ in $H(z)$.

3.8 STABILITY AND CAUSALITY

We know that the necessary and sufficient condition for a causal linear time-invariant discrete-time system to be BIBO stable is:

$$\sum_{n=0}^{\infty} |h(n)| < \infty$$

i.e. an LTI discrete-time system is BIBO stable if its impulse response is absolutely summable.

We also know that for a system to be causal, its impulse response must be equal to zero for $n < 0$ [i.e. $h(n) = 0$ for $n < 0$]. Alternately, if the system is causal, then the ROC for $H(z)$ will be outside the outermost pole.

For a causal LTI system to be stable, all the poles of $H(z)$ must lie inside the unit circle in the z-plane, i.e. for a causal LTI system to be stable, the ROC of the system function must include the unit circle.

EXAMPLE 3.42 Consider an LTI system with a system function $H(z) = \dfrac{1}{1-(1/2)z^{-1}}$. Find the difference equation. Determine the stability.

Solution: Given
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-(1/2)z^{-1}} = \frac{z}{z-(1/2)}$$

That is
$$Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z)$$

Taking inverse Z-transform on both sides (applying the time shifting property), we get the difference equation
$$y(n) - \frac{1}{2}y(n-1) = x(n)$$

The only pole of $H(z)$ is at $z = 1/2$, i.e., inside the unit cirle. So the system is stable.

EXAMPLE 3.43 A causal system is represented by
$$H(z) = \frac{z+2}{2z^2-3z+4}$$

Find the difference equation and the frequency response of the system.

Solution: Given
$$H(z) = \frac{z+2}{2z^2-3z+4}$$

As the system is causal, $H(z)$ is expressed in negative powers of z.

$\therefore$
$$H(z) = \frac{Y(z)}{X(z)}$$
$$= \frac{z+2}{2z^2-3z+4} = \frac{z^{-1}+2z^{-2}}{2-3z^{-1}+4z^{-2}}$$

i.e. $$2Y(z) - 3z^{-1}Y(z) + 4z^{-2}Y(z) = z^{-1}X(z) + 2z^{-2}X(z)$$

Taking inverse Z-transform on both sides, we have

$$2y(n) - 3y(n-1) + 4y(n-2) = x(n-1) + 2x(n-2)$$

which is the required difference equation.

Putting $z = e^{j\omega}$ in $H(z)$, we get the frequency response $H(\omega)$ of the system.

$$H(\omega) = \left.\frac{z+2}{2z^2 - 3z + 4}\right|_{z=e^{j\omega}} = \frac{e^{j\omega}+2}{2e^{j2\omega} - 3e^{j\omega} + 4}$$

$$= \frac{2 + \cos\omega + j\sin\omega}{4 + (2\cos 2\omega - 3\cos\omega) + j(2\sin 2\omega - 3\sin\omega)}$$

EXAMPLE 3.44 Determine the system function of a discrete-time system described by the difference equation

$$y(n) - \frac{1}{3}y(n-1) + \frac{1}{5}y(n-2) = x(n) - 2x(n-1)$$

Solution: Taking Z-transform on both sides of the given difference equation, we get

$$Y(z) - \frac{1}{3}z^{-1}Y(z) + \frac{1}{5}z^{-2}Y(z) = X(z) - 2z^{-1}X(z)$$

Hence the system function or transfer function of the given system is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - 2z^{-1}}{1 - (1/3)z^{-1} + (1/5)z^{-2}} = \frac{z(z-2)}{z^2 - (1/3)z + (1/5)}$$

EXAMPLE 3.45 Plot the pole-zero pattern and determine which of the following systems are stable:

(a) $y(n) = y(n-1) - 0.8y(n-2) + x(n) + x(n-2)$

(b) $y(n) = 2y(n-1) - 0.8y(n-2) + x(n) + 0.8x(n-1)$

Solution:

(a) Given $$y(n) = y(n-1) - 0.8y(n-2) + x(n) + x(n-2)$$

Taking Z-transform on both sides and neglecting the initial conditions, we have

$$Y(z) = z^{-1}Y(z) - 0.8z^{-2}Y(z) + X(z) + z^{-2}X(z)$$

i.e. $$Y(z)\,[1 - z^{-1} + 0.8z^{-2}] = X(z)(1 + z^{-2})$$

The transfer function of the system is:

$$\frac{Y(z)}{X(z)} = H(z)$$

$$= \frac{1+z^{-2}}{1-z^{-1}+0.8z^{-2}} = \frac{z^2+1}{z^2-z+0.8}$$

$$= \frac{(z+j)(z-j)}{(z-0.5-j0.74)(z-0.50+j0.74)}$$

The zeros of $H(z)$ are $z = +j1$ and $z = -j1$.
The poles of $H(z)$ are $z = 0.5 - j0.74$ and $z = 0.5 + j0.74$
The pole-zero plot is shown in Figure 3.9(a).
All the poles are inside the unit circle. Hence, the system is stable.

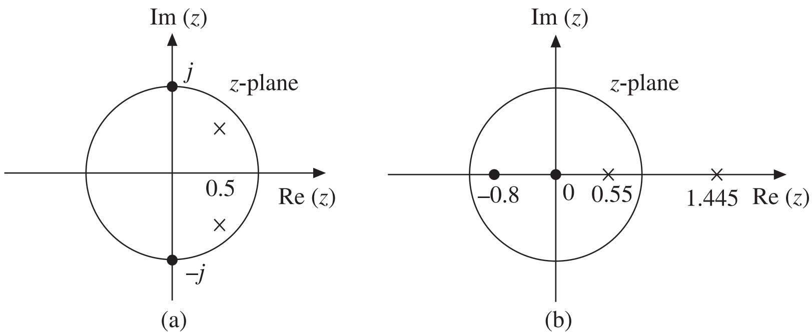


Figure 3.9 Pole-zero plots for Example 3.45.

(b) Given $\quad y(n) = 2y(n-1) - 0.8y(n-2) + x(n) + 0.8x(n-1)$

Taking Z-transform on both sides and neglecting the initial conditions, we have

$$Y(z) = 2z^{-1}Y(z) - 0.8z^{-2}Y(z) + X(z) + 0.8z^{-1}X(z)$$

i.e. $$Y(z)[1-2z^{-1}+0.8z^{-2}] = X(z)[1+0.8z^{-1}]$$

The transfer function of the system is:

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1+0.8z^{-1}}{1-2z^{-1}+0.8z^{-2}} = \frac{z(z+0.8)}{z^2-2z+0.8}$$

$$= \frac{z(z+0.8)}{(z-1.445)(z-0.555)}$$

The zeros of $H(z)$ are $z = 0$ and $z = -0.8$.
The poles of $H(z)$ are $z = 1.445$ and $z = 0.555$.
The pole-zero plot is shown in Figure 3.9(b).
One pole is outside the unit circle. Therefore, the system is unstable.

EXAMPLE 3.46 A causal system has input $x(n)$ and output $y(n)$. Find the system function, frequency response and impulse response of the system if

$$x(n) = \delta(n) + \frac{1}{6}\delta(n-1) - \frac{1}{6}\delta(n-2)$$

and
$$y(n) = \delta(n) - \frac{2}{3}\delta(n-1)$$

Also assess the stability.

Solution: Given
$$x(n) = \delta(n) + \frac{1}{6}\delta(n-1) - \frac{1}{6}\delta(n-2)$$

and
$$y(n) = \delta(n) - \frac{2}{3}\delta(n-1)$$

Taking Z-transform of the above equations, we get

$$X(z) = 1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}$$

and
$$Y(z) = 1 - \frac{2}{3}z^{-1}$$

The system function or the transfer function of the system is:

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1-(2/3)\,z^{-1}}{1+(1/6)\,z^{-1}-(1/6)\,z^{-2}} = \frac{z[z-(2/3)]}{[z-(1/3)]\,[z+(1/2)]}$$

The frequency response of the system is:

$$H(\omega) = \left.\frac{z[z-(2/3)]}{[z-(1/3)]\,[z+(1/2)]}\right|_{z=e^{j\omega}} = \frac{e^{j\omega}\,[e^{j\omega}-(2/3)]}{[e^{j\omega}-(1/3)]\,[e^{j\omega}+(1/2)]}$$

By partial fraction expansion, we have

$$\frac{H(z)}{z} = \frac{[z-(2/3)]}{[z-(1/3)][z+(1/2)]} = \frac{A}{z-(1/3)} + \frac{B}{z+(1/2)} = \frac{-2/5}{z-(1/3)} + \frac{7/5}{z+(1/2)}$$

$\therefore$
$$H(z) = -\frac{2}{5}\left[\frac{z}{z-(1/3)}\right] + \frac{7}{5}\left[\frac{z}{z+(1/2)}\right]$$

Taking inverse Z-transform on both sides, we get the impulse response as:

$$h(n) = -\frac{2}{5}\left(\frac{1}{3}\right)^n u(n) + \frac{7}{5}\left(-\frac{1}{2}\right)^n u(n)$$

Both the poles of $H(z)$ are inside the unit circle. So the system is stable.

EXAMPLE 3.47 We want to design a causal discrete-time LTI system with the property that if the input is $x(n)=\left(\frac{1}{3}\right)^n u(n)-\frac{1}{5}\left(\frac{1}{3}\right)^{n-1} u(n-1)$, then the output is $y(n)=\left(\frac{1}{2}\right)^n u(n)$. Determine the transfer function $H(z)$, the impulse response $h(n)$ and frequency response $H(\omega)$ of the system that satisfies this condition. Also assess the stability.

Solution: Given
$$x(n)=\left(\frac{1}{3}\right)^n u(n)-\frac{1}{5}\left(\frac{1}{3}\right)^{n-1} u(n-1)$$

and
$$y(n)=\left(\frac{1}{2}\right)^n u(n)$$

We want to design a causal system.

$$\therefore \qquad h(n) = 0 \quad \text{for } n < 0$$

Taking Z-transform on both sides of the above equations, we have

$$X(z)=\frac{z}{z-(1/3)}-\frac{1}{5}z^{-1}\frac{z}{z-(1/3)}=\frac{z}{z-(1/3)}-\frac{1}{5}\frac{1}{z-(1/3)}=\frac{z-(1/5)}{z-(1/3)}$$

and
$$Y(z)=\frac{z}{z-(1/2)}$$

The system function $H(z)$ is:

$$H(z)=\frac{Y(z)}{X(z)}$$

$$=\frac{z}{z-(1/2)}\div\frac{z-(1/5)}{z-(1/3)}=\frac{z[z-(1/3)]}{[z-(1/2)][z-(1/5)]}$$

By partial fraction expansion of $H(z)/z$, we get

$$\therefore \qquad \frac{H(z)}{z}=\frac{z-(1/3)}{[z-(1/2)][z-(1/5)]}=\frac{A}{z-(1/2)}+\frac{B}{z-(1/5)}=\frac{5/9}{z-(1/2)}+\frac{4/9}{z-(1/5)}$$

$$\therefore \qquad H(z)=\frac{5}{9}\left[\frac{z}{z-(1/2)}\right]+\frac{4}{9}\left[\frac{z}{z-(1/5)}\right]$$

Taking inverse Z-transform of $H(z)$, we get the impulse response $h(n)$ as:

$$h(n)=\frac{5}{9}\left(\frac{1}{2}\right)^n u(n)+\frac{4}{9}\left(\frac{1}{5}\right)^n u(n)$$

The frequency response is:

$$H(\omega) = \left.\frac{z[z-(1/3)]}{[z-(1/2)][z-(1/5)]}\right|_{z=e^{j\omega}} = \frac{e^{j\omega}[e^{j\omega}-(1/3)]}{[e^{j\omega}-(1/2)][e^{j\omega}-(1/5)]}$$

Both the poles of $H(z)$ are inside the unit circle. So the system is stable.

EXAMPLE 3.48 A causal LTI system is described by the difference equation

$$y(n) = y(n-1) + y(n-2) + x(n) + 2x(n-1)$$

Find the system function and frequency response of the system. Plot the poles and zeros and indicate the ROC. Also determine the stability and impulse response of the system.

Solution: The given difference equation is:

$$y(n) = y(n-1) + y(n-2) + x(n) + 2x(n-1)$$

Taking Z-transform on both sides, we have

$$Y(z) = z^{-1}Y(z) + z^{-2}Y(z) + X(z) + 2z^{-1}X(z)$$

i.e.

$$Y(z)(1 - z^{-1} - z^{-2}) = X(z)(1 + 2z^{-1})$$

The system function is:

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1+2z^{-1}}{1-z^{-1}-z^{-2}} = \frac{z(z+2)}{z^2-z-1} = \frac{z(z+2)}{(z-1.62)(z+0.62)}$$

The frequency response of the system is:

$$H(\omega) = \left.\frac{z(z+2)}{(z-1.62)(z+0.62)}\right|_{z=e^{j\omega}} = \frac{e^{j\omega}(e^{j\omega}+2)}{(e^{j\omega}-1.62)(e^{j\omega}+0.62)}$$

$H(z)$ has the zeros at $z = 0$ and $z = -2$.

$H(z)$ has the poles at $z = 1.62$ and $z = -0.62$. One of the pole is outside the unit circle. So the system is unstable. The poles and zeros and the ROC are shown in Figure 3.10.

To find the impulse response $h(n)$, partial fraction expansion of $H(z)/z$ gives

$$\frac{H(z)}{z} = \frac{z+2}{(z-1.62)(z+0.62)} = \frac{A}{z-1.62} + \frac{B}{z+0.62} = \frac{1.62}{z-1.62} - \frac{0.62}{z+0.62}$$

$$\therefore \qquad H(z) = 1.62\left(\frac{z}{z-1.62}\right) - 0.62\left(\frac{z}{z+0.62}\right)$$

Taking inverse Z-transform, the impulse response is:

$$h(n) = 1.62(1.62)^n u(n) - 0.62(-0.62)^n u(n)$$

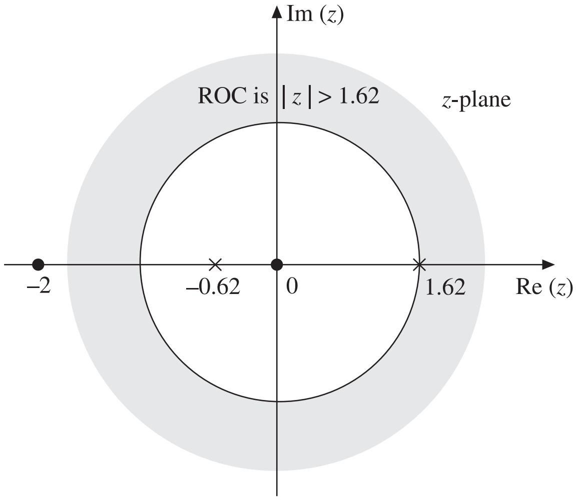


Figure 3.10 Pole-zero plot and ROC for Example 3.48.

EXAMPLE 3.49 Determine whether the following systems are both causal and stable.

(a) $H(z) = \dfrac{3 + z^{-1}}{1 + z^{-1} - (4/9)\,z^{-2}}$ (b) $H(z) = \dfrac{1 + 2z^{-1}}{1 + (6/5)\,z^{-1} + (9/25)\,z^{-2}}$

Solution:

(a) Given $H(z) = \dfrac{3 + z^{-1}}{1 + z^{-1} - (4/9)z^{-2}} = \dfrac{z(z+3)}{z^2 + z - (4/9)} = \dfrac{z(z+3)}{[z - (1/3)]\,[z + (4/3)]}$

The poles of $H(z)$ are: $z = 1/3$ and $z = -4/3$.
For a causal system to be stable, the ROC must include the unit circle. Now, for a causal $H(z)$, the ROC is $|z| > 4/3$. Since one pole is lying outside the unit circle, the given system is not both causal and stable.

(b) Given $H(z) = \dfrac{1 + 2z^{-1}}{1 + (6/5)\,z^{-1} + (9/25)\,z^{-2}} = \dfrac{z\,(z+2)}{z^2 + (6/5)z + (9/25)} = \dfrac{z(z+2)}{[z + (3/5)]^2}$

The location of the poles is at $z = -3/5$.
Since all the poles are lying inside the unit circle in the z-plane, the system is both causal and stable.

EXAMPLE 3.50 An LTI system is described by the difference equation.

$$y(n) - \frac{9}{4}y(n-1) + \frac{1}{2}y(n-2) = x(n) - 3x(n-1)$$

Specify the ROC of $H(z)$, and determine $h(n)$ for the following conditions:

(a) The system is stable.
(b) The system is causal.

Solution: Given $\quad y(n) - \frac{9}{4}y(n-1) + \frac{1}{2}y(n-2) = x(n) - 3x(n-1)$

Taking Z-transform on both sides, we have

$$Y(z) - \frac{9}{4}z^{-1}Y(z) + \frac{1}{2}z^{-2}Y(z) = X(z) - 3z^{-1}X(z)$$

i.e.
$$Y(z)\left(1 - \frac{9}{4}z^{-1} + \frac{1}{2}z^{-2}\right) = X(z)\,(1 - 3z^{-1})$$

The system function is:

$$\frac{Y(z)}{X(z)} = H(z)$$

$$= \frac{1 - 3z^{-1}}{1 - (9/4)z^{-1} + (1/2)z^{-2}} = \frac{z(z-3)}{z^2 - (9/4)z + (1/2)} = \frac{z(z-3)}{[z - (1/4)]\,(z-2)}$$

By partial fraction expansion of $H(z)/z$, we have

$$\frac{H(z)}{z} = \frac{(z-3)}{[z-(1/4)]\,(z-2)} = \frac{A}{z-(1/4)} + \frac{B}{z-2} = \frac{11}{7}\frac{z}{z-(1/4)} - \frac{4}{7}\frac{z}{z-2}$$

The system function has poles at $z = 1/4$ and $z = 2$.

(a) For the system to be stable, its ROC must include the unit circle. Therefore, ROC is $\frac{1}{4} < |z| < 2.$

Taking the inverse Z-transform of $H(z)$, we have the impulse response as:

$$h(n) = \frac{11}{7}\left(\frac{1}{4}\right)^n u(n) + \frac{4}{7}(2)^n u(-n-1)$$

Here the system is stable but non-causal.

(b) For the system to be causal, the ROC should be outside the outermost pole.

$\therefore \qquad$ ROC is $|z| > 2$

$$\therefore \qquad h(n) = \frac{11}{7}\left(\frac{1}{4}\right)^n u(n) - \frac{4}{7}(2)^n u(n)$$

Here the system is causal, but unstable.

EXAMPLE 3.51 An LTI system is described by the equation

$$y(n) = x(n) + 0.81x(n-1) - 0.81x(n-2) - 0.45y(n-2)$$

Determine the transfer function of the system. Sketch the poles and zeros on the z-plane. Assess the stability.

Solution: The given difference equation is:

$$y(n) = x(n) + 0.81x(n-1) - 0.81x(n-2) - 0.45y(n-2)$$

Taking Z-transform on both sides and neglecting initial conditions, we have

$$Y(z) = X(z) + 0.81z^{-1}X(z) - 0.81z^{-2}X(z) - 0.45z^{-2}Y(z)$$

i.e.
$$Y(z)[1 + 0.45z^{-2}] = X(z)[1 + 0.81z^{-1} - 0.81z^{-2}]$$

The transfer function of the system $H(z)$ is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 0.81z^{-1} - 0.81z^{-2}}{1 + 0.45z^{-2}} = \frac{z^2 + 0.81z - 0.81}{z^2 + 0.45}$$

Equating the denominator to zero, we have $z^2 + 0.45 = 0$, i.e. $z^2 = -0.45$

$$\therefore \qquad z = \pm j\sqrt{0.45} = \pm j0.67$$

So the poles are $z = +j0.67$ and $z = -j0.67$.

Equating the numerator to zero, we have

$$z^2 + 0.81z - 0.81 = 0$$

$$\therefore \qquad z = \frac{-0.81 \pm \sqrt{0.656 + 3.24}}{2}$$

$$= \frac{-0.81 \pm \sqrt{3.896}}{2} = -0.81 \pm 1.974 = -1.39, 0.582$$

The zeros are at $z = -1.39$ and $z = 0.58$.

$$\therefore \qquad \frac{Y(z)}{X(z)} = \frac{(z + 1.39)(z - 0.582)}{(z - j0.67)(z + j0.67)}$$

The pole-zero plot is shown in Figure 3.11.

The system is stable as both the poles are inside the unit circle.

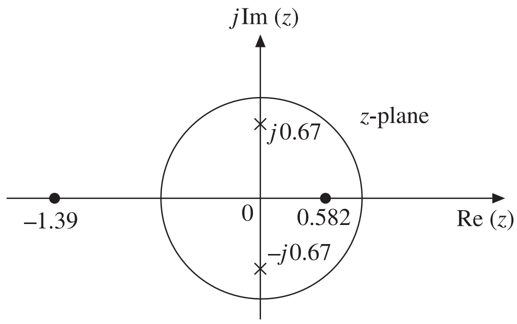


Figure 3.11 Pole-zero plot for Example 3.51.

EXAMPLE 3.52 Define stable and unstable systems. Test the condition for stability of the first order infinite impulse response (IIR) filter governed by the equation

$$y(n) = x(n) + bx(n - 1)$$

Solution: A stable system is a system for which the impulse response is absolutely summable. For a stable system all the poles of the system function must lie inside the unit circle centered at the origin of the z-plane. An unstable system is a system for which the impulse response is not absolutely summable. For an unstable system one or more poles of the system will lie either on the unit circle or outside the unit circle.

Given difference equation is:

$$y(n) = x(n) + bx(n - 1)$$

Taking Z-transform on both sides, we have

$$Y(z) = X(z) + bz^{-1} X(z)$$

$$= X(z)[1 + bz^{-1}]$$

The transfer function of the system $H(z)$ is:

$$H(z) = \frac{Y(z)}{X(z)}$$

$$= 1 + bz^{-1} = \frac{z + b}{z}$$

The system has a zero at $z = -b$ and a pole at $z = 0$. So the system is stable for all values of b.

EXAMPLE 3.53 Determine the impulse response and step response of the causal system given below and discuss on stability

$$y(n) - y(n-1) - 2y(n-2) = x(n-1) + 2x(n-2)$$

Solution: The given difference equation is:

$$y(n) - y(n-1) - 2y(n-2) = x(n-1) + 2x(n-2)$$

Taking Z-transform on both sides, we have

$$Y(z) - z^{-1}Y(z) - 2z^{-2}Y(z) = z^{-1}X(z) + 2z^{-2}X(z)$$

i.e.

$$Y(z)[1 - z^{-1} - 2z^{-2}] = X(z)[z^{-1} + 2z^{-2}]$$

The transfer function of the system $H(z)$ is:

$$H(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{z^{-1} + 2z^{-2}}{1 - z^{-1} - 2z^{-2}} = \frac{(z+2)}{z^2 - z - 2} = \frac{(z+2)}{(z-2)(z+1)}$$

The impulse response of the system is:

$$h(n) = Z^{-1}[H(z)] = Z^{-1}\left[\frac{z+2}{(z-2)(z+1)}\right] = Z^{-1}\left(\frac{A}{z-2} + \frac{B}{z+1}\right)$$

$$= Z^{-1}\left(\frac{4}{3}\frac{1}{z-2}\right) - Z^{-1}\left(\frac{1}{3}\frac{1}{z+1}\right)$$

$$= \frac{4}{3}(2)^{n-1}u(n-1) - \frac{1}{3}(-1)^{n-1}u(n-1)$$

For step response, $x(n) = u(n)$

$\therefore$ $$X(z) = \frac{z}{z-1}$$

$\therefore$ $$\text{Output } Y(z) = H(z)\,X(z) = \frac{z+2}{(z-2)(z+1)}\,\frac{z}{z-1}$$

$$\frac{Y(z)}{z} = \frac{z+2}{(z-2)\,(z+1)\,(z-1)} = \frac{A}{z-2} + \frac{B}{z+1} + \frac{C}{z-1} = \frac{4/3}{z-2} + \frac{1/6}{z+1} - \frac{3/2}{z-1}$$

$\therefore$ $$Y(z) = \frac{4}{3}\left(\frac{z}{z-2}\right) + \frac{1}{6}\left(\frac{z}{z+1}\right) - \frac{3}{2}\left(\frac{z}{z-1}\right)$$

Taking inverse Z-transform on both sides, we have the step response

$$y(n) = \frac{4}{3}(2)^n u(n) + \frac{1}{6}(-1)^n u(n) - \frac{3}{2}u(n)$$

One pole of $H(z)$ is on the unit circle and another pole is outside the unit circle. So the system is unstable.

EXAMPLE 3.54 Consider a discrete-time linear time invariant system described by the difference equation

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) + \frac{1}{3}x(n-1)$$

where $y(n)$ is the output and $x(n)$ is the input. Assuming that the system is relaxed initially, obtain the unit sample response of the system.

Solution: The given difference equation is:

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) + \frac{1}{3}x(n-1)$$

Taking Z-transform on both sides and assuming that the system is relaxed initially, i.e. neglecting the initial conditions, we have

$$Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = X(z) + \frac{1}{3}z^{-1}X(z)$$

i.e.
$$Y(z)\left(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right) = X(z)\left(1 + \frac{1}{3}z^{-1}\right)$$

The system function $H(z)$ is:

$$H(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{1 + (1/3)z^{-1}}{1 - (3/4)z^{-1} + (1/8)z^{-2}} = \frac{z[z + (1/3)]}{z^2 - (3/4)z + (1/8)}$$

Taking partial fractions of $H(z)/z$, we have

$$\frac{H(z)}{z} = \frac{[z + (1/3)]}{[z - (1/2)][z - (1/4)]} = \frac{A}{z - (1/2)} + \frac{B}{z - (1/4)} = \frac{10/3}{z - (1/2)} - \frac{7/3}{z - (1/4)}$$

$\therefore$
$$H(z) = \frac{10}{3}\frac{z}{z - (1/2)} - \frac{7}{3}\frac{z}{z - (1/4)}$$

Taking inverse Z-transform on both sides, the unit sample response of the system is:

$$h(n) = \frac{10}{3}\left(\frac{1}{2}\right)^n u(n) - \frac{7}{3}\left(\frac{1}{4}\right)^n u(n)$$

EXAMPLE 3.55 Find the

(a) Impulse response

(b) Output response for a step input applied at $n = 0$ of a discrete-time linear time-invariant system whose difference equation is given by

$$y(n) = y(n-1) + 0.5y(n-2) + x(n) + x(n-1)$$

(c) Stability of the system

Solution:

(a) The given difference equation is:

$$y(n) = y(n-1) + 0.5y(n-2) + x(n) + x(n-1)$$

Taking Z-transform on both sides and neglecting the initial conditions, we have

$$Y(z) = z^{-1}Y(z) + 0.5z^{-2}Y(z) + X(z) + z^{-1}X(z)$$

i.e.
$$Y(z)\,(1 - z^{-1} - 0.5z^{-2}) = X(z)\,(1 + z^{-1})$$

The system function $H(z)$ is:

$$H(z)=\frac{Y(z)}{X(z)}$$

$$=\frac{1+z^{-1}}{1-z^{-1}-0.5z^{-2}}=\frac{z(z+1)}{z^2-z-0.5}$$

$$z^2-z-0.5=0,\ z=\frac{1\pm\sqrt{1+2}}{2}=\frac{1}{2}\pm\sqrt{0.75}=0.5\pm 0.866=1.366,\ -0.366$$

$$\therefore \qquad H(z)=\frac{z(z+1)}{(z-1.366)\,(z+0.366)}$$

To find the impulse response $h(n)$, we have to take the inverse Z-transform of $H(z)$. Taking partial fractions of $H(z)/z$, we have

$$\frac{H(z)}{z}=\frac{z+1}{(z-1.366)(z+0.366)}=\frac{A}{z-1.366}+\frac{B}{z+0.366}=\frac{1.366}{z-1.366}-\frac{0.366}{z+0.366}$$

$$\therefore \qquad H(z)=1.366\,\frac{z}{z-1.366}-0.366\frac{z}{z+0.366}$$

Taking inverse Z-transform, the impulse response of the system is:

$$h(n)=1.366\,(1.366)^n\,u(n)-0.366\,(-0.366)^n\,u(n)$$

(b) For a unit step input, $x(n) = u(n)$

$$\therefore \qquad X(z)=\frac{z}{z-1}$$

Therefore, the output for a step input is:

$$Y(z)=\frac{z\,(z+1)\,z}{(z-1.366)\,(z+0.366)\,(z-1)}$$

Taking partial fractions of $Y(z)/z$, we have

$$\frac{Y(z)}{z}=\frac{z\,(z+1)}{(z-1.366)\,(z+0.366)\,(z-1)}=\frac{A}{z-1.366}+\frac{B}{z+0.366}+\frac{C}{z-1}$$

$$=\frac{5}{z-1.366}+\frac{0.1}{z+0.366}+\frac{4}{z-1}$$

$$\therefore \qquad Y(z)=5\frac{z}{z-1.366}+0.1\,\frac{z}{z+0.366}+4\,\frac{z}{z-1}$$

Taking inverse Z-transform on both sides, the output response for a step input is:

$$y(n) = 5(1.366)^n\, u(n) + 0.1(-0.366)^n\, u(n) + 4u(n)$$

(c) $H(z)$ has one pole outside the unit circle. So the system is unstable.

EXAMPLE 3.56 Check the stability of the filter for $H(z) = \dfrac{z^2 - z + 1}{z^2 - z + \dfrac{1}{2}}$

Solution: Given
$$H(z) = \frac{z^2 - z + 1}{z^2 - z + \dfrac{1}{2}} = \frac{z^2 - z + 1}{(z - P_1)\,(z - P_2)}$$

The poles are at
$$z = \frac{1 \pm \sqrt{1-2}}{2} = \frac{1 \pm j}{2}$$

$\therefore$
$$H(z) = \frac{z^2 - z + 1}{\left(z - \dfrac{1}{2} + j\dfrac{1}{2}\right)\left(z - \dfrac{1}{2} - j\dfrac{1}{2}\right)}$$

Since the magnitude of the poles $|z| < 1,$ both the poles are inside the unit circle and the filter is stable.

EXAMPLE 3.57 Test the stability of the system

$$y(n) - y(n-1) = x(n) + x(n-1)$$

Solution: Given
$$y(n) - y(n-1) = x(n) + x(n-1)$$

Taking Z-transform on both sides, we have

$$Y(z) - z^{-1}Y(z) = X(z) + z^{-1}X(z)$$

$$Y(z)[1 - z^{-1}] = X(z)[1 + z^{-1}]$$

$\therefore$
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-1}}{1 - z^{-1}} = \frac{z+1}{z-1}$$

The pole is at $z = 1$. So the system is unstable.

EXAMPLE 3.58 Check the stability condition for the DSP systems described by the following equations.

(a) $y(n) = a^n u(n)$

(b) $y(n) = x(n) + e^a y(n-1)$

Solution:

(a) Given $$y(n) = a^n u(n)$$

Taking Z-transform on both sides, we have

$$Y(z) = \frac{z}{z-a}$$

Here the pole is at $z = a$ and hence for the system to be stable $|a| < 1$.

(b) Given $$y(n) = x(n) + e^a y(n-1)$$

Taking Z-transform on both sides, we have

$$Y(z) = X(z) + e^a z^{-1} Y(z)$$

i.e. $$Y(z)[1 - e^a z^{-1}] = X(z)$$

or $$\frac{Y(z)}{X(z)} = H(z) = \frac{1}{1 - e^a z^{-1}} = \frac{z}{z - e^a}$$

Here the pole is at $$z = e^a$$

Hence the condition for stability is $|e^a| < 1$, i.e. $a < 0$.

EXAMPLE 3.59 Determine the impulse response of the system described by the difference equation $y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$ using Z-transform.

Solution: Given the difference equation $y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$
Taking Z-transform on both sides, we have

$$Y(z) - 3z^{-1}Y(z) - 4z^{-2}Y(z) = X(z) + 2z^{-1}X(z)$$

i.e. $$Y(z)\,(1 - 3z^{-1} - 4z^{-2}) = X(z)\,(1 + 2z^{-1})$$

The transfer function of the system is:

$$H(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{1 + 2z^{-1}}{1 - 3z^{-1} - 4z^{-2}} = \frac{z(z+2)}{z^2 - 3z - 4}$$

Taking partial fractions of $H(z)/z$, we have

$$\frac{H(z)}{z} = \frac{z+2}{z^2 - 3z - 4} = \frac{z+2}{(z-4)(z+1)} = \frac{1.2}{z-4} - \frac{0.2}{z+1}$$

$\therefore$ $$H(z) = 1.2\left(\frac{z}{z-4}\right) - 0.2\left(\frac{z}{z+1}\right)$$

Taking inverse Z-transform on both sides, we have the impulse response

$$h(n) = 1.2(4)^n\, u(n) - 0.2(-1)^n\, u(n)$$

EXAMPLE 3.60 Determine the impulse response and step response of the causal system given below and discuss on stability

$$y(n) + y(n-1) - 2y(n-2) = x(n-1) + 2x(n-2)$$

Solution: The given difference equation is

$$y(n) + y(n-1) - 2y(n-2) = x(n-1) + 2x(n-2)$$

Taking Z-transform on both sides, we have

$$Y(z) + z^{-1}Y(z) - 2z^{-2}Y(z) = z^{-1}X(z) + 2z^{-2}X(z)$$

i.e.

$$Y(z)\,[1 + z^{-1} - 2z^{-2}] = X(z)\,[z^{-1} + 2z^{-2}]$$

The system transfer function $H(z)$ is

$$H(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{z^{-1} + 2z^{-2}}{1 + z^{-1} - 2z^{-2}} = \frac{z+2}{z^2 + z - 2} = \frac{z+2}{(z+2)(z-1)} = \frac{1}{z-1}$$

Taking inverse Z-transform, the impulse response of the system is

$$h(n) = u(n-1)$$

For step response, $x(n) = u(n)$

$$\therefore \qquad X(z) = \frac{z}{z-1}$$

$\therefore$ The output $Y(z) = X(z)\,H(z) = \dfrac{z}{(z-1)}\,\dfrac{1}{z-1} = \dfrac{z}{(z-1)^2}$

Taking inverse Z-transform on both sides, the step response is

$$y(n) = nu(n-1)$$

3.9 SOLUTION OF DIFFERENCE EQUATIONS USING Z-TRANSFORMS

To solve the difference equation, first it is converted into algebraic equation by taking its Z-transform. The solution is obtained in z-domain and the time domain solution is obtained by taking its inverse Z-transform.

The system response has two components. The source free response and the forced response. The response of the system due to input alone when the initial conditions are

neglected is called the forced response of the system. It is also called the steady state response of the system. It represents the component of the response due to the driving force. The response of the system due to initial conditions alone when the input is neglected is called the free or natural response of the system. It is also called the transient response of the system. It represents the component of the response when the driving function is made zero. The response due to input and initial conditions considered simultaneously is called the total response of the system.

For a stable system, the source free component always decays with time. In fact a stable system is one whose source free component decays with time. For this reason the source free component is also designated as the transient component and the component due to source is called the steady state component.

When input is a unit impulse input, the response is called the impulse response of the system and when the input is a unit step input, the response is called the step response of the system.

EXAMPLE 3.61 A linear shift invariant system is described by the difference equation

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) + x(n-1)$$

with $y(-1) = 0$ and $y(-2) = -1$.

Find (a) the natural response of the system (b) the forced response of the system for a step input and (c) the frequency response of the system.

Solution:

(a) The natural response is the response due to initial conditions only. So make $x(n) = 0$. Then the difference equation becomes

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = 0$$

Taking Z-transform on both sides, we have

$$Y(z) - \frac{3}{4}\,[z^{-1}Y(z) + y(-1)] + \frac{1}{8}\,[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)] = 0$$

i.e. $$Y(z)\left(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right) - \frac{1}{8} = 0$$

$$\therefore\ Y(z) = \frac{1/8}{1 - (3/4)z^{-1} + (1/8)z^{-2}} = \frac{1/8z^2}{z^2 - (3/4)z + (1/8)} = \frac{1/8z^2}{[z - (1/2)]\,[z - (1/4)]}$$

The partial fraction expansion of $Y(z)/z$ gives

$$\frac{Y(z)}{z} = \frac{(1/8)z}{[z - (1/2)]\,[z - (1/4)]} = \frac{A}{z - (1/2)} + \frac{B}{z - (1/4)} = \frac{1/4}{z - (1/2)} - \frac{1/8}{z - (1/4)}$$

$$\therefore \qquad Y(z) = \frac{1}{4}\frac{z}{z-(1/2)} - \frac{1}{8}\frac{z}{z-(1/4)}$$

Taking inverse Z-transform on both sides, we get the natural response as:

$$y(n) = \frac{1}{4}\left(\frac{1}{2}\right)^n u(n) - \frac{1}{8}\left(\frac{1}{4}\right)^n u(n)$$

(b) To find the forced response due to a step input, put $x(n) = u(n)$. So we have

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = u(n) + u(n-1)$$

We know that the forced response is due to input alone. So for forced response, the initial conditions are neglected. Taking Z-transform on both sides of the above equation and neglecting the initial conditions, we have

$$Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = U(z) + z^{-1}U(z) = \frac{z}{z-1} + \frac{1}{z-1}$$

i.e.
$$Y(z)\left(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right) = \frac{z+1}{z-1}$$

$$\therefore \quad Y(z) = \frac{z+1}{(z-1)\,[1-(3/4)z^{-1} + (1/8)z^{-2})} = \frac{z^2(z+1)}{(z-1)[z^2-(3/4)z+(1/8)]}$$

$$= \frac{z^2(z+1)}{(z-1)\,[z-(1/2)]\,[z-(1/4)]}$$

Taking partial fractions of $Y(z)/z$, we have

$$\therefore \quad \frac{Y(z)}{z} = \frac{z(z+1)}{(z-1)\,[z-(1/2)]\,[z-(1/4)]} = \frac{A}{z-1} + \frac{B}{z-(1/2)} + \frac{C}{z-(1/4)}$$

$$= \frac{16/3}{z-1} - \frac{6}{z-(1/2)} + \frac{5/3}{z-(1/4)}$$

or
$$Y(z) = \frac{16}{3}\left(\frac{z}{z-1}\right) - 6\left[\frac{z}{z-(1/2)}\right] + \frac{5}{3}\left[\frac{z}{z-(1/4)}\right]$$

Taking the inverse Z-transform on both sides, we have the forced response for a step input.

$$y(n) = \frac{16}{3}u(n) - 6\left(\frac{1}{2}\right)^n u(n) + \frac{5}{3}\left(\frac{1}{4}\right)^n u(n)$$

(c) The frequency response of the system $H(\omega)$ is obtained by putting $z = e^{j\omega}$ in $H(z)$.

Here $$H(z) = \frac{Y(z)}{X(z)} = \frac{z(z+1)}{z^2 - (3/4)z + (1/8)}$$

Therefore, $$H(\omega) = \frac{e^{j\omega}(e^{j\omega}+1)}{(e^{j\omega})^2 - (3/4)e^{j\omega} + (1/8)}$$

EXAMPLE 3.62 (a) Determine the free response of the system described by the difference equation

$$y(n) - \frac{5}{6}y(n-1) + \frac{1}{6}y(n-2) = x(n) \text{ with } y(-1) = 1 \text{ and } y(-2) = 0$$

(b) Determine the forced response for an input

$$x(n) = \left(\frac{1}{4}\right)^n u(n)$$

Solution:

(a) The free response, also called the natural response or transient response is the response due to initial conditions only [i.e. make $x(n) = 0$].

So, the difference equation is:

$$y(n) - \frac{5}{6}y(n-1) + \frac{1}{6}y(n-2) = 0$$

Taking Z-transform on both sides, we get

$$Y(z) - \frac{5}{6}[z^{-1}Y(z) + y(-1)] + \frac{1}{6}[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)] = 0$$

$$Y(z)\left(1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}\right) - \frac{5}{6} + \frac{1}{6}z^{-1} = 0$$

$$\therefore \quad Y(z) = \frac{(5/6) - (1/6)z^{-1}}{1 - (5/6)z^{-1} + (1/6)z^{-2}} = \frac{5/6[z - (1/5)]z}{z^2 - (5/6)z + (1/6)} = \frac{(5/6)z[z - (1/5)]}{[z - (1/2)][z - (1/3)]}$$

Taking partial fractions of $Y(z)/z$, we have

$$\frac{Y(z)}{z} = \frac{5/6[z - (1/5)]}{[z - (1/2)][z - (1/3)]} = \frac{A}{z - (1/2)} + \frac{B}{z - (1/3)} = \frac{3/2}{z - (1/2)} - \frac{2/3}{z - (1/3)}$$

$$\therefore \quad Y(z) = \frac{3}{2}\frac{z}{z - (1/2)} - \frac{2}{3}\frac{z}{z - (1/3)}$$

Taking inverse Z-transform on both sides, we get the free response of the system as:

$$y(n) = \frac{3}{2}\left(\frac{1}{2}\right)^n u(n) - \frac{2}{3}\left(\frac{1}{3}\right)^n u(n)$$

(b) To determine the forced response, i.e. the steady state response, the initial conditions are to be neglected.

The given difference equation is:

$$y(n) - \frac{5}{6}y(n-1) + \frac{1}{6}y(n-2) = x(n) = \left(\frac{1}{4}\right)^n u(n)$$

Taking Z-transform on both sides and neglecting the initial conditions, we have

$$Y(z) - \frac{5}{6}z^{-1}Y(z) + \frac{1}{6}z^{-2}Y(z) = \frac{z}{z-(1/4)}$$

i.e.,
$$Y(z)\left(1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}\right) = \frac{z}{z-(1/4)}$$

$$\therefore \quad Y(z) = \frac{z}{z-(1/4)} \frac{1}{1-(5/6)z^{-1}+(1/6)z^{-2}} = \frac{z^3}{[z-(1/4)][z-(1/2)][z-(1/3)]}$$

Partial fraction expansion of $Y(z)/z$ gives

$$\frac{Y(z)}{z} = \frac{z^2}{[z-(1/4)][z-(1/3)][z-(1/2)]} = \frac{A}{z-(1/4)} + \frac{B}{z-(1/3)} + \frac{C}{z-(1/2)}$$

$$= \frac{3}{z-(1/4)} - \frac{8}{z-(1/3)} + \frac{6}{z-(1/2)}$$

Multiplying both sides by z, we get

$$Y(z) = 3\frac{z}{z-(1/4)} - 8\frac{z}{z-(1/3)} + 6\frac{z}{z-(1/2)}$$

Taking inverse Z-transform on both sides, the forced response of the system is:

$$y(n) = 3\left(\frac{1}{4}\right)^n u(n) - 8\left(\frac{1}{3}\right)^n u(n) + 6\left(\frac{1}{2}\right)^n u(n)$$

EXAMPLE 3.63 Find the impulse and step response of the system

$$y(n) = 2x(n) - 3x(n-1) + x(n-2) - 4x(n-3)$$

Solution: For impulse response, $x(n) = \delta(n)$

The impulse response of the system is:

$$y(n) = 2\delta(n) - 3\delta(n-1) + \delta(n-2) - 4\delta(n-3)$$

For step response, $x(n) = u(n)$

The step response of the system is:

$$y(n) = 2u(n) - 3u(n-1) + u(n-2) - 4u(n-3)$$

EXAMPLE 3.64 Solve the following difference equation

$$y(n) + 2y(n-1) = x(n)$$

with $x(n) = (1/3)^n\, u(n)$ and the initial condition $y(-1) = 1$.

Solution: The solution of the difference equation considering the initial condition and input simultaneously gives the total response of the system.

The given difference equation is:

$$y(n) + 2y(n-1) = x(n) = \left(\frac{1}{3}\right)^n u(n) \text{ with } y(-1) = 1$$

Taking Z-transform on both sides, we get

$$Y(z) + 2[z^{-1}Y(z) + y(-1)] = X(z) = \frac{1}{1-(1/3)z^{-1}}$$

Substituting the initial conditions, we have

$$Y(z)\,(1+2z^{-1}) = -2(1) + \frac{1}{1-(1/3)\,z^{-1}}$$

$$\therefore \qquad Y(z) = \frac{-2}{1+2z^{-1}} + \frac{1}{[1-(1/3)\,z^{-1}][1+2z^{-1}]}$$

$$= \frac{-2z}{z+2} + \frac{z^2}{[z-(1/3)]\,(z+2)}$$

Let $$Y_1(z) = \frac{z^2}{[z-(1/3)]\,(z+2)}$$

Taking partial fractions of $Y_1(z)/z$, we have

$$\frac{Y_1(z)}{z} = \frac{z}{[z-(1/3)]\,(z+2)} = \frac{A}{z-(1/3)} + \frac{B}{z+2} = \frac{1/7}{z-(1/3)} + \frac{6/7}{z+2}$$

Multiplying both sides by z, we have

$$Y_1(z) = \frac{1}{7}\frac{z}{z-(1/3)} + \frac{6}{7}\frac{z}{z+2}$$

$$\therefore \qquad Y(z) = -\frac{2z}{z+2} + \frac{6}{7}\frac{z}{z+2} + \frac{1}{7}\frac{z}{z-(1/3)} = -\frac{8}{7}\frac{z}{z+2} + \frac{1}{7}\frac{z}{z-(1/3)}$$

Taking inverse Z-transform on both sides, the solution of the difference equation is:

$$y(n) = -\frac{8}{7}(-2)^n u(n) + \frac{1}{7}\left(\frac{1}{3}\right)^n u(n)$$

EXAMPLE 3.65 Solve the following difference equation using unilateral Z-transform.

$$y(n) - \frac{7}{12}y(n-1) + \frac{1}{12}y(n-2) = x(n) \text{ for } n \geq 0$$

with initial conditions $y(-1) = 2,\ y(-2) = 4$ and $x(n) = \left(\frac{1}{5}\right)^n u(n)$

Solution: The solution of the difference equation gives the total response of the system (i.e., the sum of the natural (free) response and the forced response)

The given difference equation is:

$$y(n) - \frac{7}{12}y(n-1) + \frac{1}{12}y(n-2) = x(n) = \left(\frac{1}{5}\right)^n u(n)$$

with initial conditions $y(-1) = 2$ and $y(-2) = 4$.

Taking Z-transform on both sides, we have

$$Y(z) - \frac{7}{12}[z^{-1}Y(z) + y(-1)] + \frac{1}{12}[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)] = \frac{1}{1-(1/5)z^{-1}}$$

i.e. $$Y(z)\left(1 - \frac{7}{12}z^{-1} + \frac{1}{12}z^{-2}\right) = \frac{7}{12}(2) - \frac{1}{12}(2z^{-1}) - \frac{1}{12}(4) + \frac{1}{1-(1/5)z^{-1}}$$

i.e. $$Y(z)\left(1 - \frac{7}{12}z^{-1} + \frac{1}{12}z^{-2}\right) = \frac{5}{6}\left(1 - \frac{1}{5}z^{-1}\right) + \frac{1}{1-(1/5)z^{-1}}$$

$\therefore$ $$Y(z) = \frac{(5/6)[1-(1/5)z^{-1}]}{[1-(7/12)z^{-1}+(1/12)z^{-2}]} + \frac{1}{[1-(1/5)z^{-1}][1-(7/12)z^{-1}+(1/12)z^{-2}]}$$

$$= \frac{(5/6)[z-(1/5)]z}{[z-(1/4)][z-(1/3)]} + \frac{z^3}{[z-(1/5)][z-(1/4)][z-(1/3)]}$$

$$= \frac{z[(11/6)z^2 - (1/3)z + (1/30)]}{[z-(1/5)][z-(1/4)][z-(1/3)]}$$

Taking partial fractions of $Y(z)/z$, we have

$$\frac{Y(z)}{z} = \frac{A}{z-(1/5)} + \frac{B}{z-(1/4)} + \frac{C}{z-(1/3)} = \frac{6}{5}\frac{1}{z-(1/5)} + \frac{1}{8}\frac{1}{z-(1/4)} + \frac{100}{27}\frac{1}{z-(1/3)}$$

Multiplying both sides by z, we have

$$Y(z) = \frac{6}{5}\frac{z}{z-(1/5)} + \frac{1}{8}\frac{z}{z-(1/4)} + \frac{102}{27}\frac{z}{z-(1/3)}$$

Taking inverse Z-transform on both sides, the solution of the difference equation is:

$$y(n) = \frac{6}{5}\left(\frac{1}{5}\right)^n u(n) + \frac{1}{8}\left(\frac{1}{4}\right)^n u(n) + \frac{102}{27}\left(\frac{1}{3}\right)^n u(n)$$

EXAMPLE 3.66 Using Z-transform determine the response of the LTI system described by $y(n) - 2r\cos\theta\, y(n-1) + r^2 y(n-2) = x(n)$ to an excitation $x(n) = a^n u(n)$.

Solution: Taking Z-transform on both sides of the difference equation, we have

$$Y(z) - 2r\cos\theta\,[z^{-1}Y(z) + y(-1)] + r^2[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)] = X(z)$$

i.e.
$$Y(z)\,[1 - 2r\cos\theta\, z^{-1} + r^2 z^{-2}] = \frac{z}{z-a}$$

$\therefore$
$$Y(z) = \frac{z^3}{(z-a)\,(z-re^{j\theta})\,(z-re^{-j\theta})}$$

$$= \frac{a^2}{a^2 - 2ar\cos\theta + r^2}\,\frac{z}{z-a} + \frac{r^2 e^{j2\theta}}{(re^{j\theta}-a)(re^{j\theta}-re^{-j\theta})}\,\frac{z}{z-re^{j\theta}}$$
$$+ \frac{r^2 e^{-j2\theta}}{(re^{-j\theta}-a)(re^{-j\theta}-re^{j\theta})}\,\frac{z}{z-re^{-j\theta}}$$

$\therefore$
$$y(n) = \frac{a^2}{a^2 - 2ar\cos\theta + r^2}\,a^n u(n) + \frac{r^{n+1}}{\sin\theta}\left[\frac{r\sin(n+1)\theta - a\sin(n+2)\theta}{a^2 - 2ar\cos\theta + r^2}\right]u(n)$$

EXAMPLE 3.67 Determine the step response of an LTI system whose impulse response $h(n)$ is given by $h(n) = a^{-n}u(-n);\ 0 < a < 1$.

Solution: The impulse response is $h(n) = a^{-n}u(-n);\ 0 < a < 1$

$\therefore$
$$H(z) = \frac{1}{1-az} = -\frac{1}{a}\,\frac{1}{z-(1/a)}$$

We have to find the step response

$\therefore$
$$x(n) = u(n) \text{ and } H(z) = \frac{z}{z-1}$$

The step response of the system is given by

$$y(n) = x(n) * h(n)$$

$$\therefore \qquad Y(z) = X(z)\,H(z) = \left(-\frac{1}{a}\right)\frac{z}{z-1}\,\frac{1}{z-(1/a)} = \frac{1}{1-a}\left[\frac{z}{z-1} - \frac{z}{z-(1/a)}\right]$$

So the step response is

$$y(n) = \frac{1}{1-a}\left[u(n) - \left(\frac{1}{a}\right)^n u(n)\right]$$

EXAMPLE 3.68 Determine the frequency response, magnitude response and phase response for the system given by

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) - x(n-1)$$

Solution: Given $y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) - x(n-1)$

Taking Z-transform on both sides and neglecting initial conditions, we have

$$Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = X(z) - z^{-1}X(z)$$

i.e. $$Y(z)\,[1-(3/4)\,z^{-1} + (1/8)\,z^{-2}] = X(z)[1-z^{-1}]$$

The transfer function of the system

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-z^{-1}}{1-(3/4)z^{-1}+(1/8)z^{-2}} = \frac{z(z-1)}{z^2-(3/4)z+(1/8)}$$

The frequency response of the system

$$H(\omega) = \frac{e^{j\omega}(e^{j\omega}-1)}{e^{j2\omega}-(3/4)\,e^{j\omega}+(1/8)} = \frac{e^{j2\omega}-e^{j\omega}}{e^{j2\omega}-(3/4)\,e^{j\omega}+(1/8)}$$

$$= \frac{(\cos 2\omega - \cos\omega) + j(\sin 2\omega - \sin\omega)}{[\cos 2\omega - (3/4)\cos\omega + (1/8)] + j[\sin 2\omega - (3/4)\sin\omega]}$$

The magnitude response

$$\left|H(\omega)\right| = \left\{\frac{(\cos 2\omega - \cos\omega)^2 + (\sin 2\omega - \sin\omega)^2}{[\cos 2\omega - (3/4)\cos\omega + (1/8)]^2 + [\sin 2\omega - (3/4)\sin\omega]^2}\right\}^{1/2}$$

The phase response

$$\underline{|H(\omega)} = \tan^{-1}\frac{\sin 2\omega - \sin\omega}{\cos 2\omega - \cos\omega} - \tan^{-1}\frac{\sin 2\omega - (3/4)\sin\omega}{\cos 2\omega - (3/4)\cos\omega + (1/8)}$$

The magnitude response plot and phase response plot can be obtained by plotting $|H(\omega)|$ versus ω plot and $\underline{|H(\omega)}$ versus ω plots for various values of ω.

EXAMPLE 3.69 A causal LTI system is defined by the difference equation

$$2y(n) - y(n-2) = x(n-1) + 3x(n-2) + 2x(n-3)$$

Find the frequency response, magnitude response and phase response.

Solution: Given $2y(n) - y(n-2) = x(n-1) + 3x(n-2) + 2x(n-3)$

Taking Z-transform on both sides and neglecting initial conditions, we have

$$2Y(z) - z^{-2}Y(z) = z^{-1}X(z) + 3z^{-2}X(z) + 2z^{-3}X(z)$$

i.e.

$$Y(z)\,[2 - z^{-2}] = X(z)[z^{-1} + 3z^{-2} + 2z^{-3}]$$

The transfer function of the system

$$\frac{Y(z)}{X(z)} = H(z) = \frac{z^{-1} + 3z^{-2} + 2z^{-3}}{2 - z^{-2}} = \frac{[z^2 + 3z + 2]}{[2z^3 - z]}$$

The frequency response of the system

$$H(\omega) = \frac{e^{j2\omega} + 3e^{j\omega} + 2}{2e^{j3\omega} - e^{j\omega}}$$

The magnitude response of the system

$$|H(\omega)| = \left|\frac{(\cos 2\omega + 3\cos\omega + 2) + j(\sin 2\omega + 3\sin\omega)}{(2\cos 3\omega - \cos\omega) + j(2\sin 3\omega - \sin\omega)}\right|$$

$$= \left[\frac{(\cos 2\omega + 3\cos\omega + 2)^2 + (\sin 2\omega + 3\sin\omega)^2}{(2\cos 3\omega - \cos\omega)^2 + (2\sin 3\omega - \sin\omega)^2}\right]^{1/2}$$

Phase response

$$\underline{|H(\omega)} = \tan^{-1}\frac{\sin 2\omega + 3\sin\omega}{\cos 2\omega + 3\cos\omega + 2} - \tan^{-1}\frac{2\sin 3\omega - \sin\omega}{2\cos 3\omega - \cos\omega}$$

EXAMPLE 3.70 Determine the steady state response for the system with impulse response $h(n) = \left(\frac{1}{2}\right)^n u(n)$ for an input $x(n) = [\cos(\pi n)]u(n)$.

Solution: Let $y(n)$ be the steady state (forced) response of the system which is given by the convolution of $x(n)$ and $h(n)$.

Then $$y(n) = x(n) * h(n)$$

By the convolution property of Z-transforms we get

$$Y(z) = X(z)H(z)$$

$\therefore$ $$y(n) = Z^{-1}[Y(z)] = Z^{-1}[X(z)H(z)]$$

Given $$x(n) = \cos(\pi n)u(n)$$

$\therefore$ $$X(z) = \frac{z(z-\cos\pi)}{z^2 - 2z\cos\pi + 1} = \frac{z(z+1)}{z^2 + 2z + 1} = \frac{z}{(z+1)}$$

Given $$h(n) = \left(\frac{1}{2}\right)^n u(n)$$

$\therefore$ $$H(z) = \frac{z}{z-(1/2)}$$

$$Y(z) = X(z)H(z) = \frac{z}{z+1}\,\frac{z}{z-j0.5} = (0.8 - j0.4)\,\frac{z}{z+1} + (0.2 + j0.4)\,\frac{z}{z - j0.5}$$

Taking inverse Z-transform on both sides, we have

$$y(n) = (0.8 - j0.4)(-1)^n u(n) + (0.2 + j0.4)(j0.5)^n u(n)$$

EXAMPLE 3.71 Find the response of the time-invariant system with impulse response $h(n) = \{1, 2, 1, -1\}$ to an input signal $x(n) = \{1, 2, 3, 6\}$.

Solution: Let $y(n)$ be the response or output of the system. We know that the response is given by the convolution of input $x(n)$ and impulse response $h(n)$.

i.e. $$y(n) = x(n) * h(n)$$

$\therefore$ $$Y(z) = X(z)H(z)$$

Given $$x(n) = \{1, 2, 3, 6\} = \delta(n) + 2\delta(n-1) + 3\delta(n-2) + 6\delta(n-3)$$

$\therefore$ $$X(z) = 1 + 2z^{-1} + 3z^{-2} + 6z^{-3}$$

Given $$h(n) = \{1, 2, 1, 1\} = \delta(n) + 2\delta(n-1) + \delta(n-2) - \delta(n-3)$$

$\therefore$ $$H(z) = 1 + 2z^{-1} + z^{-2} - z^{-3}$$

$\therefore$ $$Y(z) = X(z)H(z) = (1 + 2z^{-1} + 3z^{-2} + 6z^{-3})(1 + 2z^{-1} + z^{-2} - z^{-3})$$

$$= 1 + 4z^{-1} + 8z^{-2} + 8z^{-3} + 3z^{-4} - 2z^{-5} - z^{-6}$$

Taking inverse Z-transform on both sides, we have

$$y(n) = \delta(n) + 4\delta(n-1) + 8\delta(n-2) + 8\delta(n-3) + 3\delta(n-4) - 2\delta(n-5) - \delta(n-6)$$

i.e. $y(n) = \{1, 4, 8, 8, 3, -2, -1\}$

EXAMPLE 3.72 The step response of an LTI system is

$$s(n) = \left(\frac{1}{3}\right)^{n-2} u(n+2)$$

Find the impulse response of the system.

Solution: We have $s(n) = h(n) * u(n)$

$$\therefore \qquad S(z) = H(z)U(z) = H(z)\frac{z}{z-1}$$

Given
$$s(n) = \left(\frac{1}{3}\right)^{n-2} u(n+2)$$

$$S(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{3}\right)^{n-2} u(n+2)\, z^{-n} = 3^2 \sum_{n=-2}^{\infty} \left(\frac{1}{3z}\right)^{n}$$

$$= 3^2 \frac{\left(\dfrac{1}{3z}\right)^{-2}}{1-\dfrac{1}{3z}} = \frac{3^4 z^2}{1-\dfrac{1}{3}z^{-1}} = \frac{81z^3}{\left(z-\dfrac{1}{3}\right)}$$

The system function $H(z)$ is

$$H(z) = S(z)\frac{z-1}{z} = \frac{81z^3}{\left(z-\dfrac{1}{3}\right)}\frac{z-1}{z}$$

$$= \frac{81z^2(z-1)}{\left(z-\dfrac{1}{3}\right)} = \frac{81z^3}{z-\dfrac{1}{3}} - \frac{81z^2}{z-\dfrac{1}{3}}$$

$$= 81z^2 \frac{z}{z-\dfrac{1}{3}} - 81z \frac{z}{z-\dfrac{1}{3}}$$

The impulse response of the system is

$$h(n) = 81\left(\frac{1}{3}\right)^{n+2} u(n+2) - 81\left(\frac{1}{3}\right)^{n+1} u(n+1)$$

$$= 9\left(\frac{1}{3}\right)^{n} u(n+2) - 27\left(\frac{1}{3}\right)^{n} u(n+1)$$

EXAMPLE 3.73 Consider a causal linear shift-invariant system with system function

$$H(z) = \frac{1-a^{-1}z^{-1}}{1-az^{-1}}$$

where a is real. Determine the range of values of a for which the system is stable? Show analytically that this system is an all pass system.

Solution: Given $$H(z) = \frac{1-a^{-1}z^{-1}}{1-az^{-1}} = \frac{z-\frac{1}{a}}{z-a}$$

The system function $H(z)$ has one pole at $z = a$ and a zero at $z = 1/a$. The ROC of $H(z)$ is exterior to the circle of radius a. For a stable filter all poles must be inside the unit circle. Therefore for the system to be stable $|z| > |a| < 1$.

For $|a| < 1$, the frequency response is

$$H(\omega) = \left|\frac{1-a^{-1}z^{-1}}{1-az^{-1}}\right|_{z=e^{j\omega}} = \frac{1-a^{-1}e^{-j\omega}}{1-ae^{-j\omega}}$$

$$= \frac{a-e^{-j\omega}}{a(1-ae^{-j\omega})} = \frac{a-(\cos\omega - j\sin\omega)}{a(1-a\cos\omega + ja\sin\omega)}$$

$$= \frac{a-\cos\omega + j\sin\omega}{a(1-a\cos\omega + ja\sin\omega)}$$

$$|H(\omega)|^2 = \frac{(a-\cos\omega)^2 + \sin^2\omega}{a^2\{(1-a\cos\omega)^2 + a^2\sin^2\omega\}}$$

$$= \frac{a^2 - 2a\cos\omega + \cos^2\omega + \sin^2\omega}{a^2(1-2a\cos\omega + a^2\cos^2\omega + a^2\sin^2\omega)}$$

$$= \frac{1+a^2-2a\cos\omega}{a^2(1+a^2-2a\cos\omega)} = \frac{1}{a^2}$$

That is $|H(\omega)| = \dfrac{1}{a}$ for all values of ω. Therefore the system is an all pass system.

3.10 DECONVOLUTION USING Z-TRANSFORM

Deconvolution is used to find the input $x(n)$ applied to the system once the output $y(n)$ and the impulse response $h(n)$ of the system are known. The Z-transform also can be used for deconvolution operation.

We know that

$$Y(z) = X(z)H(z) \quad \text{or} \quad X(z) = \frac{Y(z)}{H(z)}$$

where $Y(z)$, $X(z)$ and $H(z)$ are the Z-transforms of output, input and impulse response, respectively. If $y(n)$ and $h(n)$ are given, we can determine their Z-transforms $Y(z)$ and $H(z)$. Knowing $Y(z)$ and $H(z)$, we can determine $X(z)$ and knowing $X(z)$, we can determine the input $x(n)$. Thus, the deconvolution is reduced to the procedure of evaluating an inverse Z-transform.

EXAMPLE 3.74 Find the input $x(n)$ of the system, if the impulse response $h(n)$ and the output $y(n)$ are as given below:

$$h(n) = \{2, 1, 0, -1, 3\}; \quad y(n) = \{2, -5, 1, 1, 6, -11, 6\}$$

Solution: Given $\quad h(n) = \{2, 1, 0, -1, 3\}$

$$\therefore \quad H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n}$$

$$= \sum_{n=0}^{4} h(n)z^{-n} = 2 + z^{-1} - z^{-3} + 3z^{-4}$$

$$y(n) = \{2, -5, 1, 1, 6, -11, 6\}$$

$$\therefore \quad Y(z) = \sum_{n=-\infty}^{\infty} y(n)z^{-n}$$

$$= \sum_{n=0}^{6} y(n)z^{-n}$$

$$= 2 - 5z^{-1} + z^{-2} + z^{-3} + 6z^{-4} - 11z^{-5} + 6z^{-6}$$

$$\therefore \quad X(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{2 - 5z^{-1} + z^{-2} + z^{-3} + 6z^{-4} - 11z^{-5} + 6z^{-6}}{2 + z^{-1} - z^{-3} + 3z^{-4}}$$

$$
\begin{array}{r|l}
 & 1-3z^{-1}+2z^{-2} \\
\hline
2+z^{-1}-z^{-3}+3z^{-4} & 2-5z^{-1}+z^{-2}+z^{-3}+6z^{-4}-11z^{-5}+6z^{-6} \\
 & 2+z^{-1}-z^{-3}+3z^{-4} \\
\cline{2-2}
 & -6z^{-1}+z^{-2}+2z^{-3}+3z^{-4}-11z^{-5}+6z^{-6} \\
 & -6z^{-1}-3z^{-2}+3z^{-4}-9z^{-5} \\
\cline{2-2}
 & 4z^{-2}+2z^{-3}-2z^{-5}+6z^{-6} \\
 & 4z^{-2}+2z^{-3}-2z^{-5}+6z^{-6} \\
\cline{2-2}
 & 0
\end{array}
$$

$\therefore$ $X(z) = 1 - 3z^{-1} + 2z^{-2}$

Taking inverse Z-transform, we have input

$$x(n) = \delta(n) - 3\delta(n-1) + 2\delta(n-2)$$

i.e. $x(n) = \{1, -3, 2\}$

3.11 RELATION BETWEEN s-PLANE AND z-PLANE

Let $x(t)$ be a continuous signal and $x^*(t)$ be the sampled version of $x(t)$. Let the sampling period be T.

$$\therefore \quad x^*(t) = \sum_{n=0}^{\infty} x(nT)\delta(t-nT)$$

The Laplace transform of $x^*(t)$ can be written as

$$X^*(s) = \sum_{n=0}^{\infty} x(nT)e^{-nTs}$$

The Z-transform of $x^*(t)$ is

$$X(z) = \sum_{n=0}^{\infty} x(nT)z^{-n}$$

Therefore, the relation between s-plane and z-plane can be described by the equation

$$z = e^{sT}$$

Let $z = re^{j\omega}$ and $s = \sigma + j\omega$

$$\therefore \quad z = re^{j\omega} = e^{(\sigma+j\omega)T} = e^{\sigma T}e^{j\omega T}$$

$$\therefore \qquad |z| = |r| = e^{\sigma T} \quad \text{and} \quad \angle z = \omega T$$

If $\sigma = 0$, $|z| = 1$. For $\sigma < 0$, $|z| < 1$ and for $\sigma > 0$, $|z| > 1$

When $\sigma = 0$, $|z| = 1$. So the $j\omega$ axis of s-plane maps into the unit circle. The left half of the s-plane where $\sigma < 0$ maps into the inside of the unit circle ($|z| < 1$). The right half of the s-plane where $\sigma > 0$ maps into the outside of the unit circle ($|z| > 1$). The mapping is as shown in Figure 3.12.

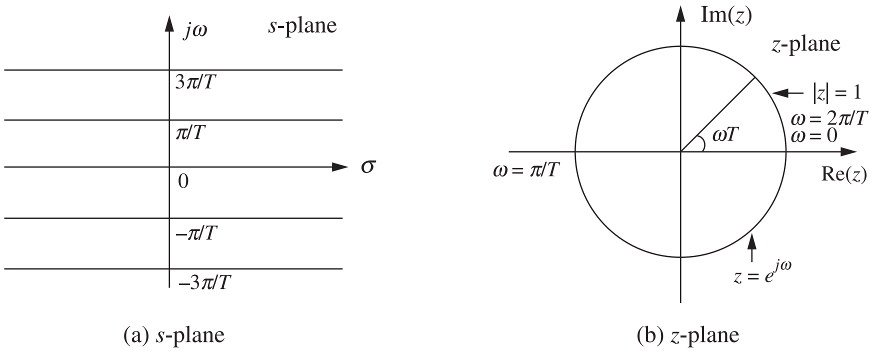


Figure 3.12 Relation between *s*-plane and *z*-plane.

From Figure 3.12 we can observe that a single horizontal strip of width $2\pi/T$, from $\omega = -\pi/T$ to $\omega = \pi/T$ completely maps into the inside of the unit circle and the horizontal strips between π/T and $3\pi/T$ between $-\pi/T$ and $-3\pi/T$ (In general between $(2n-1)\pi/T$ and $(2n+1)\pi/T$ where $n = 0, \pm1, \pm2, ...$) are mapped again into the inside of the unit circle. Thus, many points in the s-plane are mapped into a single point in the z-plane, causing aliasing effect. That is when we sample two sinusoidal signals of the frequencies which differ by a multiple of the sampling frequency, we cannot distinguish between the results.

SHORT QUESTIONS WITH ANSWERS

1. How are discrete-time systems analysed using Z-transforms?
Ans. Discrete-time systems are described by difference equations. The difference equations, which are in time domain, are converted into algebraic equations in z-domain using Z-transforms. Those algebraic equations are manipulated and result is obtained in z-domain. Using inverse Z-transform, the result is converted into time domain.

2. Define Z-transform.
Ans. The Z-transform of a bilateral discrete-time signal $x(n)$ is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

The Z-transform of a unilateral discrete-time signal $x(n)$ is defined as:

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

where z is a complex variable given by $z = re^{j\omega}$ and r is the radius of a circle which decides the ROC.

3. What are the advantages of Z-transform?

Ans. Advantages of Z-transform are:

1. They convert the difference equations of a discrete-time system into linear algebraic equations so that the analysis becomes easy and simple.
2. Convolution in time domain is converted into multiplication in z-domain.
3. Z-transform exists for most of the signals for which DTFT does not exist.

4. What is the condition for Z-transform to exist?

Ans. The condition for the Z-transform to exist is the sum $\sum_{n=-\infty}^{\infty} |x(n) r^{-n}| < \infty$, i.e. $x(n) r^{-n}$ must be absolutely summable.

5. What is the relation between discrete-time Fourier transform and Z-transform?

Ans. The relation between Z-transform and discrete-time Fourier transform is: The Z-transform of $x(n)$ is same as the discrete-time Fourier transform of $x(n)\, r^{-n}$, i.e. $Z[x(n)] = \text{DTFT}\,[x(n) r^{-n}]$.

6. When are the Z-transform and discrete-time Fourier transform same?

Ans. In general, $Z[x(n)] = \text{DTFT}\,[x(n) r^{-n}]$

when $r = 1$, $Z[x(n)] = \text{DTFT}\,[x(n)]$, i.e. the discrete-time Fourier transform is the Z-transform evaluated along the unit circle centred at the origin of the z-plane.

7. How do you get the DTFT from the Z-transform?

Ans. The DTFT can be obtained from the Z-transform by substituting $z = e^{j\omega}$ in $X(z)$.

8. What is ROC of Z-transform?

Ans. The range of values of $|z|$ for which $X(z)$ converges is called ROC of $X(z)$.

9. What is the ROC of an infinite duration causal sequence?

Ans. The ROC of an infinite duration causal sequence is $|z| > \alpha$, i.e. it is the exterior of a circle of radius α where $z = \alpha$ is the largest pole in $X(z)$.

10. What is the ROC of an infinite duration non-causal sequence?

Ans. The ROC of an infinite duration non-causal sequence is $|z| < \alpha$, i.e. it is the interior of a circle of radius α where $z = \alpha$ is the smallest pole in $X(z)$.

11. What is the ROC of an infinite duration two-sided sequence?

Ans. The ROC of an infinite duration two-sided sequence is a ring in z-plane or the Z-transform does not exist at all.

12. What is the ROC of the sum of two or more sequences?

Ans. The ROC of the sum of two or more sequences is equal to the intersection of the ROCs of these sequences.

13. What is the ROC of a finite duration positive time sequence?
Ans. The ROC of a finite duration positive time sequence is the entire z-plane except at $z = 0$.

14. What is the ROC of a finite duration negative time sequence?
Ans. The ROC of a finite duration negative time sequence is the entire z-plane except at $z = \infty$.

15. What is the ROC of a finite duration two-sided sequence?
Ans. The ROC of a finite duration two-sided sequence is the entire z-plane except at $z = 0$ and $z = \infty$.

16. Define transfer function of a discrete-time system.
Ans. The transfer function of a discrete-time system is defined as the ratio of the Z-transform of the output to the Z-transform of the input when the initial conditions are neglected.

The transfer function of a discrete-time system is also defined as the Z-transform of the impulse response of the system.

17. What is the necessary and sufficient condition for the stability of discrete-time systems?
Ans. The necessary and sufficient condition for the stability of discrete-time systems is its impulse response must be absolutely summable, i.e. $\sum_{n=0}^{\infty} |h(n)| < \infty$.

REVIEW QUESTIONS

1. Derive the relation between Z-transform and DTFT.

2. Distinguish between one-sided and two-sided Z-transforms. What are their applications?

3. Derive the relation between Laplace and Z-transforms.

4. Write the properties of ROC of $X(z)$.

5. State and prove initial and final value theorems of Z-transforms.

6. Define inverse Z-transform. Explain in detail the different methods of finding the inverse Z-transform.

7. Explain how the analysis of discrete-time time-invariant systems can be obtained using convolution properties of Z-transforms.

8. Prove that for causal sequences, the ROC is the exterior of a circle of radius r.

FILL IN THE BLANKS

1. The Z-transform converts _______ equations into _______ equations.

2. The Z-transform plays the same role for _______ systems as that played by Laplace transform for _______ systems.

3. The range of values of z for which $X(z)$ converges is called the _______.
4. The Z-transform of $x(n)$ is same as the discrete-time Fourier transform of _______.
5. The DTFT is same as the Z-transform when _______.
6. The ROC of the sum of two or more sequences is equal to the _______ of the ROCs of those sequences.
7. The ROC of a causal sequence is the _______ of a circle of radius α.
8. The ROC of a non-causal sequence is the _______ of a circle of radius α.
9. If $X(z)$ is rational, its ROC is bounded by _______ or extends upto _______.
10. Initial value theorem states that for a causal signal $x(0)$ = _______.
11. Final value theorem states that for a causal signal $x(\infty)$ = _______.
12. We cannot obtain a _______ sided sequence by long division.
13. The frequency response of a system is obtained by substituting _______ in $H(z)$.
14. For a causal LTI system to be stable, all the poles of $H(z)$ must lie _______ in the z-plane.
15. For a causal LTI system to be stable, the ROC of the system function must include the _______.
16. The response of the system due to input alone, when the initial conditions are neglected is called the _______ of the system.
17. The response of the system due to initial conditions alone, when the input is neglected is called the _______ of the system.
18. The response due to input and initial conditions considered simultaneously is called the _______ of the system.
19. The output due to unit sample sequence is called the _______ of the system.

OBJECTIVE TYPE QUESTIONS

1. The ROC of Z-transform of a sequence $\left(\frac{4}{5}\right)^n u(n) - \left(\frac{5}{4}\right)^n u(n)$ must be

   (a) $|z| < \frac{4}{5}$ (b) $|z| > \frac{5}{4}$ (c) $\frac{4}{5} < |z| < \frac{5}{4}$ (d) $\frac{5}{4} < |z| < \frac{4}{5}$

2. The inverse Z-transform of $2 + 3z^{-1} + 4z$ is

   (a) $\begin{bmatrix} 2, 3, 4 \\ \uparrow \quad \end{bmatrix}$ (b) $\begin{bmatrix} 3, 4, 2 \\ \quad\quad \uparrow \end{bmatrix}$ (c) $\begin{bmatrix} 4, 2, 3 \\ \uparrow \end{bmatrix}$ (d) $\begin{bmatrix} 3, 2, 4 \\ \uparrow \end{bmatrix}$

3. Which sequence cannot be the inverse Z-transform of $\left(\dfrac{1}{1-3z^{-1}} - \dfrac{1}{1-4z^{-1}}\right)$?

(a) $3^n u(n) - 4^n u(n)$ (b) $-3^n u(-n-1) + 4^n u(n-1)$

(c) $3^n u(n) + 4^n u(-n-1)$ (d) $-3^n u(-n-1) - 4^n u(n)$

4. Which sequence cannot be the inverse Z-transform of $\left[\dfrac{1}{1-(1/3)z^{-1}} - \dfrac{1}{1-(1/4)z^{-1}}\right]$?

(a) $\left(\dfrac{1}{3}\right)^n u(n) - \left(\dfrac{1}{4}\right)^n u(n)$ (b) $-\left(\dfrac{1}{3}\right)^n u(-n-1) + \left(\dfrac{1}{4}\right)^n u(n-1)$

(c) $\left(\dfrac{1}{3}\right)^n u(n) + \left(\dfrac{1}{4}\right)^n u(-n-1)$ (d) $-\left(\dfrac{1}{3}\right)^n u(-n-1) - \left(\dfrac{1}{4}\right)^n u(n)$

5. A system is described by $H(z) = \dfrac{z(z+1)}{(z-2)(z+2)}$. The initial value of the system is

(a) 1 (b) –1/4 (c) –4 (d) ∞

6. Which one is the ROC of a sequence $x(n) = a^n u(n) - b^n u(-n-1)$, where $a > b$?

(a) $|z| > a$ (b) $|z| < b$ (c) $b < |z| < a$ (d) none

7. The only signal whose ROC is the entire z-plane is

(a) $\delta(n)$ (b) $u(n)$ (c) $r(n)$ (d) a^n

8. If $x(n) \xleftrightarrow{\text{ZT}} X(z)$, then the initial value theorem states that $x(0)$ =

(a) $\underset{z\to 1}{\text{Lt}}\,(z-1)\,X(z)$ (b) $\underset{z\to 0}{\text{Lt}}\,X(z)$

(c) $\underset{z\to \infty}{\text{Lt}}\,X(z)$ (d) $\underset{z\to \infty}{\text{Lt}}\,zX(z)$

9. If $x(n) \xleftrightarrow{\text{ZT}} X(z)$, then the final value theorem states that $x(\infty)$ =

(a) $\underset{z\to 1}{\text{Lt}}\,(z-1)\,X(z)$ (b) $\underset{z\to 0}{\text{Lt}}\,X(z)$

(c) $\underset{z\to \infty}{\text{Lt}}\,X(z)$ (d) $\underset{z\to \infty}{\text{Lt}}\,zX(z)$

10. The Z-transform of a signal with $X(s) = (1/s)$ is

(a) $\dfrac{1}{1-z^{-1}}$ (b) $\dfrac{1}{1-z}$ (c) $\dfrac{1}{1+z^{-1}}$ (d) $\dfrac{z}{1-z}$

PROBLEMS

1. Find the Z-transform and ROC of the following sequences:

(a) $x_1(n) = \{\underset{\uparrow}{2}, 1, 3, -4, 1, 2\}$ (b) $x_2(n) = \{1, 3, -2, 0, 2, \underset{\uparrow}{4}\}$

(c) $x_3(n) = \{2, 4, 1, \underset{\uparrow}{0}, 1, 3, 5\}$

2. Using properties of Z-transform, find the Z-transform of the following sequences.

(a) $x_1(n) = nu(n-1)$ (b) $x_2(n) = n^2 u(n)$

(c) $x_3(n) = n\left(\frac{1}{2}\right)^n u(n)$ (d) $x_4(n) = 2^n \cos 3n\, u(n)$

(e) $x_5(n) = \left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4}n\right) u(n)$ (f) $x_6(n) = \begin{cases} 0; & n < 0 \\ 1; & 0 \le n \le 9 \\ 0; & n > 9 \end{cases}$

(g) $x_7(n) = \left[3\left(\frac{4}{5}\right)^n - \left(\frac{2}{3}\right)^{2n}\right] u(n)$ (h) $x_8(n) = \left(\frac{1}{3}\right)^n u(-n)$

(i) $x_9(n) = \left(\frac{1}{2}\right)^n [u(n) - u(n-8)]$ (j) $x_{10}(n) = 3(2)^n u(-n)$

(k) $x_{11}(n) = n^2 \left(\frac{1}{3}\right)^n u(n-3)$

3. Using power series expansion, find the inverse Z-transform of

(a) $X(z) = \dfrac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$; ROC; $|z| > 1$

(b) $X(z) = \dfrac{1}{1 - 15z^{-1} + 0.5z^{-2}}$; ROC; $|z| < \dfrac{1}{2}$

4. Find the inverse Z-transform of

(a) $X(z) = \dfrac{1}{(1 + z^{-1})^2 (1 - z^{-1})}$; ROC; $|z| > 1$

(b) $X(z) = \dfrac{1 - (1/4)z^{-1}}{1 + (5/6)z^{-1} + (1/6)z^{-2}}$; ROC; $|z| > \dfrac{1}{2}$

(c) $X(z) = \dfrac{1 - (1/3)z^{-1}}{1 - (1/9)z^{-2}}$; ROC; $|z| > \dfrac{1}{3}$

(d) $X(z) = \dfrac{z^3 + z^2 + (3/2)z + (1/2)}{z^3 + (3/2)z^2 + (1/2)z}$; ROC; $|z| > \dfrac{1}{2}$

(e) $X(z) = \dfrac{1 - (1/2)z^{-1}}{1 + (3/4)z^{-1} + (1/8)z^{-2}}$; ROC; $|z| > \dfrac{1}{2}$

(f) $X(z) = \dfrac{4 - 3z^{-1} + 3z^2}{(1 + 2z^{-1})(1 - 3z^{-1})^2}$

(g) $X(z) = \dfrac{3 - 2z^{-1} + z^2}{(1 - z^{-1})[1 - (1/3)z^{-1}]^2}$

5. Using partial fraction expansion method, obtain all possible inverse Z-transforms of the following $X(z)$:

(a) $X(z) = \dfrac{(1/4)z}{[z - (1/2)][z - (1/4)]}$ (b) $\dfrac{z^2 - 3z}{z^3 + (3/2)z - 1}$

(c) $X(z) = \dfrac{1 + 2z^{-1} + z^{-2}}{1 - (3/2)z^{-1} + (1/2)z^{-2}}$ (d) $X(z) = \dfrac{4 - 3z^{-1} + 3z^{-2}}{(z + 2)(z - 3)^2}$

6. Using convolution theorem, find the inverse Z-transform for

(a) $X(z) = \dfrac{z}{(z - 1)^2}$ (b) $X(z) = \dfrac{z}{(z - 1)^3}$

7. Determine the inverse Z-transform of

$$X(z) = \frac{1}{1 - z^{-3}}$$

8. A causal discrete-time LTI system is to be designed with the property that if the input is $x(n) = \left(\dfrac{1}{2}\right)^n u(n) - \left(\dfrac{1}{4}\right)\left(\dfrac{1}{2}\right)^{n-1} u(n-1)$, then the output is $y(n) = \left(\dfrac{1}{3}\right)^n u(n)$.

Determine the impulse response $h(n)$ and the system function $H(z)$ of that system.

9. A system has impulse response $h(n) = \left(\dfrac{1}{2}\right)^n u(n)$. Determine the input to the system if the output is given by $y(n) = \dfrac{1}{3}u(n) + \dfrac{2}{3}\left(-\dfrac{1}{2}\right)^n u(n)$.

10. Determine whether the system described below is causal and stable.

(a) $H(z) = \dfrac{2z + 1}{z^2 + z - (5/16)}$ (b) $H(z) = \dfrac{1 + 2z^{-1}}{1 + (14/8)z^{-1} + (49/64)z^{-2}}$

11. Find the natural response of the system described by the difference equation $y(n) - \frac{1}{4}y(n-1) - \frac{1}{8}y(n-2) = x(n) + x(n-1)$ with $y(-1) = 0$ and $y(-2) = 1$.

12. Determine the forced response of the system described by the difference equation

$$y(n) - \frac{5}{6}y(n-1) + \frac{1}{6}y(n-2) = x(n)$$

if input $x(n) = 2^n u(n)$, $y(-1) = 2$ and $y(-2) = 1$.

13. Solve the following difference equation with $x(n) = u(n)$ and the initial condition $y(-1) = 1$.

14. Solve the following difference equation for the given initial conditions and input

$$y(n) - \frac{1}{9}y(n-2) = x(n-1)$$

with $y(-1) = 0$, $y(-2) = 1$, and $x(n) = 3u(n)$.

15. Solve the following difference equation using unilateral Z-transform:

$$y(n) - \frac{3}{2}y(n-1) + \frac{1}{2}y(n-2) = x(n) \text{ for } n \ge 0$$

with initial conditions $y(-1) = 4$, $y(-2) = 10$ and $x(n) = (1/4)^n u(n)$.

16. Determine the step response of the system

$$y(n) = \alpha y(n-1) + x(n)$$

with the initial condition $y(-1) = 1$, $-1 < \alpha < 1$.

17. Find the impulse response and step response of the following system:

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n)$$

18. Find the output $y(n)$ of an LTI discrete-time system specified by the following equation:

$$y(n) - \frac{3}{2}y(n-1) + \frac{1}{2}y(n-2) = 2x(n) + \frac{3}{2}x(n-1)$$

if the initial conditions are $y(-1) = 0$, $y(-2) = 1$ and $x(n) = (1/4)^n u(n)$.

19. Find the response of

$$y(n) + y(n-1) - 2y(n-2) = u(n-1) + 2u(n-2)$$

due to $y(-1) = 0.5$, $y(-2) = 0.25$.

20. Solve the following difference equation

$$y(n) - y(n-1) + \frac{1}{4}\,y(n-2) = x(n)$$

where $x(n) = 2\left(\frac{1}{8}\right)^n u(n);\ y(-1) = 2$ and $y(-2) = 1$

21. A causal system is represented by the following difference equation

$$y(n) + \frac{1}{4}y(n-1) = x(n) + \frac{1}{2}x(n-1)$$

(a) Find the system function $H(z)$ and give the corresponding ROC.
(b) Find the unit sample response of the system.
(c) Find the frequency response $H(\omega)$ and determine its magnitude and phase.

22. Find the output of the system whose input and output are related by $y(n) = 7y(n-1) - 12y(n-2) + 2x(n) - x(n-2)$ for the input $x(n) = u(n)$.

MATLAB PROGRAMS

Program 3.1

% Z-transform and Inverse Z-transform of given signals

```
clc; clear all; close all;
syms n wo
% first signal
a=n+1;
disp('The input equation is')
disp(a)
b=ztrans(a);
disp('The z-transform is')
disp(b)
c=iztrans(b);
disp('The inverse z-transform is')
disp(c)
% second signal
a1=cos(wo*n);
disp('The input equation is')
disp(a1)
b1=ztrans(a1);
disp('The z-transform is')
disp(b1)
c1=iztrans(b1);
disp('The inverse z-transform is')
disp(c1)
```

Output:

```
The input equation is
n + 1
The z-transform is
z/(z – 1) + z/(z – 1)^2
The inverse z-transform is
n + 1
```

The input equation is
cos(n*wo)
The z-transform is
(z*(z – cos(wo)))/(z^2 – 2*cos(wo)*z + 1)
The inverse z-transform is
cos(n*acos(cos(wo)))

Program 3.2

% Finding the residues of Z^3/((z–0.5)*(z–0.75)*(z–1))

```
clc; clear all; close all;
syms z
d=(z–0.5)*(z–0.75)*(z–1);% The denominator of F(z)
a1=collect(d);
den=sym2poly(a1);
num=[0 1 0 0];
[num1,den1]=residue(num,den);
fprintf('r1 = %4.2f \t', num(1)); fprintf('p1 = %4.2f \t', den(1))
fprintf('r2 = %4.2f \t', num(2)); fprintf('p2 = %4.2f \t', den(2))
fprintf('r3 = %4.2f \t', num(3)); fprintf('p3 = %4.2f \t', den(3))
```

Output:

```
r1 = 0.00    p1 = 1.00
r2 = 1.00    p2 = –2.25
r3 = 0.00    p3 = 1.63
```

Program 3.3

% Inverse Z-transform by the polynomial division method

x(z)=(1+2*z^(–1)+z^(–2))/(1–z^(–1)+0.3561*z^(–2))

```
clc; clear all; close all;
b=[1 2 1];
a=[1 –1 0.3561];
n=5; %number of  power series points
b=[b zeros(1,n–1)];
[h,om]=deconv(b,a);
disp('The terms of inverse z-transforms')
disp(h)
```

Output:

```
The terms of  inverse z-transforms

   1.0000    3.0000    3.6439    2.5756    1.2780
```

Program 3.4

% Inverse z-transform for the cascaded form using polynomial division method

%x(Z)=[N1(Z)*N2(Z)*N3(Z)]/[D1(z)*D2(z)*D3(Z)]
%N1(z)=1–0.22346Z^(–1)+z^(–2)
%N2(z)=1–0.437883Z^(-1)+z^(–2)
%N3(z)=1+z^(–1)
%D1(z)=1–1.433509Z^(–1)+0.85811z^(–2)
%D2(z)=1–1.293601Z^(–1)+0.556929z^(–2)
%D3(z)=1–0.612159Z^(–1)

```
clc; clear all; close all;
n=5;% number of power series points
n1=[1 –0.22346 1];
n2=[1 –0.4377883 1];
n3=[1 1 0];
d1=[1 –1.433509 0.85811];
```

```
d2=[1 –1.293601 0.556929];
d3=[1 –0.612159 0];
b=[n1; n2; n3];
a=[d1; d2; d3];
[b,a]=sos2tf([b a]);
b=[b zeros(1,n–1)];
[x r]=deconv(b,a);
disp('The first five values of inverse Z-transform are')
disp(x)
```

Output:

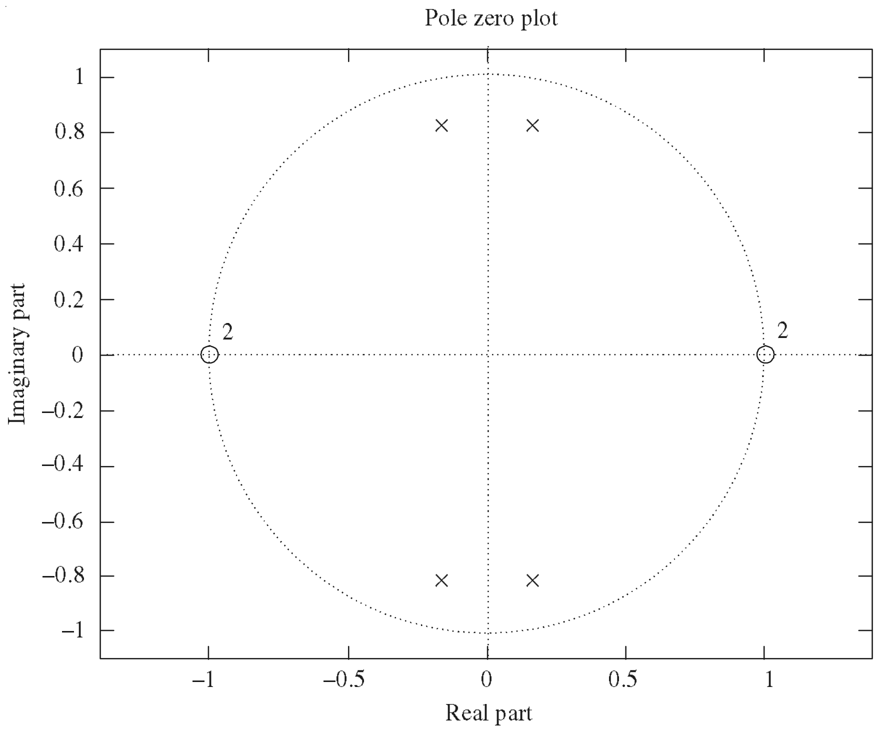
The first five values of inverse Z-transform are

```
1.0000    3.6780    8.7796   16.4987   24.8055
```

Program 3.5

```
% Pole-zero plot of a Butterworth band pass filter
clc; clear all; close all;
fs=1000;%sampling frequency
alphap=3;
alphas=20;
wp=[200/500 300/500];
ws=[50/500 450/500];
[n wc]=buttord(wp,ws,alphap,alphas);
[z p k]=butter(n,wp);
zplane(z,p)
title('Pole Zero plot')
```

Output:



Program 3.6

% Convolution using Z-transform

```
clc; clear all; close all;
x1=[2 1 0 –1 3];%x1=2+z^(–1)–z^(–3)+3*z^(–4)
x2=[1 –3 2];%x2=1–3*z^(–1)+2*z^(–2)
x3=conv(x1,x2)  % x3=x1*x2;
```

Output:

```
x3 =

    [2   –5    1    1    6   –11    6]
```

4

System Realization

4.1 INTRODUCTION

Systems may be continuous-time systems or discrete-time systems. Discrete-time systems may be FIR (Finite Impulse Response) systems or IIR (Infinite Impulse Response) systems. FIR systems are the systems whose impulse response has finite number of samples and IIR systems are systems whose impulse response has infinite number of samples. Realization of a discrete-time system means obtaining a network corresponding to the difference equation or transfer function of the system. In this chapter, various methods of realization of discrete-time systems are discussed.

4.2 REALIZATION OF DISCRETE-TIME SYSTEMS

To realize a discrete-time system, the given difference equation in time domain is to be converted into an algebraic equation in z-domain, and each term of that equation is to be represented by a suitable element (a constant multiplier or a delay element). Then using adders, all the elements representing various terms of the equation are to be connected to obtain the output. The symbols of the basic elements used for constructing the block diagram of a discrete-time system (adder, constant multiplier and unit delay element) are shown in Figure 4.1.

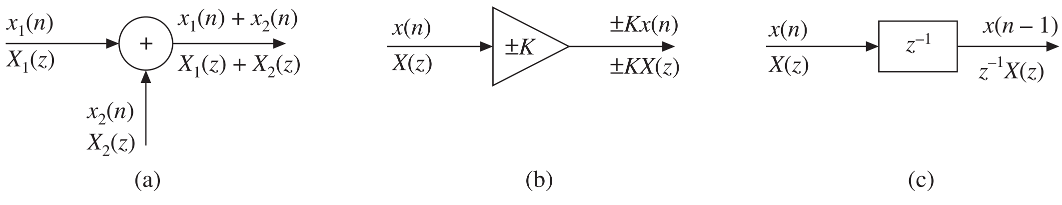


Figure 4.1 (a) Adder (b) Constant multiplier and (c) Unit delay element.

Adder: An adder is used to add two or more signals. The output of adder is equal to the sum of all incoming signals.

Constant multiplier: A constant multiplier is used to multiply the signals by a constant. The output of the multiplier is equal to the product of the input signal and the constant of the multiplier.

Unit delay element: A unit delay element is used to delay the signal passing through it by one sampling time.

EXAMPLE 4.1 Construct the block diagram for the discrete-time systems whose input-output relations are described by the following difference equations:

(a) $y(n) = 0.7x(n) + 0.3x(n-1)$

(b) $y(n) = 0.5y(n-1) + 0.8x(n) + 0.4x(n-1)$

Solution:

(a) Given $$y(n) = 0.7x(n) + 0.3x(n-1)$$

The system may be realized by using the difference equation directly or by using the Z-transformed version of that. The individual terms of the given difference equation are $0.7x(n)$ and $0.3x(n-1)$. They are represented by the basic elements as shown in Figure 4.2.

Alternatively

Taking Z-transform on both sides of the given difference equation, we have

$$Y(z) = 0.7X(z) + 0.3z^{-1}X(z)$$

The individual terms of the above equation are: $0.7X(z)$ and $0.3z^{-1}X(z)$. They are represented by the basic elements as shown in Figure 4.2.

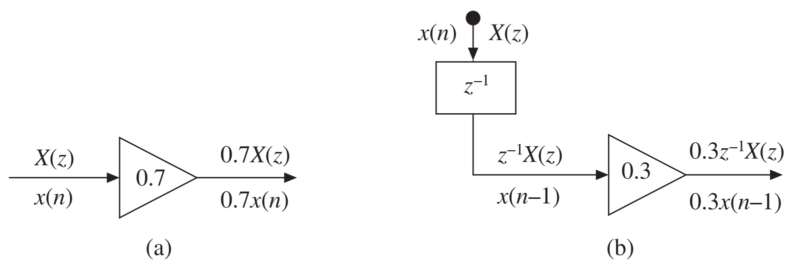


Figure 4.2 Block diagram representation of (a) $0.7X(z)$ and (b) $0.3z^{-1}X(z)$.

The input to the system is $X(z)$ [or $x(n)$] and the output of the system is $Y(z)$ [or $y(n)$]. The above elements are connected as shown in Figure 4.3 to get the output $Y(z)$ [or $y(n)$].

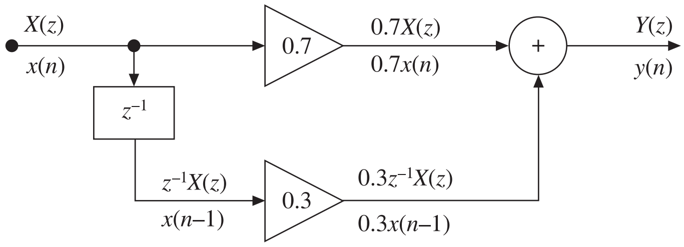


Figure 4.3 Realization of system described by $y(n) = 0.7x(n) + 0.3x(n-1)$.

(b) Given $\quad y(n) = 0.5y(n-1) + 0.8x(n) + 0.4x(n-1)$

The individual terms of the above equations are $0.5y(n-1)$, $0.8x(n)$ and $0.4x(n-1)$. They are represented by the basic elements as shown in Figure 4.4.

Alternatively

Taking Z-transform on both sides of the given difference equation, we have

$$Y(z) = 0.5z^{-1}Y(z) + 0.8X(z) + 0.4z^{-1}X(z)$$

The individual terms of the above equation are $0.5z^{-1}Y(z)$, $0.8X(z)$ and $0.4z^{-1}X(z)$. They are represented by the basic elements as shown in Figure 4.4.

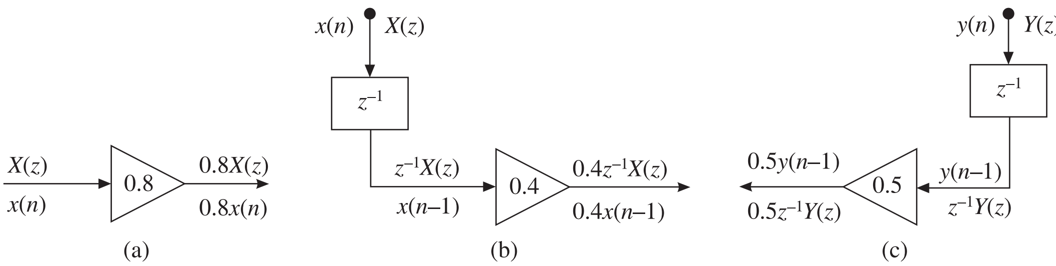


Figure 4.4 Block diagram representation of (a) $0.8X(z)$ (b) $0.4z^{-1}X(z)$ and (c) $0.5z^{-1}Y(z)$.

The input to the system is $X(z)$[or $x(n)$] and the output of the system is $Y(z)$[or $y(n)$]. The above elements are connected as shown in Figure 4.5 to get the output $Y(z)$[or $y(n)$].

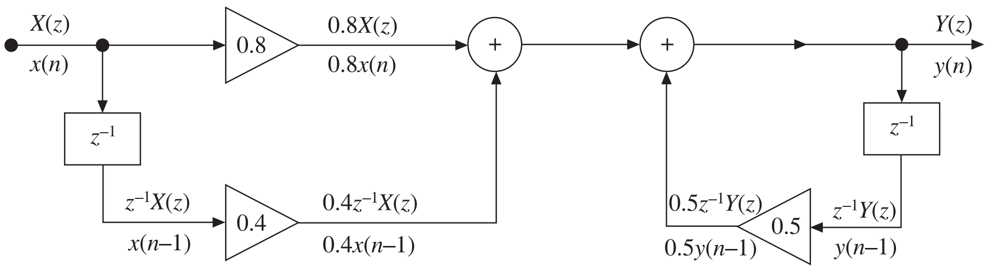


Figure 4.5 Realization of the system described by $y(n) = 0.5y(n-1) + 0.8x(n) + 0.4x(n-1)$.

Discrete-time LTI systems may be divided into two types: IIR systems (those that have an infinite duration impulse response) and FIR systems (those that have a finite duration impulse response).

4.3 STRUCTURES FOR REALIZATION OF IIR SYSTEMS

IIR systems are systems whose impulse response has infinite number of samples. They are designed by using all the samples of the infinite duration impulse response. The convolution formula for IIR systems is given by

$$y(n) = \sum_{k=0}^{\infty} h(k)\,x(n-k)$$

Since this weighted sum involves the present and all the past input samples, we can say that the IIR system has an infinite memory.

A system whose output $y(n)$ at time n depends on the present input and any number of past values of input and output is called a recursive system. The past outputs are

$$y(n-1),\ y(n-2),\ y(n-3),\ \ldots$$

Hence, for recursive system, the output $y(n)$ is given by

$$y(n) = \mathrm{F}[y(n-1), y(n-2), \ldots, y(n-N), x(n), x(n-1), \ldots, x(n-M)]$$

In recursive system, in order to compute $y(n_0)$, we need to compute all the previous values $y(0)$, $y(1)$, $y(2)$, ..., $y(n_0 - 1)$ before calculating $y(n_0)$. Hence, output of recursive system has to be computed in order $[y(0), y(1), y(2), \ldots]$.

Transfer function of IIR system

In general, an IIR system is described by the difference equation

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

i.e. in general, IIR systems are those in which the output at any instant of time depends not only on the present and past inputs but also on the past outputs. Hence, in general, an IIR system is of recursive type.

On taking Z-transform of the above equation for $y(n)$, we get

$$Y(z) = -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z)$$

i.e.

$$Y(z) + \sum_{k=1}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

The system function or the transfer function of the IIR system is:

$$\frac{Y(z)}{X(z)} = H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

The above equations for $Y(z)$ and $H(z)$ can be viewed as a computational procedure (or algorithm) to determine the output sequence $y(n)$ from the input sequence $x(n)$. The computations in the above equation can be arranged into various equivalent sets of difference equations with each set of equations defining a computational procedure or algorithm for implementing the system.

For each set of equations, we can construct a block diagram consisting of delays, adders and multipliers. Such block diagrams are referred to as realization of the system or equivalently as structure for realizing the system.

The main advantage of re-arranging the sets of difference equations (i.e. the main criteria for selecting a particular structure) is to reduce the computational complexity, memory requirements and finite word length effects in computations.

So the factors that influence the choice of structure for realization of LTI system are: computational complexity, memory requirements and finite word length effects in computations.

Computational complexity refers to the number of arithmetic operations required to compute the output value $y(n)$ for the system.

Memory requirements refer to the number of memory locations required to store the system parameters, past inputs and outputs and any intermediate computed values.

Finite-word-length effects or finite precision effects refer to the quantization effects that are inherent in any digital implementation of the system either in hardware or in software.

Although the above three factors play a major role in influencing our choice of the realization of the system, other factors such as whether the structure lends itself to parallel processing or whether the computations can be pipelined may play a role in selecting a specific structure.

The different types of structures for realizing IIR systems are:

1. Direct form-I structure
2. Direct form-II structure
3. Transposed form structure
4. Cascade form structure
5. Parallel form structure
6. Lattice structure
7. Ladder structure

4.3.1 Direct Form-I Structure

Direct form-I realization of an IIR system is nothing, but the direct implementation of the difference equation or transfer function. It is the simplest and most straight forward realization structure available.

The difference equation governing the behaviour of an IIR system is

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

i.e. $y(n) = -a_1 y(n-1) - a_2 y(n-2) - \cdots - a_N y(n-N) + b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M)$

On taking the Z-transform of the above equation for $y(n)$, we get

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \cdots - a_N z^{-N} Y(z) + b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_M z^{-M} X(z)$$

The equation for $Y(z)$ [or $y(n)$] can be directly represented by a block diagram as shown in Figure 4.6 and this structure is called Direct form-I structure. This structure uses separate delays (z^{-1}) for input and output samples. Hence, for realizing this structure more memory is required. The direct form structure provides a direct relation between time domain and z-domain equations.

The structure shown in Figure 4.6 is called a ***non-canonical structure*** because the number of delay elements used is more than the order of the difference equation.

If the IIR system is more complex, that is of higher order, then introduce an intermediate variable $W(z)$ so that

$$W(z) = \sum_{k=0}^{M} b_k z^{-k} X(z) = b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_M z^{-M} X(z)$$

or $$w(n) = \sum_{k=0}^{M} b_k x(n-k) = b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-m)$$

$\therefore$ $$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \cdots + W(z)$$

or $$y(n) = -a_1 y(n-1) - a_2 y(n-2) - \cdots + w(n)$$

So, the direct form-I structure is in two parts. The first part contains only zeros [that is, the input components either $x(n)$ or $X(z)$] and the second part contains only poles [that is, the output components either $y(n)$ or $Y(z)$]. In direct form-I, the zeros are realized first and poles are realized second.

Limitations of direct form-I

- Since the number of delay elements used in direct form-I is more than (double) the order of the difference equation, it is not effective.
- It lacks hardware flexibility.
- There are chances of instability due to the quantization noise.

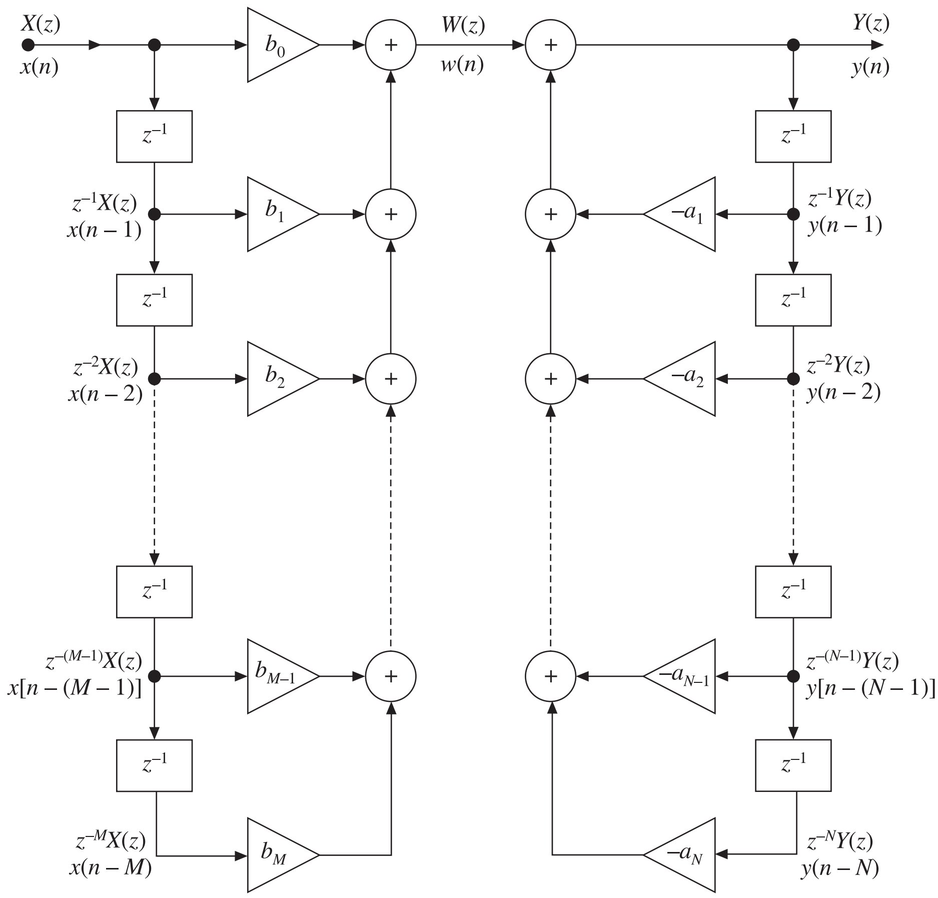


Figure 4.6 Direct form-I structure.

4.3.2 Direct Form-II Structure

The Direct form-II structure is an alternative to direct form-I structure. It is more advantageous to use direct form-II technique than direct form-I, because it uses less number of delay elements than the direct form-I structure.

In direct form-II, an intermediate variable is introduced and the given transfer function is split into two, one containing only poles and the other containing only zeros. The poles [that is, the output components $y(n)$ or $Y(z)$ which is the denominator part of the transfer function] are realized first and the zeros [that is, the input components either $x(n)$ or $X(z)$, which is the numerator part of the transfer function] second.

If the coefficient of the present output sample $y(n)$ or the non-delay constant at denominator is non unity, then transform it to unity. The systematic procedure is given as follows:

Consider the general difference equation governing an IIR system

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

i.e.
$$y(n) = -a_1 y(n-1) - a_2 y(n-2) - a_3 y(n-3) - \cdots - a_N y(n-N)$$
$$+ b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \cdots + b_M x(n-M)$$

On taking Z-transform of the above equation and neglecting initial conditions, we get

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \cdots - a_N z^{-N} Y(z) + b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_M z^{-M} X(z)$$

i.e. $$Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + \cdots + a_N z^{-N} Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_M z^{-M} X(z)$$

i.e. $$Y(z)[1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}] = X(z)[b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}]$$

i.e. $$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

Let $$\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \frac{W(z)}{X(z)}$$

where $$\frac{W(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

and $$\frac{Y(z)}{W(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}$$

On cross multiplying the above equations, we get

$$W(z) + a_1 z^{-1} W(z) + a_2 z^{-2} W(z) + \cdots + a_N z^{-N} W(z) = X(z)$$

$\therefore$ $$W(z) = X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z) - \cdots - a_N z^{-N} W(z)$$

and $$Y(z) = b_0 W(z) + b_1 z^{-1} W(z) + b_2 z^{-2} W(z) + \cdots + b_M z^{-M} W(z)$$

The realization of an IIR system represented by these equations in direct form-II is shown in Figure 4.7.

Advantage of the direct form-II over the direct form-I

The number of delay elements used in direct form-II is less than that of direct form-I.

Limitations of direct form-II

- It also lacks hardware flexibility
- There are chances of instability due to the quantization noise

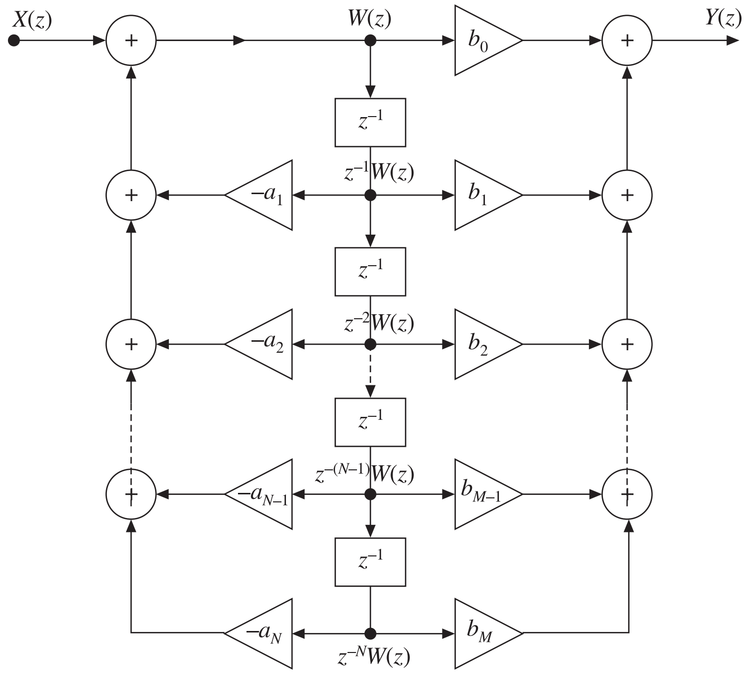


Figure 4.7 Direct form-II structure of IIR system for $M = N$.

Since the number of delay elements used in direct form-II is the same as that of the order of the difference equation, direct form-II is called a ***canonical structure.***

The comparison of direct form-I and direct form-II structures is given in Table 4.1

TABLE 4.1 Comparison of direct form-I and direct form-II structures

Direct form-I structure	*Direct form-II structure*
This realization uses separate delays (memory) for both the input and output signal samples.	This realization uses a single delay (memory) for both the input and output signal samples.
For the $(M-1)$th or $(N-1)$th order IIR system, direct form-I requires $M+N-1$ multipliers, $M+N-2$ adders and $M+N-2$ delays.	For the $(M-1)$th or $(N-1)$th order IIR system, direct form-II requires $M+N-1$ multipliers, $M+N-2$ adders and max $[(M-1), (N-1)]$ delays.
It is also called non-canonical, because it requires more number of delays.	It is called canonical, because it requires a minimum number of delays.
It is not efficient in terms of memory requirements compared to direct form-II.	It is more efficient in terms of memory requirements.
Direct form-I can be viewed as two linear time-invariant systems in cascade. The first one is non-recursive and the second one recursive.	Direct form-II can also be viewed as two linear time-invariant systems in cascade. The first one is recursive and the second one non-recursive.

Conversion of direct form-I structure to direct form-II structure

The direct form-I structure can be converted to direct form-II structure by considering the direct form-I structure as cascade of two systems $H_1(z)$ and $H_2(z)$ as shown in Figure 4.8(a). By linearity property, the order of cascading can be interchanged as shown in Figure 4.8(b).

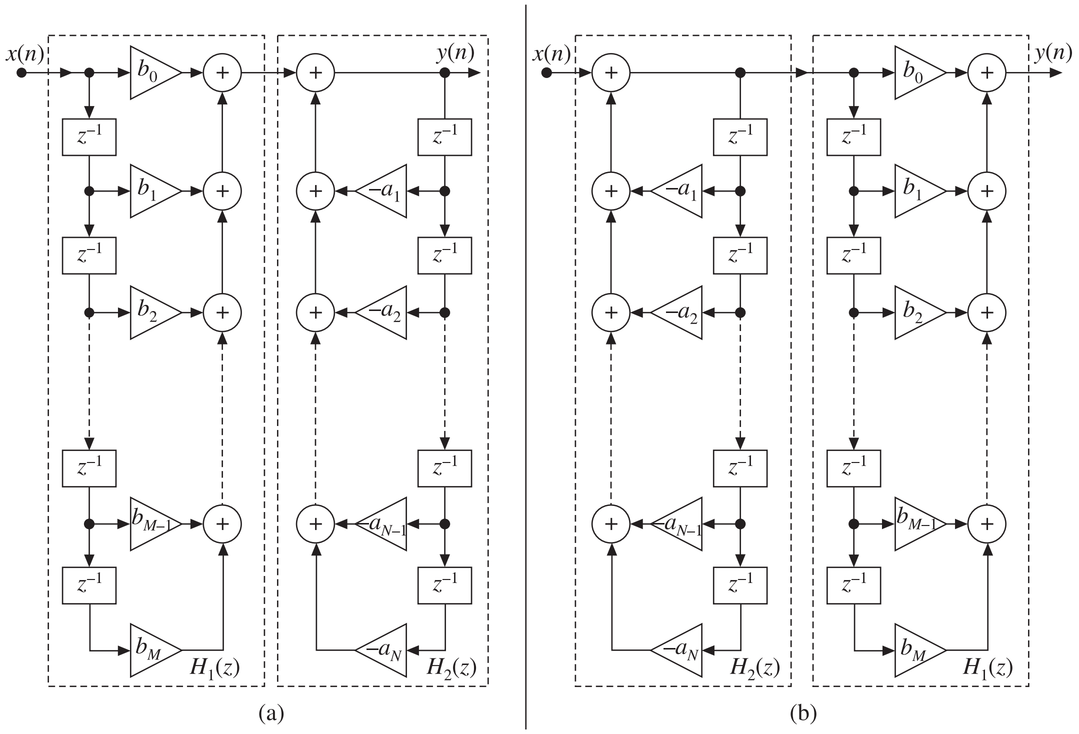


Figure 4.8 (a) Direct form-I structure as cascade of two systems (b) Direct form-I structure after interchanging the order of cascading.

In Figure 4.8(b), we can observe that the inputs to the delay elements in $H_1(z)$ and $H_2(z)$ are the same and so the outputs of the delay elements in $H_1(z)$ and $H_2(z)$ are same. Therefore, instead of having separate delays for $H_1(z)$ and $H_2(z)$, a single set of delays can be used. Hence, the delays can be merged to combine the cascaded systems to a single system. The resultant structure will be direct form-II structure as that of Figure 4.7. The process of converting direct form-I structure to direct form-II structure is shown in Figure 4.9.

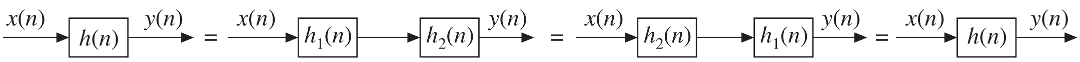


Figure 4.9 Conversion of direct form-I structure to direct form-II.

4.3.3 Transposed Form Structure Realization of IIR System

It is practically true that if we reverse the direction of all the branch transmittances and interchange the input and the output in the structure or signal flow graph, the system remains unchanged. The transposed structure or transpose form or reverse structure is obtained by reversing the direction of all branch transmittances and interchanging the input and output in the direct form structure.

The transposed structure remains valid, provided:

1. The branch transmittances are untouched.
2. The direction of all the branches in the structure is reversed.
3. The roles of the input and output are reversed.

By these steps, the system function remains unchanged.

Procedure to realize transposed form structure of IIR system

1. First realize the given difference equation or transfer function by using the direct form structure.
2. Reverse or transpose the direction of signal flow and interchange the input and output nodes.
3. Replace the junction points by adders and adders by junction points.
4. Fold the structure, which is the transposed form realization of an IIR system.

In general, the transposed structure realization of an IIR system is advantageous only if it is implemented over a direct form-II structure, because the number of components used or the number of additions and multiplications get reduced when it is used over a direct form-II structure. However, the higher the order of the systems, the better would be its advantages. It has no advantage for a direct form-I structure.

The realization of a general *N*th order IIR system in Direct form-I is shown in Figure 4.6. Its transposed version is shown in Figure 4.10(a).

The realization of a general *N*th order IIR system in Direct form-II is shown in Figure 4.7. Its transposed version is shown in Figure 4.10(b).

EXAMPLE 4.2 Realize an FIR system

$$y(n) + 2y(n-1) + 3y(n-2) = 4x(n) + 5x(n-1) + 6x(n-2)$$

using the transposed form structure.

Solution: Taking Z-transform on both sides of the given difference equation and neglecting initial conditions, we get

$$Y(z) + 2z^{-1}Y(z) + 3z^{-2}Y(z) = 4X(z) + 5z^{-1}X(z) + 6z^{-2}X(z)$$

Therefore, the transfer function of the given IIR system is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{4 + 5z^{-1} + 6z^{-2}}{1 + 2z^{-1} + 3z^{-2}}$$

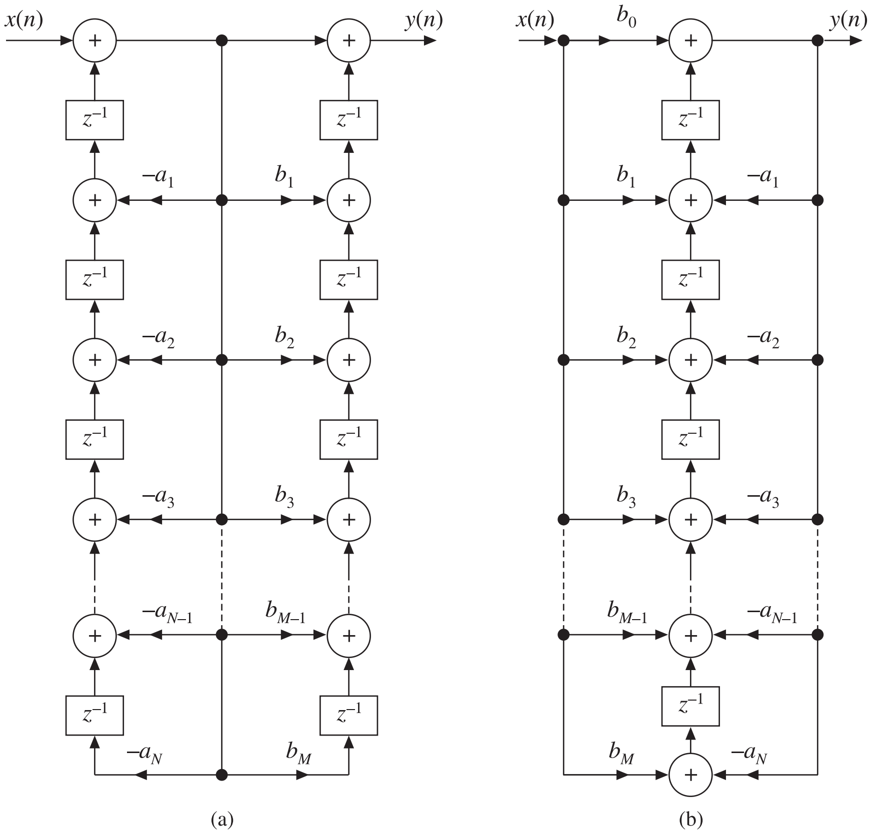


Figure 4.10 (a) General transposed structure realization of IIR system through direct form-I, (b) General transposed structure realization of IIR system through direct form-II.

The direct form-II realization structure, the recovered realization structure and the transposed form realization structure of this system are shown in Figure 4.11[(a), (b) and (c) respectively].

$$a_1 = 2, a_2 = 3, b_0 = 4, b_1 = 5, b_2 = 6$$

4.3.4 Cascade Form Realization

The cascade form structure is nothing, but a cascaded or series interconnection of the sub transfer functions or sub system functions which are realized by using the direct form structures (either direct form-I or direct form-II or a combination of both).

Hence, in cascade form realization, the given transfer function $H(z)$ is expressed as a product of a number of second order or first order sections as indicated below:

$$H(z) = \frac{Y(z)}{X(z)} = \prod_{i=1}^{k} H_i(z)$$

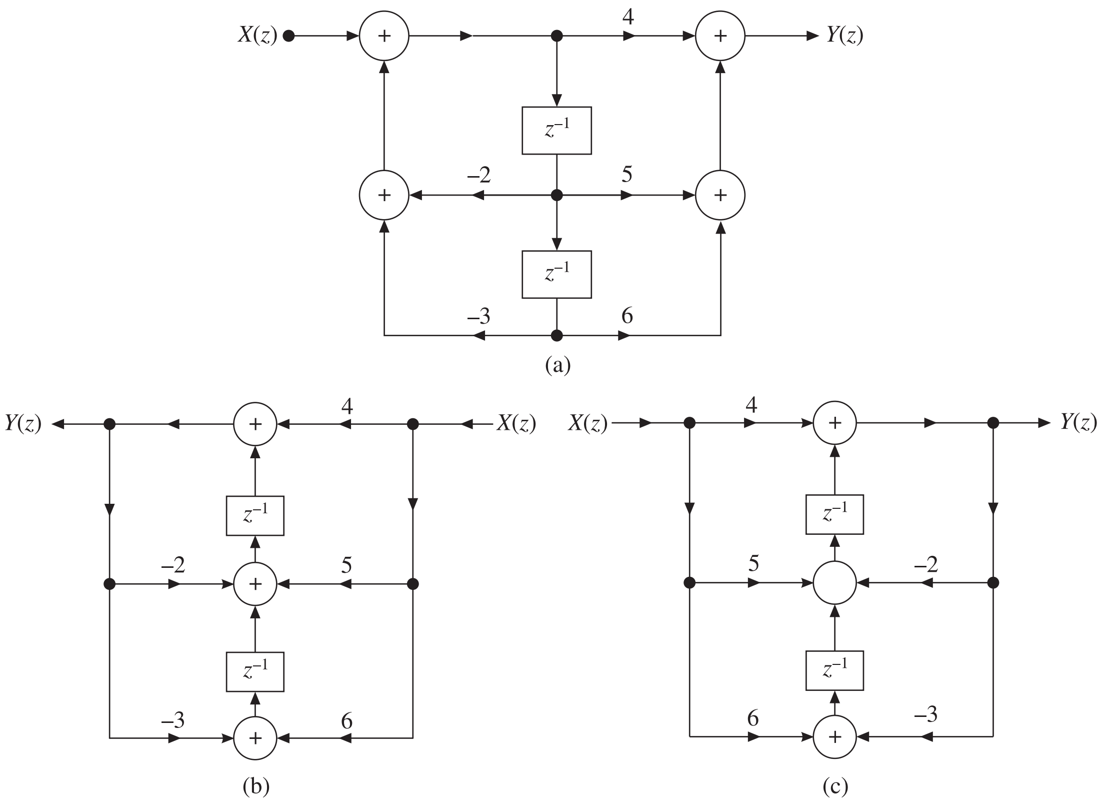


Figure 4.11 (a) Direct form-II realization (b) Recovered realization structure and (c) Transposed realization structure (Example 4.2).

where
$$H_i(z) = \frac{C_{0i} + C_{1i}z^{-1} + C_{2i}z^{-2}}{d_{0i} + d_{1i}z^{-1} + d_{2i}z^{-2}} \quad \text{[second order section]}$$

or
$$H_i(z) = \frac{C_{0i} + C_{1i}z^{-1}}{d_{0i} + d_{1i}z^{-1}} \quad \text{[first order section]}$$

Each of these sections is realized separately and all of them are connected in cascade (series). Therefore, the cascade form realization is also called a series structure in which one sub transfer function is the input to the other transfer function and so on.

The cascade form realization is shown in Figure 4.12.

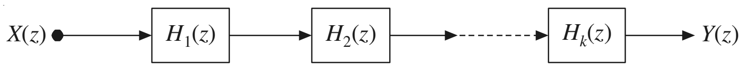


Figure 4.12 Cascade realization of IIR system.

The difficulties in cascade structure are:

1. Decision of pairing poles and zeros.
2. Deciding the order of cascading the first and second order sections.
3. Scaling multipliers should be provided between individual sections to prevent the filter variables from becoming too large or too small.

4.3.5 Parallel Form Realization

Parallel form structure is nothing, but the parallel connection of sub-transfer functions or sub-system functions, which is decomposed by using the partial fraction method.

In parallel form realization, by partial fraction expansion, the transfer function $H(z)$ is expressed as a sum of first and second order sections.

$$H(z) = \frac{Y(z)}{X(z)} = C + \sum_{i=1}^{k} H_i(z)$$

where, $$H_i(z) = \frac{C_{0i} + C_{1i}z^{-1}}{d_{0i} + d_{1i}z^{-1} + d_{2i}z^{-2}} \quad \text{[second order section]}$$

or $$H_i(z) = \frac{C_{0i}}{d_{0i} + d_{1i}z^{-1}} \quad \text{[first order section]}$$

Each first order and second order section is realized either in direct form-I structure or in direct form-II structure and the individual sections are connected in parallel to obtain the over all system as shown in Figure 4.13. As the filter operation is performed in parallel, i.e. the processing is performed simultaneously, the parallel form structure is used for high speed filtering application.

The difficulty with this method is expressing the transfer function in partial fraction form is not easy for higher order systems.

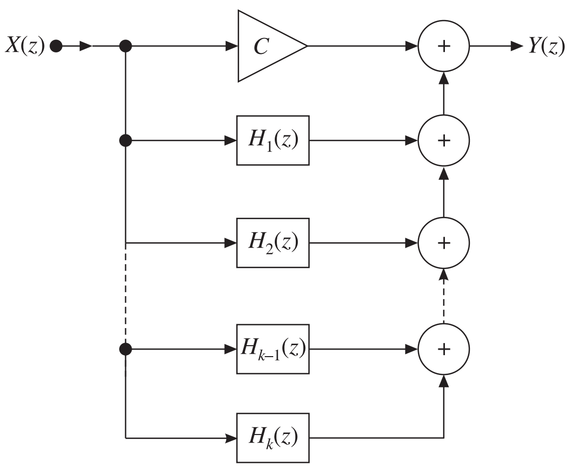


Figure 4.13 Parallel form realization of IIR system.

4.3.6 Lattice Structure Realization of IIR Systems

The IIR system consists of both zeros and poles. Therefore, the poles and zeros will be considered as separate sub-transfer functions in cascade, that is

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \left(\sum_{k=0}^{M} b_k z^{-k}\right)\left(\frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}\right) = H_z(z)\,H_p(z)$$

where $$H_z(z) = \sum_{k=0}^{M} b_k z^{-k} = \text{Zeros}$$

$$H_p(z) = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \text{Poles}$$

The above equation for $H_p(z)$ can be rewritten as

$$H_p(z) = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{Y_p(z)}{X_p(z)}$$

or $$X_p(z) = Y_p(z) + \sum_{k=1}^{N} a_k z^{-k} Y_p(z)$$

Taking the inverse Z-transform on both sides and rearranging it, we get

$$y_p(n) = x_p(n) - \sum_{k=1}^{N} a_k y_p(n-k)$$

The output of the sub-transfer function $H_z(z)$ is the input to the sub-transfer function $H_p(z)$. So $y_z(n) = x_p(n)$.

While realizing the IIR system using the lattice structure, the zeroes, i.e. $H_z(z)$ should be realized first and then the poles, i.e. $H_p(z)$ which is realized in cascade.

For the realization of the poles, a lattice structure consists of two paths, $x_p(n)$ and $x'_p(n)$ through which the input $x_p(n)$ or $y_z(n)$ is processed. However, those two different paths are opposite in direction to each other. A single-stage lattice structure for a pole is shown in Figure 4.14.

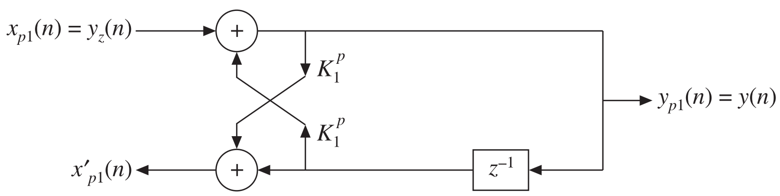


Figure 4.14 Single-stage lattice structures for a pole.

The output from the single-stage lattice structure is

$$y(n) = x_{p1}(n) + K_1^p y(n-1)$$

and the feedback response is

$$x'_{p1}(n) = K_1^p y(n) + y(n-1)$$

Similarly, the two-stage lattice structure for a pole is shown in Figure 4.15.

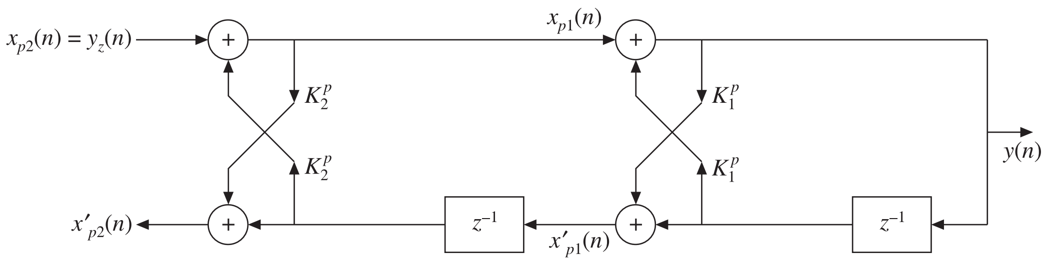


Figure 4.15 Two-stage lattice structure for a pole.

The intermediate output of the two-stage lattice structure is

$$x_{p1}(n) = x_{p2}(n) + K_2^p x'_{p1}(n-1)$$

Substituting the value of $x'_{p1}(n)$ in the expression for $x_p(n)$, we have

$$\begin{aligned} x_{p1}(n) &= x_{p2}(n) + K_2^p [K_1^p y(n-1) + y(n-2)] \\ &= x_{p2}(n) + K_1^p K_2^p y(n-1) + K_2^p y(n-2) \end{aligned}$$

Substituting this value of $x_{p1}(n)$ in the expression for $y(n)$, we get

$$\begin{aligned} y(n) &= x_{p2}(n) + K_1^p K_2^p y(n-1) + K_2^p y(n-2) + K_1^p y(n-1) \\ &= x_{p2}(n) + K_1^p [1 + K_2^p] y(n-1) + K_2^p y(n-2) \end{aligned}$$

Therefore, the overall lattice structure for a second order IIR system (both poles and zeros are of second order) can be in general, realized as shown in Figure 4.16.

In a similar fashion, the $(M-1)$th order or $(N-1)$th order IIR system can be realized by using the lattice structure.

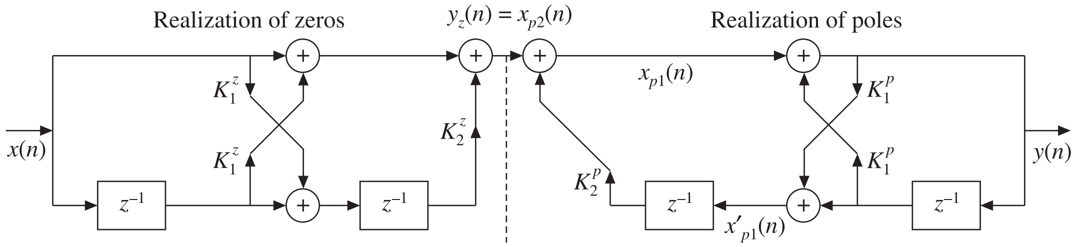


Figure 4.16 A general realization of second order IIR system using the all-zero-all-pole lattice structure.

Procedure to realize the lattice structure of IIR systems

1. Find the order of the difference equation and compare the coefficients with the reflection coefficients $K_1^p, K_2^p, K_3^p, \ldots$ and $K_1^z, K_2^z, K_3^z, \ldots$ of the same order lattice structure output. This applies both for the poles and zeros, separately.
2. Assign the calculated values of the reflection coefficients $K_1^p, K_2^p, K_3^p, \ldots$ and $K_1^z, K_2^z, K_3^z, \ldots$ and construct the structures.
3. Cascade the structures of both zeroes and poles.

EXAMPLE 4.3 Determine the lattice coefficients corresponding to an IIR filter described by $y(n) - \frac{2}{5}y(n-1) + \frac{1}{5}y(n-2) = x(n) + \frac{1}{4}x(n-1)$ and realize it.

Solution: The given system described by the difference equation

$$y(n) = \frac{2}{5}y(n-1) - \frac{1}{5}y(n-2) + x(n) + \frac{1}{4}x(n-1)$$

has first order zeroes and second order poles. Hence, the proposed lattice structure is given in Figure 4.17.

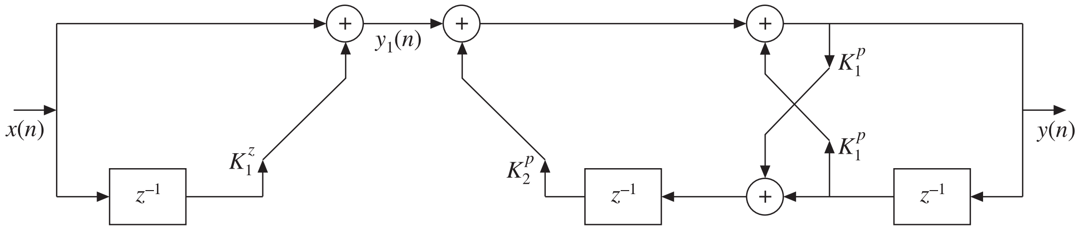


Figure 4.17 A proposed lattice structure (Example 4.3).

The output $y(n)$ from Figure 4.17 is

$$y(n) = y_1(n) + K_1^p[1 + K_2^p]y(n-1) + K_2^p y(n-2)$$

i.e.
$$y(n) = x(n) + K_1^z x(n-1) + K_1^p[1 + K_2^p]y(n-1) + K_2^p y(n-2)$$

Comparing this equation with the given difference equation, we have

$$K_1^z = \frac{1}{4},\ K_2^p = -\frac{1}{5}$$

and

$$K_1^p[1 + K_2^p] = \frac{2}{5}$$

or

$$K_1^p = \frac{2/5}{1-(1/5)} = \frac{1}{2}$$

Hence, the lattice structure realization of the given IIR filter is shown in Figure 4.18.

The realization obtained in Figure 4.18 is called all-zero-all-pole lattice structure realization.

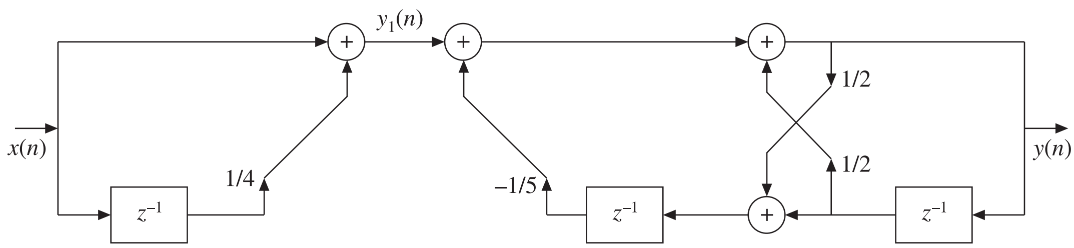


Figure 4.18 All-zero-all-pole lattice structure (Example 4.3).

4.3.7 Ladder Structure Realization of IIR Systems

Ladder structure realization is possible only for an IIR system. In this structure, the numerator polynomial will be divided by the denominator polynomial sequentially and the result substituted at the ladder fashion shown below. Two cases are considered for the realization of the ladder structure.

CASE-I: If the negative order of the numerator polynomial is more than the negative order of the denominator polynomial, that is,

$$H(z) = \frac{b_{N+1}z^{-(N+1)} + \cdots + b_3 z^{-3} + b_2 z^{-2} + b_1 z^{-1} + b_0}{a_N z^{-N} + \cdots + a_3 z^{-3} + a_2 z^{-2} + a_1 z^{-1} + a_0}$$

then, the transfer function of a ladder structure is shown below.

$$H(z) = \alpha_1 z^{-1} + \cfrac{1}{\beta_1 + \cfrac{1}{\alpha_2 z^{-1} + \cfrac{1}{\beta_2 + \cfrac{1}{\alpha_3 z^{-1} + \cfrac{1}{\beta_3 + \cdots}}}}}$$

The ladder structure realization of the above equation for $H(z)$ is shown in Figure 4.19(a).

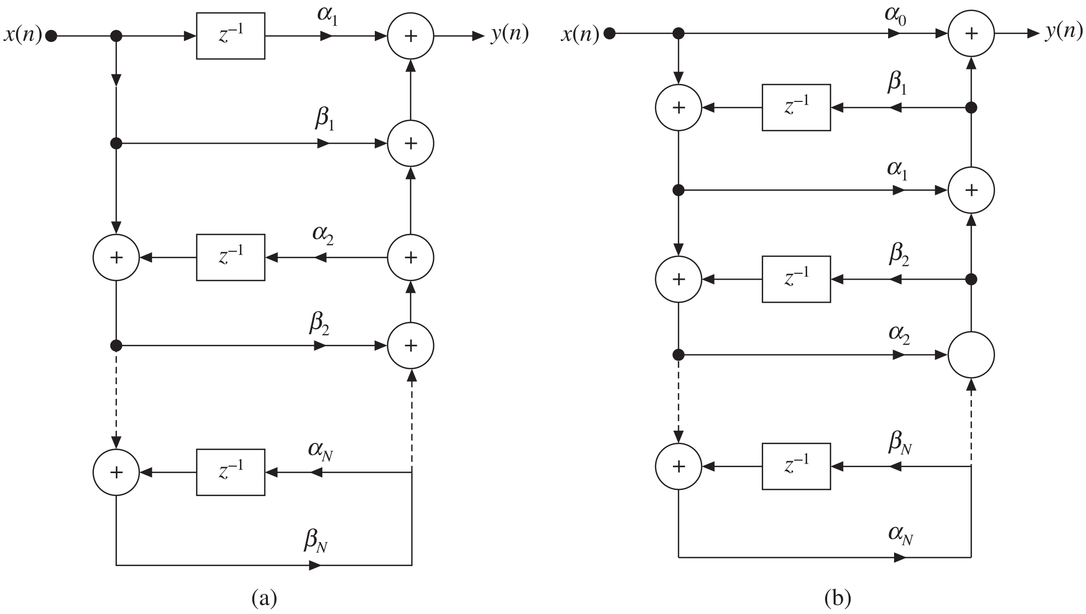


Figure 4.19 (a) Ladder structure realization of IIR system for case-I (b) Ladder structure realization of IIR system for case-II.

CASE-II: If the order of both the numerator and denominator polynomials is equal, that is

$$H(z) = \frac{b_N z^{-N} + \cdots + b_3 z^{-3} + b_2 z^{-2} + b_1 z^{-1} + b_0}{a_N z^{-N} + \cdots + a_3 z^{-3} + a_2 z^{-2} + a_1 z^{-1} + a_0}$$

then, the transfer function for a ladder structure is shown below.

$$H(z) = \alpha_0 + \cfrac{1}{\beta_1 z^{-1} + \cfrac{1}{\alpha_1 + \cfrac{1}{\beta_2 z^{-1} + \cfrac{1}{\alpha_2 + \cfrac{1}{\beta_3 z^{-1} + \cdots}}}}}$$

The ladder structure realization of the above equation for $H(z)$ is shown in Figure 4.19(b).

Procedure to realize the ladder structure of IIR systems

1. Express the numerator and denominator polynomials of the transfer function in descending order of negative powers of z and assess whether it falls under case-I or case-II.

2. Divide the numerator polynomial by the denominator polynomial sequentially and find the quotient at each division.
3. Compare the quotient obtained with that of the general transfer function for the corresponding ladder structure and find the corresponding parameters α and β.
4. Realize the ladder structure with the obtained values of parameters.

EXAMPLE 4.4 Realize the IIR filter

$$H(z) = \frac{3z^2 + 5z + 4}{z^2 + 6z + 8}$$

using ladder structure.

Solution: Given $H(z) = \dfrac{3z^2 + 5z + 4}{z^2 + 6z + 8} = \dfrac{3 + 5z^{-1} + 4z^{-2}}{1 + 6z^{-1} + 8z^{-2}} = \dfrac{4z^{-2} + 5z^{-1} + 3}{8z^{-2} + 6z^{-1} + 1}$

Here, the negative order of the numerator and denominator is equal. So it falls under case-II.
Performing sequential division operation.

$8z^{-2} + 6z^{-1} + 1)\ 4z^{-2} + 5z^{-1} + 3\ (1/2$

$4z^{-1} + 3z^{-1} + 1/2$

$2z^{-1} + 5/2)\ 8z^{-2} + 6z^{-1} + 1\ (4z^{-1}$

$8z^{-2} + 10z^{-1}$

$-4z^{-1} + 1)\ 2z^{-1} + 5/2\ (-1/2$

$2z^{-1} - 1/2$

$3)\ -4z^{-1} + 1\ (-4/3z^{-1}$

$-4z^{-1}$

$1)\ 3\ (3$

3

0

Comparing with $H(z)$ of case-II, we get

$$\alpha_0 = \frac{1}{2}, \beta_1 = 4, \alpha_1 = -\frac{1}{2}, \beta_2 = -\frac{4}{3} \text{ and } \alpha_2 = 3$$

Hence, the required transfer function for realization is

$$H(z) = \frac{1}{2} + \cfrac{1}{4z^{-1} + \cfrac{1}{-\cfrac{1}{2} + \cfrac{1}{-\cfrac{4}{3}z^{-1} + \cfrac{1}{3}}}}$$

Thus, the realization of the given IIR filter using the ladder form structure is shown in Figure 4.20.

EXAMPLE 4.5 Realize the IIR filter

$$H(z) = \frac{5z^3 + 3z^2 + 4z + 2}{z[2z^2 + 3z + 1]}$$

using ladder structure

Solution: Given

$$H(z) = \frac{5z^3 + 3z^2 + 4z + 2}{z[2z^2 + 3z + 1]} = \frac{z^3[5 + 3z^{-1} + 4z^{-2} + 2z^{-3}]}{z^3[2 + 3z^{-1} + z^{-2}]} = \frac{2z^{-3} + 4z^{-2} + 3z^{-1} + 5}{z^{-2} + 3z^{-1} + 2}$$

Here the negative order of the numerator polynomial is greater than that of the denominator. So it falls under case-I.

Performing the sequential division operation

$z^{-2} + 3z^{-1} + 2)\ 2z^{-3} + 4z^{-2} + 3z^{-1} + 5\ (2z^{-1}$

$2z^{-3} + 6z^{-2} + 4z^{-1}$

$-2z^{-2} - z^{-1} + 5)\ z^{-2} + 3z^{-1} + 2\ (-1/2$

$z^{-2} + 1/2z^{-1} - 5/2$

$5/2z^{-1} + 9/2)\ -2z^{-2} - z^{-1} + 5\ (-4/5\,z^{-1}$

$-2z^{-2} - 18/5\,z^{-1}$

$13/5z^{-1} + 5)\ 5/2z^{-1} + 9/2\ (25/26$

$5/2z^{-1} + 125/26$

$-8/13)\ 13/5z^{-1} + 5\ (-169/40z^{-1}$

$13/5z^{-1}$

$5)\ -8/13\ (-8/65$

$-8/13$

0

Comparing with $H(z)$ of the case-I, we get

$$\alpha_1 = 2,\ \beta_1 = -\frac{1}{2},\ \alpha_2 = -\frac{4}{5},\ \beta_2 = \frac{25}{26},\ \alpha_3 = -\frac{169}{40},\ \beta_3 = -\frac{8}{65}$$

Hence, the required transfer function for realization is

$$H(z) = 2z^{-1} + \cfrac{1}{-\cfrac{1}{2} + \cfrac{1}{-\cfrac{4}{5}z^{-1} + \cfrac{1}{\cfrac{25}{26} + \cfrac{1}{-\cfrac{169}{40}z^{-1} + \cfrac{1}{-\cfrac{8}{65}}}}}}$$

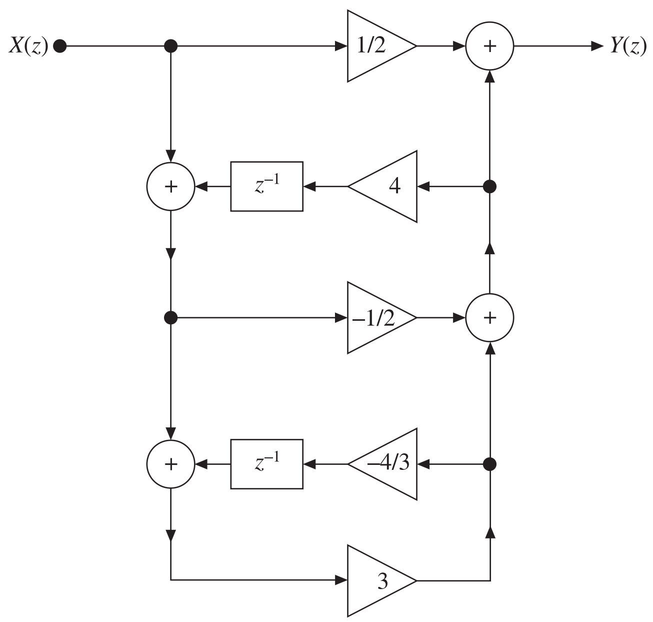


Figure 4.20 Ladder structure realization (Example 4.4).

Thus, the realization of the given IIR filter using the ladder form structure is shown in Figure 4.21.

EXAMPLE 4.6 Obtain the direct form-I, direct form-II, cascade and parallel form realizations of the LTI system governed by the equation

$$y(n) = -\frac{13}{12}y(n-1) - \frac{9}{24}y(n-2) - \frac{1}{24}y(n-3) + x(n) + 4x(n-1) + 3x(n-2)$$

Solution:

Direct form-I

Given $y(n) = -\frac{13}{12}y(n-1) - \frac{9}{24}y(n-2) - \frac{1}{24}y(n-3) + x(n) + 4x(n-1) + 3x(n-2)$

Taking Z-transform on both sides, we get

$$Y(z) = -\frac{13}{12}z^{-1}Y(z) - \frac{9}{24}z^{-2}Y(z) - \frac{1}{24}z^{-3}Y(z) + X(z) + 4z^{-1}X(z) + 3z^{-2}X(z)$$

The direct form-I structure can be obtained from the above equation as shown in Figure 4.22.

Direct form-II

Taking Z-transform of the given difference equation, we have

$$Y(z) = -\frac{13}{12}z^{-1}Y(z) - \frac{9}{24}z^{-2}Y(z) - \frac{1}{24}z^{-3}Y(z) + X(z) + 4z^{-1}X(z) + 3z^{-2}X(z)$$

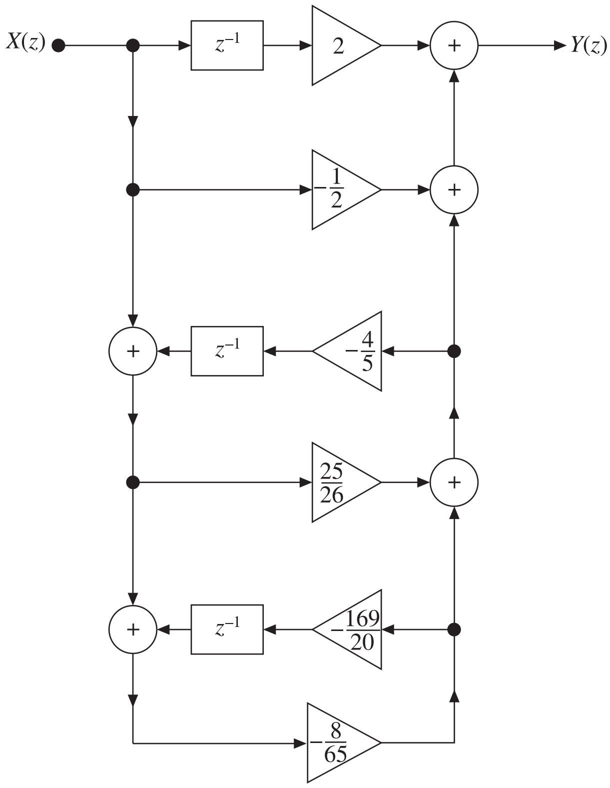


Figure 4.21 Ladder structure realization (Example 4.5).

i.e. $$Y(z) + \frac{13}{12} z^{-1} Y(z) + \frac{9}{24} z^{-2} Y(z) + \frac{1}{24} z^{-3} Y(z) = X(z) + 4z^{-1} X(z) + 3z^{-2} X(z)$$

i.e. $$Y(z)\left[1 + \frac{13}{12} z^{-1} + \frac{9}{24} z^{-2} + \frac{1}{24} z^{-3}\right] = X(z)\,[1 + 4z^{-1} + 3z^{-2}]$$

Therefore, the transfer function of the system is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 4z^{-1} + 3z^{-2}}{1 + (13/12)\,z^{-1} + (9/24)\,z^{-2} + (1/24)\,z^{-3}}$$

Let $$\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)}\,\frac{W(z)}{X(z)}$$

where $$\frac{W(z)}{X(z)} = \frac{1}{1 + (13/12)\,z^{-1} + (9/24)\,z^{-2} + (1/24)\,z^{-3}}$$

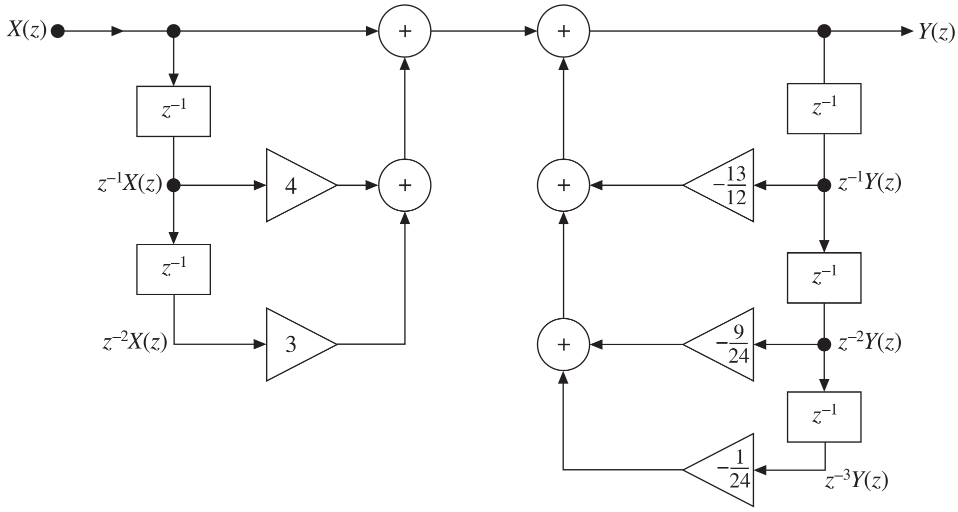


Figure 4.22 Direct form-I realization structure (Example 4.6).

and $$\frac{Y(z)}{W(z)} = 1 + 4z^{-1} + 3z^{-2}$$

On cross multiplying the above equations, we get

$$W(z)\left[1 + \frac{13}{12}z^{-1} + \frac{9}{24}z^{-2} + \frac{1}{24}z^{-3}\right] = X(z)$$

i.e. $$W(z) = X(z) - \frac{13}{12}z^{-1}W(z) - \frac{9}{24}z^{-2}W(z) - \frac{1}{24}z^{-3}W(z)$$

and $$Y(z) = W(z) + 4z^{-1}W(z) + 3z^{-2}W(z)$$

The above equations for $W(z)$ and $Y(z)$ can be realized by a direct form-II structure as shown in Figure 4.23.

Cascade form

The transfer function is $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1 + 4z^{-1} + 3z^{-2}}{1 + (13/12)z^{-1} + (9/24)z^{-2} + (1/24)z^{-3}}$

Factorizing the numerator and denominator, we have

$$H(z) = \frac{(1 + z^{-1})(1 + 3z^{-1})}{[1 + (1/2)z^{-1}][1 + (1/3)z^{-1}][1 + (1/4)z^{-1}]}$$

Since there are three first order factors in the denominator of $H(z)$, $H(z)$ can be expressed as product of three sections.

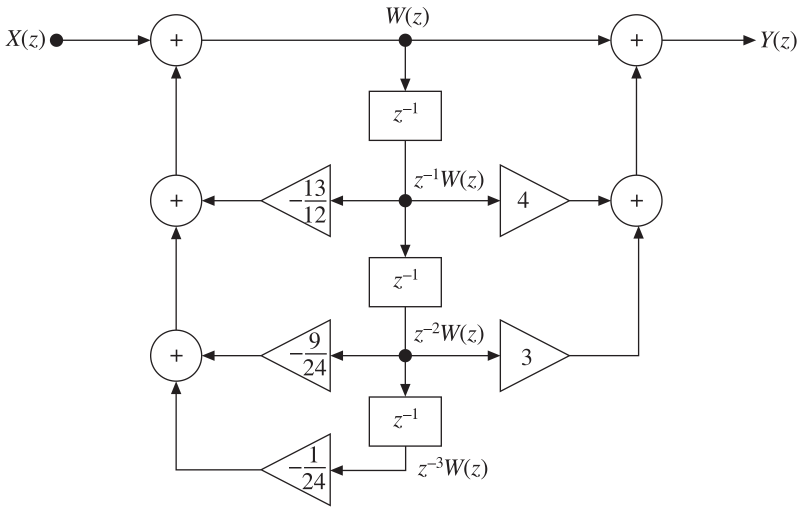


Figure 4.23 Direct form-II realization structure (Example 4.6).

Let $$H(z) = H_1(z) \quad H_2(z) \quad H_3(z)$$

where $$H_1(z) = \frac{1+z^{-1}}{1+(1/2)z^{-1}},\ H_2(z) = \frac{1+3z^{-1}}{1+(1/3)z^{-1}} \quad \text{and} \quad H_3(z) = \frac{1}{1+(1/4)z^{-1}}$$

The transfer function $H_1(z)$ can be realized in direct form-II structure as shown in Figure 4.24(a).

Let $$H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{Y_1(z)}{W_1(z)} \cdot \frac{W_1(z)}{X(z)} = \frac{1+z^{-1}}{1+(1/2)z^{-1}}$$

where $$\frac{W_1(z)}{X(z)} = \frac{1}{1+(1/2)z^{-1}} \quad \text{and} \quad \frac{Y_1(z)}{W_1(z)} = 1+z^{-1}$$

$\therefore$ $$W_1(z) = X(z) - \frac{1}{2}z^{-1}W_1(z)$$

and $$Y_1(z) = W_1(z) + z^{-1}W_1(z)$$

The transfer function $H_2(z)$ can be realized in direct form-II structure as shown in Figure 4.24(b).

Let $$H_2(z) = \frac{Y_2(z)}{Y_1(z)} = \frac{Y_2(z)}{W_2(z)} \cdot \frac{W_2(z)}{Y_1(z)} = \frac{1+3z^{-1}}{1+(1/3)z^{-1}}$$

where $$\frac{W_2(z)}{Y_1(z)} = \frac{1}{1+(1/3)z^{-1}} \quad \text{and} \quad \frac{Y_2(z)}{W_2(z)} = 1+3z^{-1}$$

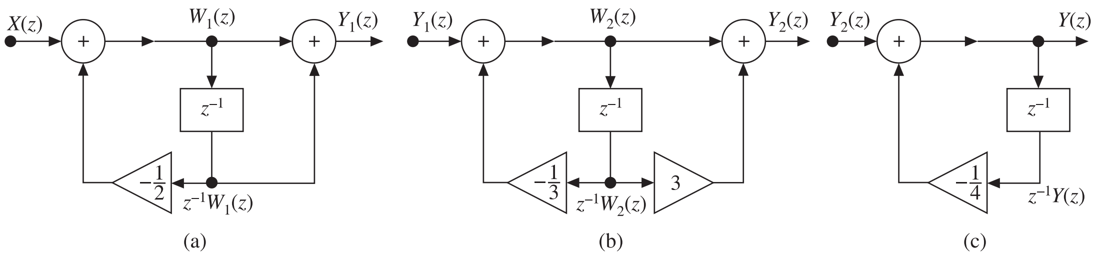


Figure 4.24 Direct form-II structure of (a) $H_1(z)$, (b) $H_2(z)$ and (c) $H_3(z)$ (Example 4.6).

$\therefore$
$$W_2(z) = Y_1(z) - \frac{1}{3}z^{-1}W_2(z)$$

and
$$Y_2(z) = W_2(z) + 3z^{-1}W_2(z)$$

The transfer function $H_3(z)$ can be realized in direct form-II structure as shown in Figure 4.24(c).

Let
$$H_3(z) = \frac{Y(z)}{Y_2(z)} = \frac{1}{1 + (1/4)z^{-1}}$$

$\therefore$
$$Y(z) = Y_2(z) - \frac{1}{4}z^{-1}Y(z)$$

The cascade structure of the given system is obtained by connecting the individual sections shown in Figures 4.24(a), (b) and (c) in cascade as shown in Figure 4.25.

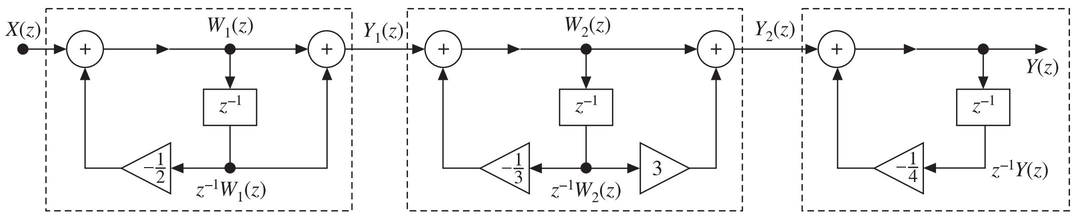


Figure 4.25 Cascade realization of the system (Example 4.6).

Parallel form

Consider the equation $H(z) = \dfrac{(1 + z^{-1})(1 + 3z^{-1})}{[1 + (1/2)z^{-1}][1 + (1/3)z^{-1}][1 + (1/4)z^{-1}]}$

By partial fraction expansion, we have

$$H(z) = \frac{A}{1 + (1/2)z^{-1}} + \frac{B}{1 + (1/3)z^{-1}} + \frac{C}{1 + (1/4)z^{-1}}$$

where, the coefficients A, B and C are

$$A = \frac{(1+z^{-1})(1+3z^{-1})}{\left(1+\frac{1}{3}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}\Bigg|_{z^{-1}=-2} = \frac{(1-2)(1-6)}{\left(1-\frac{2}{3}\right)\left(1-\frac{2}{4}\right)} = \frac{5}{\frac{1}{3}\cdot\frac{1}{2}} = 30$$

$$B = \frac{(1+z^{-1})(1+3z^{-1})}{\left(1+\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}\Bigg|_{z^{-1}=-3} = \frac{(1-3)(1-9)}{\left(1-\frac{3}{2}\right)\left(1-\frac{3}{4}\right)} = \frac{16}{-\frac{1}{2}\cdot\frac{1}{4}} = -128$$

$$C = \frac{(1+z^{-1})(1+3z^{-1})}{\left(1+\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)}\Bigg|_{z^{-1}=-4} = \frac{(1-4)(1-12)}{(1-2)\left(1-\frac{4}{3}\right)} = \frac{33}{-1\cdot-\frac{1}{3}} = 99$$

$\therefore$
$$H(z) = \frac{30}{1+(1/2)z^{-1}} - \frac{128}{1+(1/3)z^{-1}} + \frac{99}{1+(1/4)z^{-1}}$$

Let
$$H(z) = \frac{Y(z)}{X(z)}$$

$\therefore$
$$\frac{Y(z)}{X(z)} = \frac{30}{1+(1/2)z^{-1}} - \frac{128}{1+(1/3)z^{-1}} + \frac{99}{1+(1/3)z^{-1}}$$

$\therefore$
$$Y(z) = \frac{30}{1+(1/2)z^{-1}}X(z) - \frac{128}{1+(1/3)z^{-1}}X(z) + \frac{99}{1+(1/4)z^{-1}}X(z)$$

Let $Y(z) = Y_1(z) + Y_2(z) + Y_3(z)$

where $Y_1(z) = \dfrac{30}{1+(1/2)z^{-1}}X(z)$; $Y_2(z) = -\dfrac{128}{1+(1/3)z^{-1}}X(z)$ and $Y_3(z) = \dfrac{99}{1+(1/4)z^{-1}}X(z)$

Let
$$H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{30}{1+(1/2)z^{-1}};\ H_2(z) = \frac{Y_2(z)}{X(z)} = -\frac{128}{1+(1/2)z^{-1}}$$

and
$$H_3(z) = \frac{Y_3(z)}{X(z)} = \frac{99}{1+(1/4)z^{-1}}$$

The transfer function $H_1(z)$ can be realized in direct form-I structure as shown in Figure 4.26(a).

Let
$$H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{30}{1+(1/2)z^{-1}}$$

On cross multiplying and rearranging, we get

$$Y_1(z) = 30X(z) - \frac{1}{2}z^{-1}Y(z)$$

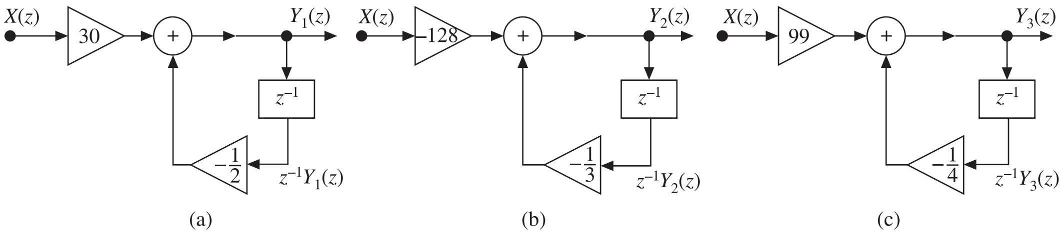


Figure 4.26 Direct form-I structure of (a) $H_1(z)$, (b) $H_2(z)$ and (c) $H_3(z)$ (Example 4.6).

The transfer function $H_2(z)$ can be realized in direct form-I structure as shown in Figure 4.26(b).

Let
$$H_2(z) = \frac{Y_2(z)}{X(z)} = -\frac{128}{1+(1/3)z^{-1}}$$

On cross multiplying and rearranging, we get

$$Y_2(z) = -128\,X(z) - \frac{1}{3}z^{-1}Y_2(z)$$

The transfer function $H_3(z)$ can be realized in direct form-I structure as shown in Figure 4.26(c).

Let
$$H_3(z) = \frac{Y_3(z)}{X(z)} = \frac{99}{1+(1/4)z^{-1}}$$

On cross multiplying and rearranging, we get

$$Y_3(z) = 99\,X(z) - \frac{1}{4}z^{-1}Y_3(z)$$

The overall structure is obtained by connecting the individual sections in Figures 4.26(a), (b) and (c) in parallel as shown in Figure 4.27.

EXAMPLE 4.7 Find the direct form-I and direct form-II realizations of a discrete-time system represented by the transfer function

$$H(z) = \frac{3z^3 - 5z^2 + 9z - 3}{[z-(1/2)]\,[z^2 - z + (1/3)]}$$

Solution: Let
$$H(z) = \frac{Y(z)}{X(z)}$$

where $Y(z)$ = Output and $X(z)$ = input

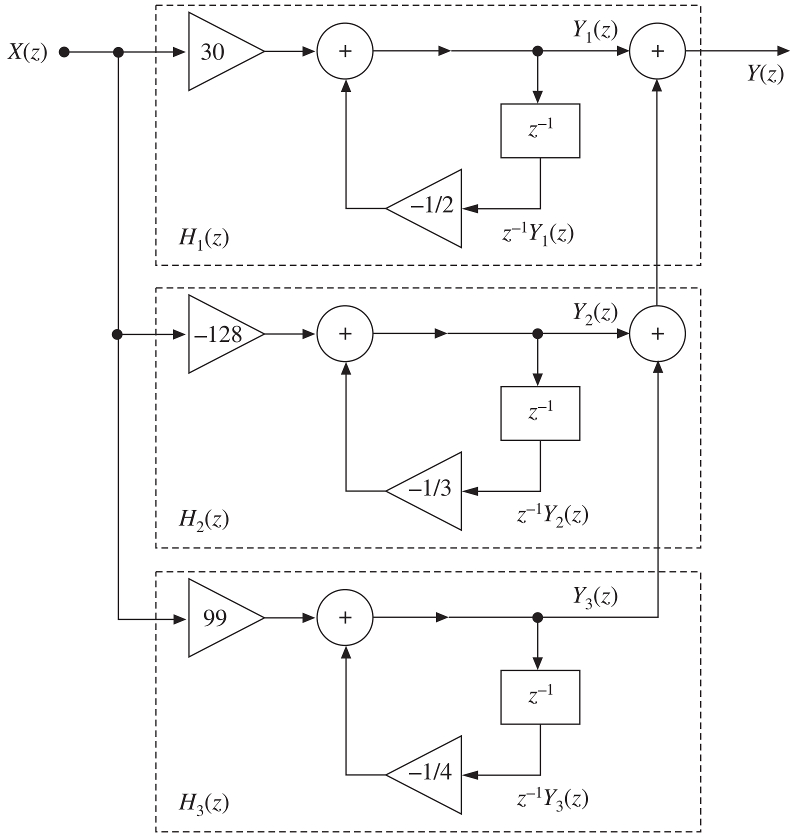


Figure 4.27 Parallel form realization (Example 4.6).

$$\therefore \qquad H(z) = \frac{Y(z)}{X(z)} = \frac{3z^3 - 5z^2 + 9z - 3}{[z - (1/2)]\,[z^2 - z + (1/3)]} = \frac{3z^3 - 5z^2 + 9z - 3}{z^3 - (3/2)z^2 + (5/6)\,z - (1/6)}$$

$$= \frac{z^3[3 - 5z^{-1} + 9z^{-2} - 3z^{-3}]}{z^3[1 - (3/2)\,z^{-1} + (5/6)\,z^{-2} - (1/6)\,z^{-3}]} = \frac{3 - 5z^{-1} + 9z^{-2} - 3z^{-3}}{1 - (3/2)\,z^{-1} + (5/6)\,z^{-2} - (1/6)\,z^{-3}}$$

Direct form-I

On cross multiplying the above equation for $Y(z)/X(z)$, we have

$$Y(z) - \frac{3}{2}z^{-1}Y(z) + \frac{5}{6}z^{-2}Y(z) - \frac{1}{6}z^{-3}Y(z) = 3X(z) - 5z^{-1}X(z) + 9z^{-2}X(z) - 3z^{-3}X(z)$$

$$\therefore \quad Y(z) = \frac{3}{2}z^{-1}Y(z) - \frac{5}{6}z^{-2}Y(z) + \frac{1}{6}z^{-3}Y(z) + 3X(z) - 5z^{-1}X(z) + 9z^{-2}X(z) - 3z^{-3}X(z)$$

The direct form-I structure of the above equation for $Y(z)$ can be obtained as shown in Figure 4.28.

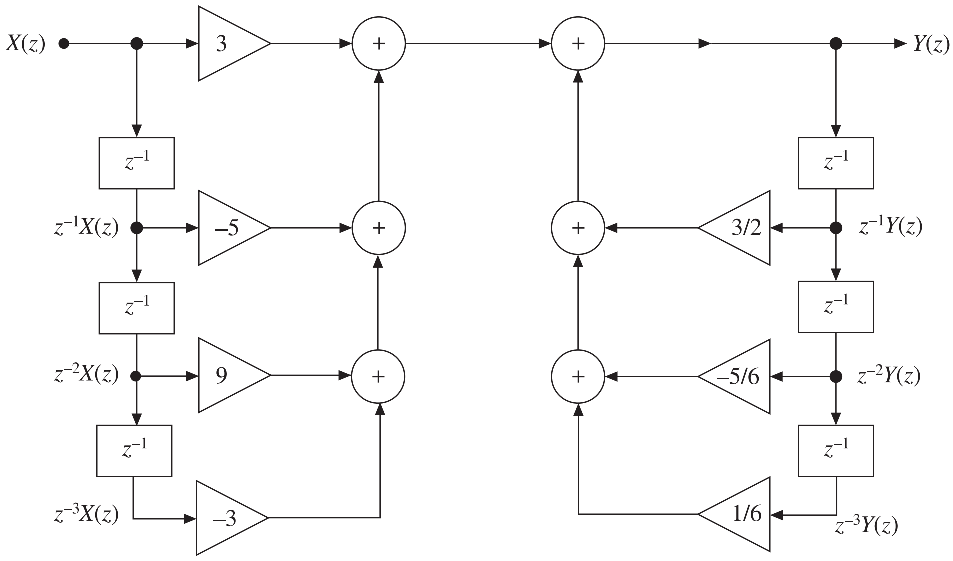


Figure 4.28 Direct form-I structure (Example 4.7).

Direct form-II

The equation for $H(z)$ is $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{3-5z^{-1}+9z^{-2}-3z^{-3}}{1-(3/2)z^{-1}+(5/6)z^{-2}-(1/6)z^{-3}}$

Let $$H(z) = \frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)} = \frac{3-5z^{-1}+9z^{-2}-3z^{-3}}{1-(3/2)z^{-1}+(5/6)z^{-2}-(1/6)z^{-3}}$$

where $$\frac{W(z)}{X(z)} = \frac{1}{1-(3/2)z^{-1}+(5/6)z^{-2}-(1/6)z^{-3}}$$

and $$\frac{Y(z)}{W(z)} = 3-5z^{-1}+9z^{-2}-3z^{-3}$$

On cross multiplying the above equations, we get

$$W(z) - \frac{3}{2}z^{-1}W(z) + \frac{5}{6}z^{-2}W(z) - \frac{1}{6}z^{-3}W(z) = X(z)$$

i.e. $$W(z) = X(z) + \frac{3}{2}z^{-1}W(z) - \frac{5}{6}z^{-2}W(z) + \frac{1}{6}z^{-3}W(z)$$

and $$Y(z) = 3W(z) - 5z^{-1}W(z) + 9z^{-2}W(z) - 3z^{-3}W(z)$$

The above equations for $Y(z)$ and $W(z)$ can be realized by a direct form-II structure as shown in Figure 4.29.

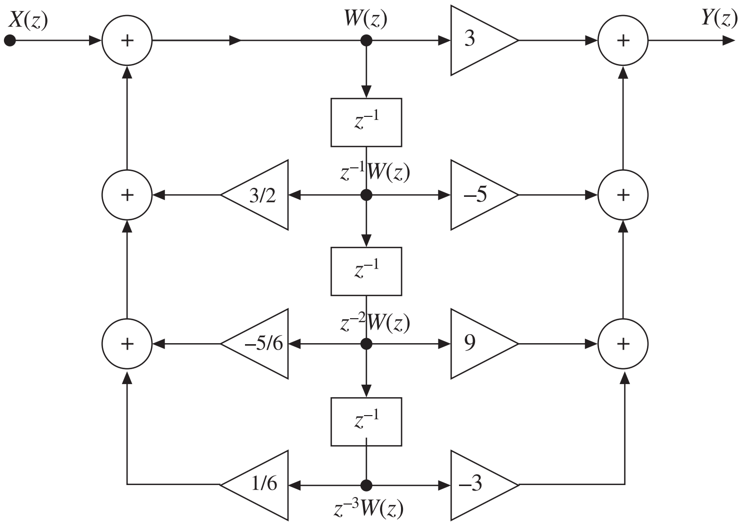


Figure 4.29 Direct form-II structure (Example 4.7).

EXAMPLE 4.8 Find the digital network in direct and transposed form for the system described by the difference equation

$$y(n) = 2x(n) + 0.3x(n-1) + 0.5x(n-2) - 0.7y(n-1) - 0.9y(n-2)$$

Solution: Given difference equation is:

$$y(n) = 2x(n) + 0.3x(n-1) + 0.5x(n-2) - 0.7y(n-1) - 0.9y(n-2)$$

Taking Z-transform on both sides, we have

$$Y(z) = 2X(z) + 0.3z^{-1}X(z) + 0.5z^{-2}X(z) - 0.7z^{-1}Y(z) - 0.9z^{-2}Y(z)$$

Direct form

The direct form-I digital network can be realized using the above equation for $Y(z)$ as shown in Figure 4.30.

Transposed form

On rearranging the equation for $Y(z)$, we get

$$Y(z) + 0.7z^{-1}Y(z) + 0.9z^{-2}Y(z) = 2X(z) + 0.3z^{-1}X(z) + 0.5z^{-2}X(z)$$

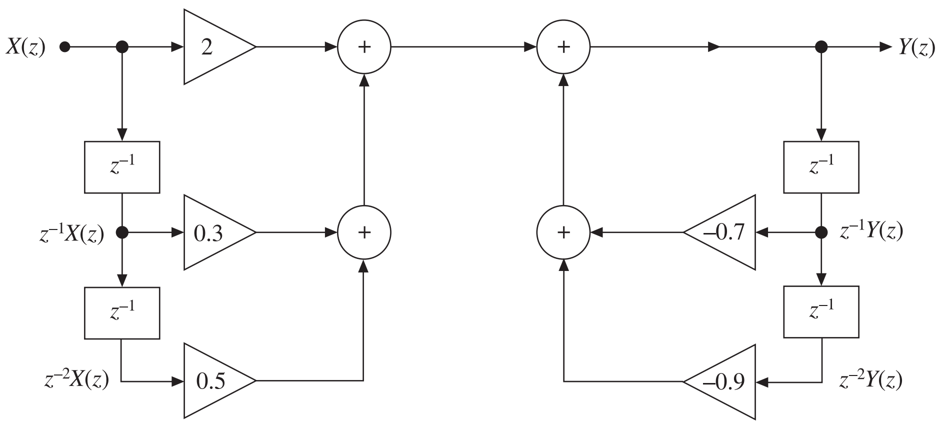


Figure 4.30 Direct form structure (Example 4.8).

i.e. $$Y(z)[1+0.7z^{-1}+0.9z^{-2}] = X(z)[2+0.3z^{-1}+0.5z^{-2}]$$

$\therefore$ $$H(z) = \frac{Y(z)}{X(z)} = \frac{2+0.3z^{-1}+0.5z^{-2}}{1+0.7z^{-1}+0.9z^{-2}}$$

Let $$\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)}\cdot\frac{W(z)}{X(z)} = \frac{2+0.3z^{-1}+0.5z^{-2}}{1+0.7z^{-1}+0.9z^{-2}}$$

where $\dfrac{W(z)}{X(z)} = \dfrac{1}{1+0.7z^{-1}+0.9z^{-2}}$ and $\dfrac{Y(z)}{W(z)} = 2+0.3z^{-1}+0.5z^{-2}$

Cross multiplying the above equations, we get

$$W(z)+0.7z^{-1}W(z)+0.9z^{-2}W(z) = X(z)$$

i.e. $$W(z) = X(z) - 0.7z^{-1}W(z) - 0.9z^{-2}W(z)$$

and $$Y(z) = 2W(z) + 0.3z^{-1}W(z) + 0.5z^{-2}W(z)$$

The transposed form of digital network is realized using equations for $W(z)$ and $Y(z)$ as shown in Figure 4.31. The recovered realization structure and the transposed forms structure obtained from the direct form-II are shown in Figure 4.31[(b) and (c)].

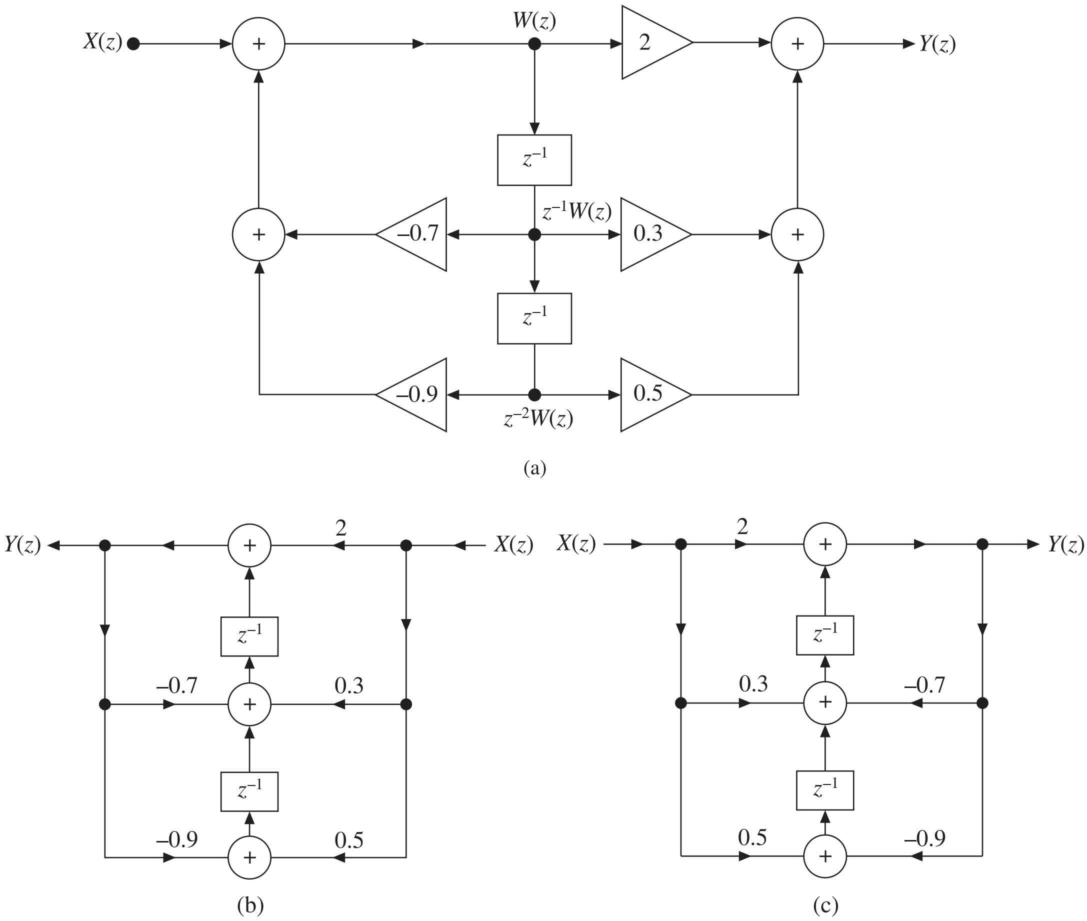


Figure 4.31 (a) Direct form-II structure (b) Recovered realization structure (c) Transposed form structure (Example 4.8).

EXAMPLE 4.9 Realize the given system in cascade and parallel forms

$$H(z) = \frac{1 + (1/3)z^{-1}}{[1 - (1/2)z^{-1} + (1/3)z^{-2}][1 - (1/3)z^{-1} + (1/2)z^{-2}]}$$

Solution:

Cascade form

Let us realize the system as cascade of two second order sections.

$$H(z) = \frac{1 + (1/3)z^{-1}}{1 - (1/2)z^{-1} + (1/3)z^{-2}} \; \frac{1}{1 - (1/3)z^{-1} + (1/2)z^{-2}} = H_1(z)\; H_2(z)$$

where $$H_1(z) = \frac{1 + (1/3)z^{-1}}{1 - (1/2)z^{-1} + (1/3)z^{-2}}$$

and $$H_2(z) = \frac{1}{1 - (1/3)z^{-1} + (1/2)z^{-2}}$$

Let $$H_1(z) = \frac{Y_1(z)}{X_1(z)} = \frac{Y_1(z)}{W_1(z)}\frac{W_1(z)}{X_1(z)} = \frac{1 + (1/3)z^{-1}}{1 - (1/2)z^{-1} + (1/3)z^{-2}}$$

where $$\frac{W_1(z)}{X_1(z)} = \frac{1}{1 - (1/2)z^{-1} + (1/3)z^{-2}}$$

and $$\frac{Y_1(z)}{W_1(z)} = 1 + \frac{1}{3}z^{-1}$$

On cross multiplying the above equations, we get

$$W_1(z) - \frac{1}{2}z^{-1}W_1(z) + \frac{1}{3}z^{-2}W_1(z) = X_1(z)$$

i.e. $$W_1(z) = X_1(z) + \frac{1}{2}z^{-1}W_1(z) - \frac{1}{3}z^{-2}W_1(z)$$

and $$Y_1(z) = W_1(z) + \frac{1}{3}z^{-1}W_1(z)$$

Using the above equations for $W_1(z)$ and $Y_1(z)$, the system $H_1(z)$ can be realized in direct form-II structure as shown in Figure 4.32(a).

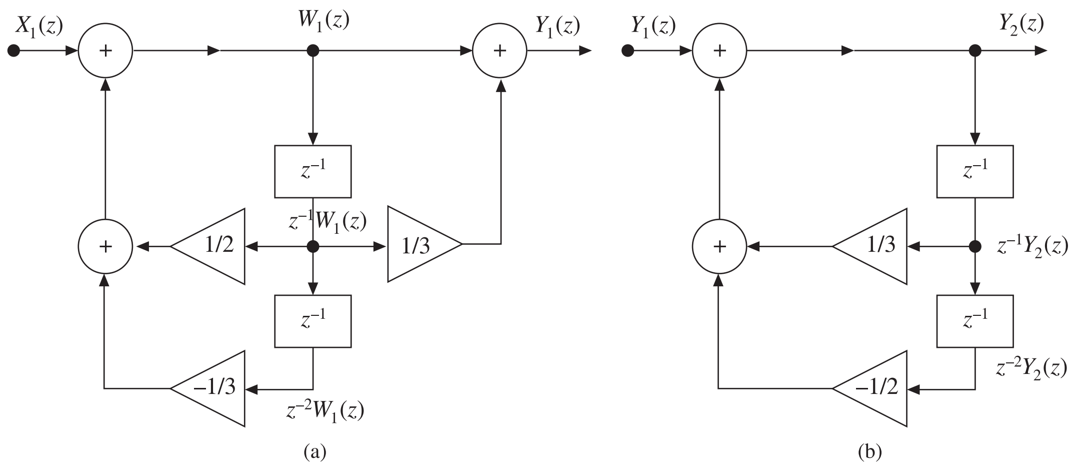


Figure 4.32 Direct form-II structure of (a) $H_1(z)$ and (b) $H_2(z)$ (Example 4.9).

Let
$$H_2(z) = \frac{Y_2(z)}{Y_1(z)} = \frac{1}{1 - \frac{1}{3}z^{-1} + \frac{1}{2}z^{-2}}$$

On cross multiplying, we get

$$Y_2(z) - \frac{1}{3}z^{-1}Y_2(z) + \frac{1}{2}z^{-2}Y_2(z) = Y_1(z)$$

i.e.
$$Y_2(z) = Y_1(z) + \frac{1}{3}z^{-1}Y_2(z) - \frac{1}{2}z^{-2}Y_2(z)$$

The above equation for $Y_2(z)$ can be realized in direct form-II structure as shown in Figure 4.32(b).

The cascade structure of $H(z)$ is obtained by connecting the structures of $H_1(z)$ and $H_2(z)$ in cascade as shown in Figure 4.33.

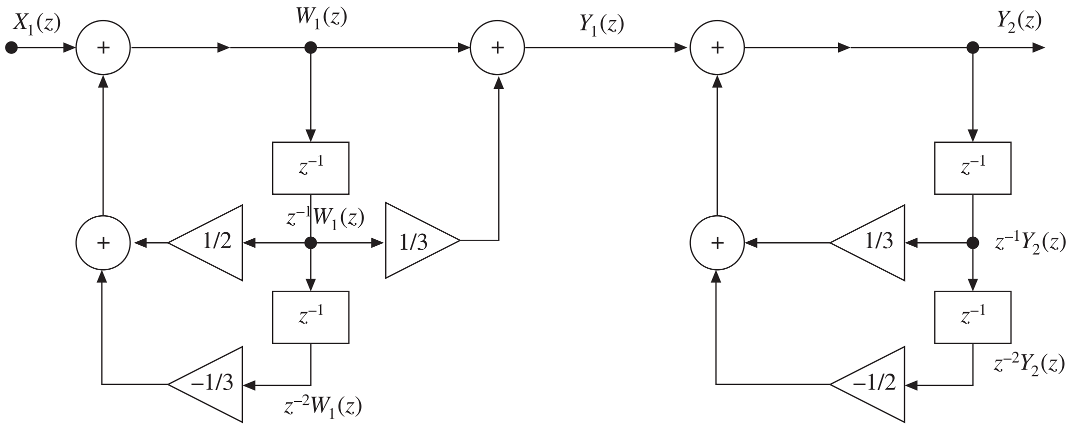


Figure 4.33 Cascade structure of $H(z)$ (Example 4.9).

Parallel realization

Given
$$H(z) = \frac{1 + (1/3)z^{-1}}{[1 - (1/2)z^{-1} + (1/3)z^{-2}]\,[1 - (1/3)z^{-1} + (1/2)z^{-2}]}$$

$$= \frac{z^{-1}[z + (1/3)]}{z^{-4}[z^2 - (1/2)z + (1/3)][z^2 - (1/3)z + (1/2)]}$$

$$= \frac{z^3[z + (1/3)]}{[z^2 - (1/2)z + (1/3)]\,[z^2 - (1/3)z + (1/2)]}$$

$$= \frac{z[z^3 + (1/3)z^2]}{[z^2 - (1/2)z + (1/3)][z^2 - (1/3)z + (1/2)]}$$

By partial fraction expansion, we can write

$$\frac{H(z)}{z} = \frac{[z^3 + (1/3)z^2]}{[z^2 - (1/2)z + (1/3)]\,[z^2 - (1/3)z + (1/2)]} = \frac{Az + B}{z^2 - (1/2)z + (1/3)} + \frac{Cz + D}{z^2 - (1/3)z + (1/2)}$$

On cross multiplying the above equation, we get

$$z^3 + (1/3)z^2 = (Az + B)[z^2 - (1/3)z + (1/2)] + (Cz + D)\,[z^2 - (1/2)z + (1/3)]$$

$$= Az^3 - \frac{1}{3}Az^2 + \frac{1}{2}Az + Bz^2 - \frac{1}{3}Bz + \frac{1}{2}B + Cz^3 - \frac{1}{2}Cz^2 + \frac{1}{3}Cz + Dz^2 - \frac{1}{2}Dz + \frac{1}{3}D$$

$$= (A + C)z^3 + (B + D - \frac{1}{3}A - \frac{1}{2}C)\,z^2 + \left(\frac{1}{2}A - \frac{1}{3}B + \frac{1}{3}C - \frac{1}{2}D\right)z + \left(\frac{1}{2}B + \frac{1}{3}D\right)$$

On equating the coefficients on both sides and solving for A, B, C and D, we get

$$A = \frac{13}{11},\ B = -\frac{14}{11},\ C = -\frac{2}{11} \text{ and } D = \frac{21}{11}$$

$$\therefore \qquad \frac{H(z)}{z} = \frac{(13/11)z - (14/11)}{z^2 - (1/2)z + (1/3)} + \frac{-(2/11)z + (21/11)}{z^2 - (1/3)z + (1/2)}$$

$$\therefore \qquad H(z) = \frac{(13/11)z^2 - (14/11)z}{z^2 - (1/2)z + (1/3)} + \frac{-(2/11)z^2 + (21/11)z}{z^2 - (1/3)z + (1/2)}$$

$$= \frac{z^2[(13/11) - (14/11)z^{-1}]}{z^2[1 - (1/2)z^{-1} + (1/3)z^{-2}]} + \frac{z^2[-(2/11) + (21/11)z^{-1}]}{z^2[1 - (1/3)z^{-1} + (1/2)z^{-2}]}$$

So $$H(z) = \frac{Y(z)}{X(z)} = \frac{(13/11) - (14/11)z^{-1}}{1 - (1/2)z^{-1} + (1/3)z^{-2}} + \frac{-(2/11) + (21/11)z^{-1}}{1 - (1/3)z^{-1} + (1/2)z^{-2}}$$

$$\therefore \qquad Y(z) = \frac{(13/11) - (14/11)z^{-1}}{1 - (1/2)z^{-1} + (1/3)z^{-2}}X(z) + \frac{-(2/11) + (21/11)z^{-1}}{1 - (1/3)z^{-1} + (1/2)z^{-2}}X(z)$$

Let $$Y(z) = Y_1(z) + Y_2(z)$$

where $$Y_1(z) = \frac{(13/11) - (14/11)z^{-1}}{1 - (1/2)z^{-1} + (1/3)z^{-2}}X(z)$$

and $$Y_2(z) = \frac{-(2/11) + (21/11)z^{-1}}{1 - (1/3)z^{-1} + (1/2)z^{-2}}X(z)$$

Let $$H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{Y_1(z)}{W_1(z)} \cdot \frac{W_1(z)}{X(z)} = \frac{(13/11) - (14/11)z^{-1}}{1 - (1/2)z^{-1} + (1/3)z^{-2}}$$

where $$\frac{W_1(z)}{X(z)} = \frac{1}{1-(1/2)z^{-1}+(1/3)z^{-2}}$$

and $$\frac{Y_1(z)}{W_1(z)} = \frac{13}{11} - \frac{14}{11}z^{-1}$$

On cross multiplying the above equations, we get

$$W_1(z) - \frac{1}{2}z^{-1}W_1(z) + \frac{1}{3}z^{-2}W_1(z) = X(z)$$

i.e. $$W_1(z) = X(z) + \frac{1}{2}z^{-1}W_1(z) - \frac{1}{3}z^{-2}W_1(z)$$

and $$Y_1(z) = \frac{13}{11}W_1(z) - \frac{14}{11}z^{-1}W_1(z)$$

The direct form-II structure of system $H_1(z)$ can be realized using the above equations for $W_1(z)$ and $Y_1(z)$ as shown in Figure 4.34(a).

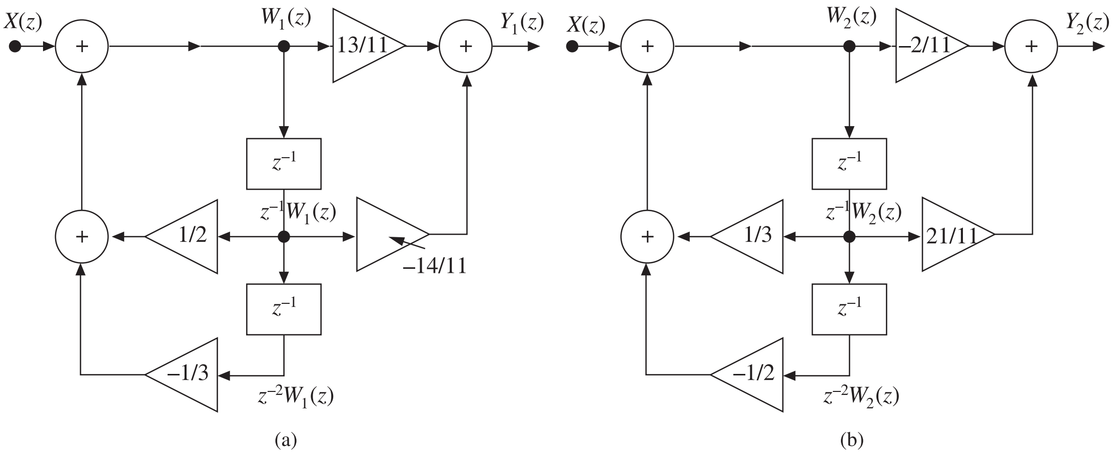


Figure 4.34 Direct form-II structure of (a) $H_1(z)$ and (b) $H_2(z)$ (Example 4.9).

Let $$H_2(z) = \frac{Y_2(z)}{X(z)} = \frac{Y_2(z)}{W_2(z)} \cdot \frac{W_2(z)}{X(z)} = \frac{-(2/11)+(21/11)z^{-1}}{1-(1/3)z^{-1}+(1/2)z^{-2}}$$

where $$\frac{W_2(z)}{X(z)} = \frac{1}{1-(1/3)z^{-1}+(1/2)z^{-2}}$$

and $$\frac{Y_2(z)}{W_2(z)} = -\frac{2}{11} + \frac{21}{11}z^{-1}$$

On cross multiplying the above equations, we get

$$W_2(z) - \frac{1}{3}z^{-1}W_2(z) + \frac{1}{2}z^{-2}W_2(z) = X(z)$$

i.e.

$$W_2(z) = X(z) + \frac{1}{3}z^{-1}W_2(z) - \frac{1}{2}z^{-2}W_2(z)$$

and

$$Y_2(z) = -\frac{2}{11}W_2(z) + \frac{21}{11}z^{-1}W_2(z)$$

The direct form-II structure of the system $H_2(z)$ can be realized using the above equations for $W_2(z)$ and $Y_2(z)$ as shown in Figure 4.34(b).

The parallel form structure of $H(z)$ is obtained by connecting the direct form-II structures of $H_1(z)$ and $H_2(z)$ in parallel as shown in Figure 4.35.

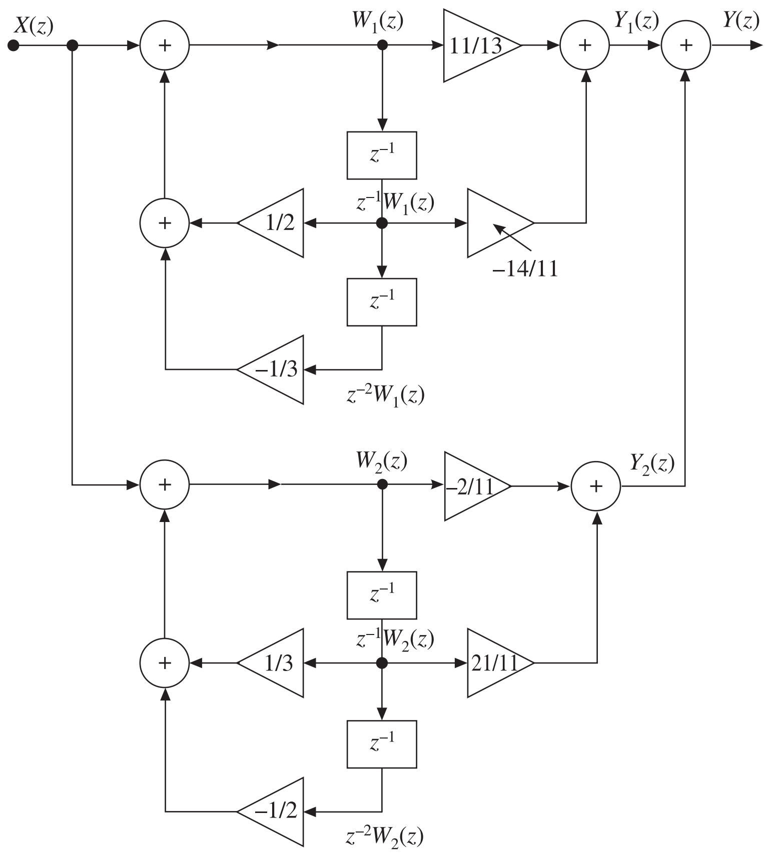


Figure 4.35 Parallel form realization of system $H(z)$ (Example 4.9).

EXAMPLE 4.10 An LTI system is described by the equation

$$y(n) + 2y(n-1) - y(n-2) = x(n)$$

Determine the cascade and parallel realization structures of the system

Solution: Given $$y(n) + 2y(n-1) - y(n-2) = x(n)$$

Taking Z-transform on both sides, we have

$$Y(z) + 2z^{-1}Y(z) - z^{-2}Y(z) = X(z)$$

i.e. $$Y(z)\,[1 + 2z^{-1} - z^{-2}] = X(z)$$

Therefore, the transfer function of the system is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 + 2z^{-1} - z^{-2}} = \frac{1}{(1 - 0.414z^{-1})(1 + 2.414z^{-1})}$$

Let $$H(z) = H_1(z)H_2(z)$$

where $$H_1(z) = \frac{1}{1 - 0.414z^{-1}} \text{ and } H_2(z) = \frac{1}{1 + 2.414z^{-1}}$$

Cascade realization

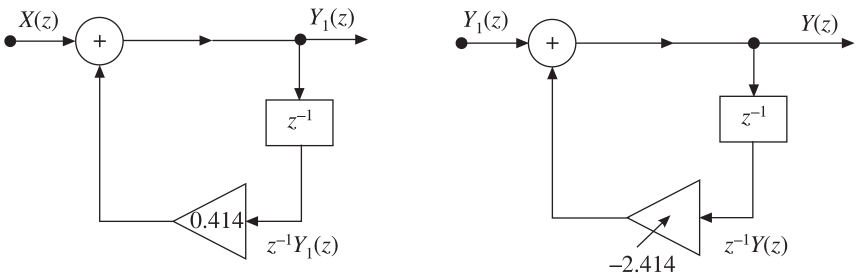
Let $$H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{1}{1 - 0.414z^{-1}}$$

On cross multiplying, we get

$$Y_1(z) - 0.414z^{-1}Y_1(z) = X(z)$$

i.e. $$Y_1(z) = X(z) + 0.414z^{-1}Y_1(z)$$

The direct form-I structure of $H_1(z)$ is obtained using the above equation for $Y_1(z)$ as shown in Figure 4.36(a).



(a) (b)

Figure 4.36 Direct form-I structure of (a) $H_1(z)$ and (b) $H_2(z)$ (Example 4.10).

Let $$H_2(z) = \frac{Y(z)}{Y_1(z)} = \frac{1}{1 + 2.414z^{-1}}$$

On cross multiplying, we get

$$Y(z) + 2.414z^{-1}Y(z) = Y_1(z)$$

i.e. $$Y(z) = Y_1(z) - 2.414z^{-1}Y(z)$$

The direct form-I structure of $H_2(z)$ is obtained using the above equation for $Y(z)$ as shown in Figure 4.36(b).

The cascade structure is obtained by connecting the direct form structures of $H_1(z)$ and $H_2(z)$ as shown in Figure 4.37.

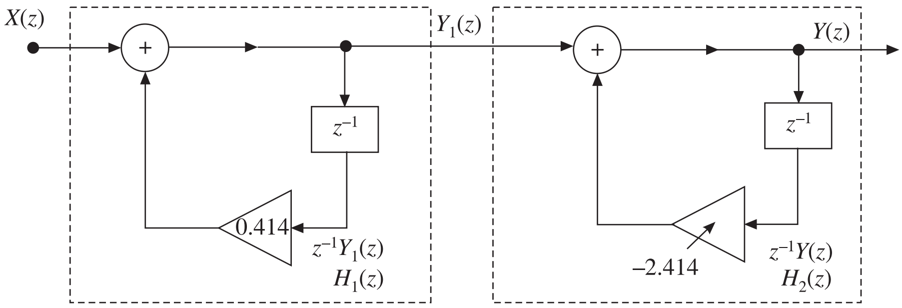


Figure 4.37 Cascade structure of $H(z)$ (Example 4.10).

Parallel realization

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{(1 - 0.414z^{-1})(1 + 2.414z^{-1})} = \frac{A}{1 - 0.414z^{-1}} + \frac{B}{1 + 2.414z^{-1}}$$

where $$A = (1 - 0.414z^{-1})\,H(z)\Big|_{z^{-1} = \frac{1}{0.414}} = \frac{1}{1 + 2.414 \times \frac{1}{0.414}} = 0.146$$

$$B = (1 + 2.414z^{-1})\;H(z)\Big|_{z^{-1} = -\frac{1}{2.414}} = \frac{1}{1 + 0.414 \times \frac{1}{2.414}} = 0.853$$

$\therefore$ $$\frac{Y(z)}{X(z)} = \frac{0.146}{1 - 0.414z^{-1}} + \frac{0.853}{1 + 2.414z^{-1}}$$

$\therefore$ $$Y(z) = \frac{0.146}{1 - 0.414z^{-1}}X(z) + \frac{0.853}{1 + 2.414z^{-1}}X(z)$$

Let $$Y(z) = Y_1(z) + Y_2(z)$$

where $$Y_1(z) = \frac{0.146}{1 - 0.414z^{-1}} X(z) \quad \text{and} \quad Y_2(z) = \frac{0.853}{1 + 2.414z^{-1}} X(z)$$

$\therefore$ $$\frac{Y_1(z)}{X(z)} = H_1(z) = \frac{0.146}{1 - 0.414z^{-1}} \quad \text{and} \quad \frac{Y_2(z)}{X(z)} = H_2(z) = \frac{0.853}{1 + 2.414z^{-1}}$$

Realizing $H_1(z)$ and $H_2(z)$ in direct form-I

$$\frac{Y_1(z)}{X(z)} = \frac{0.146}{1 - 0.414z^{-1}}$$

$\therefore$ $$Y_1(z) - 0.414z^{-1}Y_1(z) = 0.146X(z)$$

i.e. $$Y_1(z) = 0.146X(z) + 0.414z^{-1}Y_1(z)$$

Realization of $H_1(z)$ in direct form-I structure is shown in Figure 4.38(a).

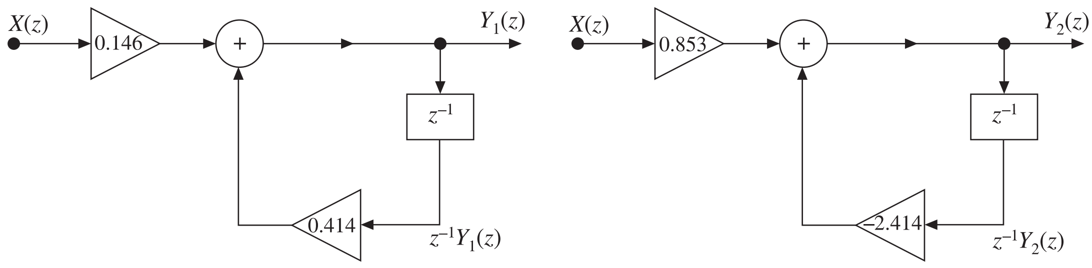


Figure 4.38 Direct form-I structure of (a) $H_1(z)$ and (b) $H_2(z)$ (Example 4.10).

$$\frac{Y_2(z)}{X(z)} = \frac{0.853}{1 + 2.414z^{-1}}$$

$\therefore$ $$Y_2(z) + 2.414z^{-1}Y_2(z) = 0.853X(z)$$

i.e. $$Y_2(z) = 0.853X(z) - 2.414z^{-1}Y_2(z)$$

Realization of $H_2(z)$ in direct form-I structure is shown in Figure 4.38(b).

The parallel structure is obtained by connecting the direct form-I structures of $H_1(z)$ and $H_2(z)$ in parallel as shown in Figure 4.39.

EXAMPLE 4.11 Obtain the cascade realization of the system

$$H(z) = \frac{3 + 2z^{-1} + z^{-2}}{[1 + (1/3)z^{-1}][1 - (1/3)z^{-1}][1 + (1/3)z^{-1}]}$$

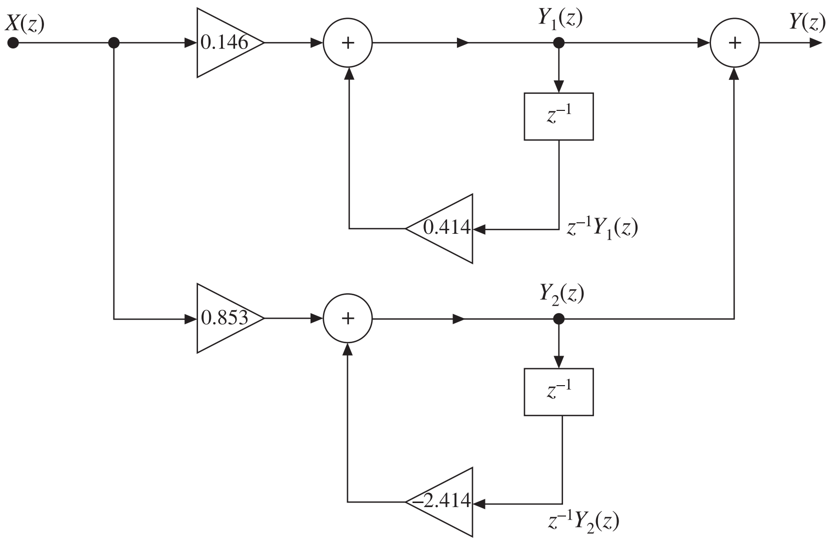


Figure 4.39 Parallel form realization of $H(z)$ (Example 4.10).

Solution: Given $$H(z) = \frac{3 + 2z^{-1} + z^{-2}}{[1 + (1/3)z^{-1}][1 - (1/3)z^{-1}][1 + (1/3)z^{-1}]}$$

The roots of the numerator polynomial

$$z^{-1} = \frac{-2 \pm \sqrt{4 - 12}}{6}$$

are complex conjugate. Hence, $H(z)$ can be realized as cascade of one first order and one second order system.

$$\therefore \qquad H(z) = \frac{1}{1 - (1/3)z^{-1}} \; \frac{3 + 2z^{-1} + z^{-2}}{[1 + (1/3)z^{-1}][(1 + (1/3)z^{-1}]}$$

$$= \frac{1}{1 - (1/3)z^{-1}} \; \frac{3 + 2z^{-1} + z^{-2}}{1 + (2/3)z^{-1} + (1/9)z^{-2}}$$

Let $$H(z) = H_1(z) \cdot H_2(z)$$

where $$H_1(z) = \frac{1}{1 - (1/3)z^{-1}} \quad \text{and} \quad H_2(z) = \frac{3 + 2z^{-1} + z^{-2}}{1 + (2/3)z^{-1} + (1/9)z^{-2}}$$

Let $$H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{1}{1 - (1/3)z^{-1}}$$

On cross multiplying, we get $Y_1(z) - (1/3)z^{-1}Y_1(z) = X(z)$

i.e. $$Y_1(z) = X(z) + (1/3)\,z^{-1}Y_1(z)$$

The direct form-II structure of $H_1(z)$ can be obtained from the equation for $Y_1(z)$ as shown in Figure 4.40(a).

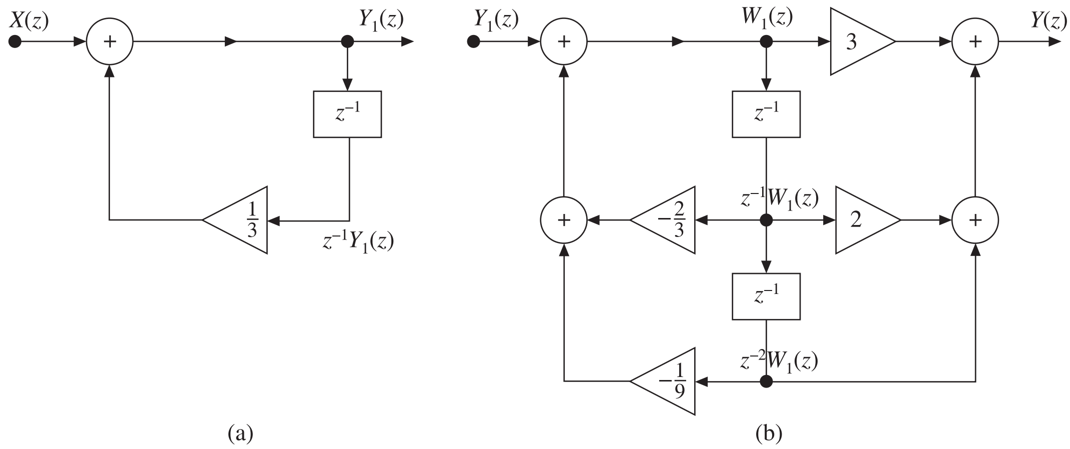


Figure 4.40 Direct form structure of (a) $H_1(z)$ and (b) $H_2(z)$ (Example 4.11).

Let $$H_2(z) = \frac{Y(z)}{Y_1(z)} \quad \frac{3 + 2z^{-1} + z^{-2}}{1 + (2/3)\,z^{-1} + (1/9)\,z^{-2}}$$

and $$\frac{Y(z)}{Y_1(z)} = \frac{Y(z)}{W_1(z)}\;\frac{W_1(z)}{Y_1(z)}$$

where $$\frac{W_1(z)}{Y_1(z)} = \frac{1}{1 + (2/3)\,z^{-1} + (1/9)\,z^{-2}} \quad \text{and} \quad \frac{Y(z)}{W_1(z)} = 3 + 2z^{-1} + z^{-2}$$

On cross multiplying the above equations, we get

$$W_1(z) + \frac{2}{3}z^{-1}W_1(z) + \frac{1}{9}z^{-2}W_1(z) = Y_1(z)$$

i.e. $$W_1(z) = Y_1(z) - \frac{2}{3}z^{-1}W_1(z) - \frac{1}{9}z^{-2}W_1(z)$$

and $$Y(z) = 3W_1(z) + 2z^{-1}W_1(z) + z^{-2}W_1(z)$$

The direct form-II structure of $H_2(z)$ can be obtained by using the above equations for $W_1(z)$ and $Y(z)$ as shown in Figure 4.30(b).

The cascade realization of $H(z)$ is obtained by connecting the direct form-II structures of $H_1(z)$ and $H_2(z)$ as shown in Figure 4.41.

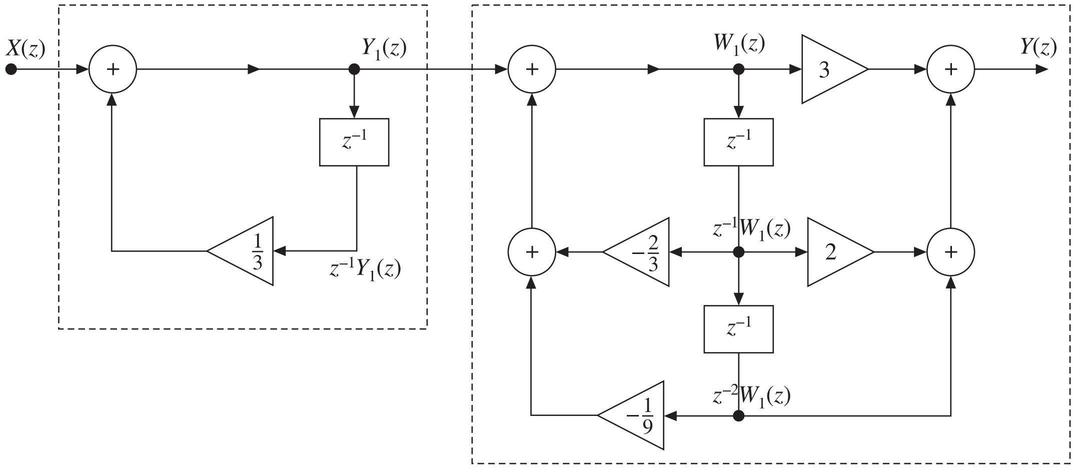


Figure 4.41 Cascade realization of $H(z)$ (Example 4.11).

EXAMPLE 4.12 The transfer function of a system is given by

$$H(z) = \frac{(1+z^{-1})^3}{[1-(1/2)z^{-1}][1+z^{-1}+(1/3)z^{-2}]}$$

Realize the system in cascade and parallel structures.

Solution:
Cascade realization

Given
$$H(z) = \frac{(1+z^{-1})^3}{[1-(1/2)z^{-1}][1+z^{-1}+(1/3)z^{-2}]}$$

The roots of the quadratic factor $1 + z^{-1} + (1/3)z^{-2}$ in the denominator

$$z^{-1} = \frac{-1 \pm \sqrt{1-(4/3)}}{2}$$

are complex conjugate. Hence, the system has to be realized as the cascading of one first order section and one second order section.

$$\therefore \qquad H(z) = \frac{(1+z^{-1})}{[1-(1/2)z^{-1}]} \; \frac{(1+z^{-1})^2}{[1+z^{-1}+(1/3)z^{-2}]}$$

$$= \frac{(1+z^{-1})}{[1-(1/2)z^{-1}]} \; \frac{(1+2z^{-1}+z^{-2})}{[1+z^{-1}+(1/3)z^{-2}]}$$

Let $$H(z) = H_1(z) H_2(z)$$

where $H_1(z) = \dfrac{(1 + z^{-1})}{[1 - (1/2) z^{-1}]}$ and $H_2(z) = \dfrac{(1 + 2z^{-1} + z^{-2})}{[1 + z^{-1} + (1/3) z^{-2}]}$

Let $$H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{1 + z^{-1}}{1 - (1/2) z^{-1}}$$

Let $$\frac{Y_1(z)}{X(z)} = \frac{Y_1(z)}{W_1(z)} \frac{W_1(z)}{X(z)} = \frac{1 + z^{-1}}{1 - (1/2) z^{-1}}$$

where $\dfrac{W_1(z)}{X(z)} = \dfrac{1}{1 - (1/2) z^{-1}}$ and $\dfrac{Y_1(z)}{W_1(z)} = 1 + z^{-1}$

On cross multiplying the above equations, we get

$$W_1(z) - \frac{1}{2} z^{-1} W_1(z) = X(z)$$

i.e. $$W_1(z) = X(z) + \frac{1}{2} z^{-1} W_1(z)$$

and $$Y_1(z) = W_1(z) + z^{-1} W_1(z)$$

The direct form-II structure of $H_1(z)$ can be realized using the above equations for $W_1(z)$ and $Y_1(z)$ as shown in Figure 4.42(a).

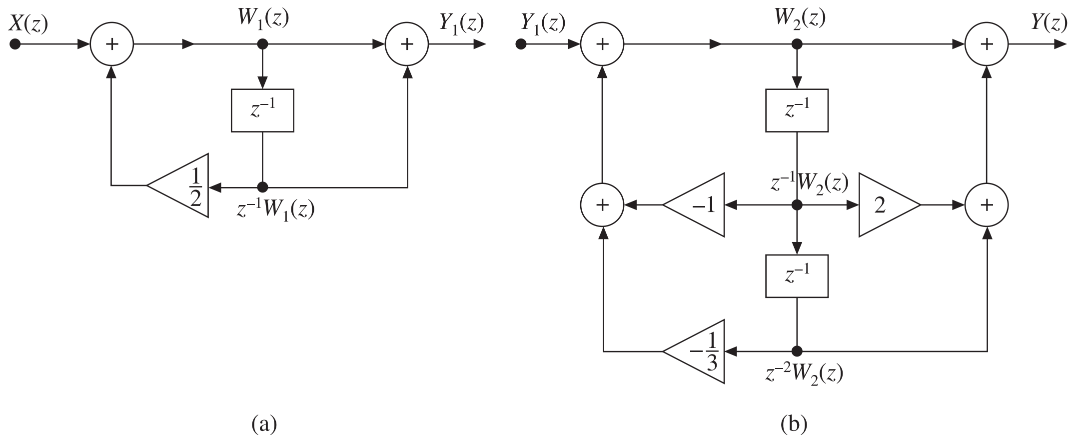


Figure 4.42 Direct form-II structure of (a) $H_1(z)$ and (b) $H_2(z)$ (Example 4.12).

Let $$H_2(z) = \frac{Y(z)}{Y_1(z)} = \frac{1 + 2z^{-1} + z^{-2}}{1 + z^{-1} + (1/3) z^{-2}}$$

and
$$\frac{Y(z)}{Y_1(z)} = \frac{Y(z)}{W_2(z)}\frac{W_2(z)}{Y_1(z)}$$

where $\dfrac{W_2(z)}{Y_1(z)} = \dfrac{1}{1 + z^{-1} + (1/3)z^{-2}}$ and $\dfrac{Y(z)}{W_2(z)} = 1 + 2z^{-1} + z^{-2}$

On cross multiplying the above equations, we get

$$W_2(z) + z^{-1}W_2(z) + \frac{1}{3}z^{-2}W_2(z) = Y_1(z)$$

i.e.
$$W_2(z) = Y_1(z) - z^{-1}W_2(z) - \frac{1}{3}z^{-2}W_2(z)$$

and
$$Y(z) = W_2(z) + 2z^{-1}W_2(z) + z^{-2}W_2(z)$$

The direct form-II structure of $H_2(z)$ can be realized using the above equations for $W_2(z)$ and $Y(z)$ as shown in Figure 4.42(b).

The cascade realization of $H(z)$ is obtained by connecting the direct form-II structures of $H_1(z)$ and $H_2(z)$ in cascade as shown in Figure 4.43.

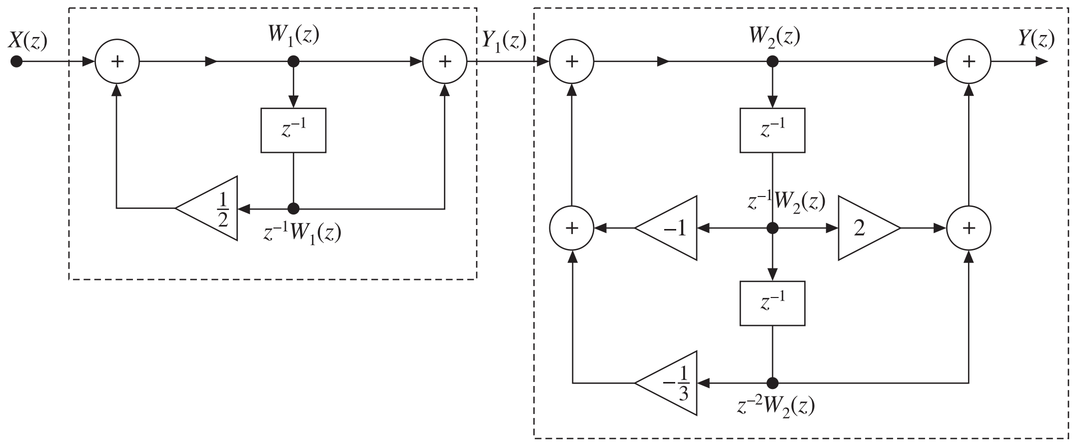


Figure 4.43 Cascade realization of $H(z)$ (Example 4.12).

Parallel realization

Given
$$H(z) = \frac{(1 + z^{-1})^3}{[1 - (1/2)z^{-1}][1 + z^{-1} + (1/3)z^{-2}]}$$

$$= \frac{z^{-3}(z+1)^3}{z^{-1}\left(z - \frac{1}{2}\right)z^{-2}\left(z^2 + z + \frac{1}{3}\right)} = \frac{z^3 + 3z^2 + 3z + 1}{z^3 + \frac{1}{2}z^2 - \frac{1}{6}z - \frac{1}{6}}$$

$$z^3 + \frac{1}{2}z^2 - \frac{1}{6}z - \frac{1}{6} \enclose{longdiv}{z^3 + 2z^2 + 3z + 1} \; (1$$

$$\frac{z^3 + \frac{1}{2}z^2 - \frac{1}{6}z - \frac{1}{6}}{\frac{5}{2}z^2 + \frac{19}{6}z + \frac{7}{6}}$$

$$\therefore \qquad H(z) = 1 + \frac{(5/2)z^2 + (19/6)z + (7/6)}{z^3 + (1/2)z^2 - (1/6)z - (1/6)} = 1 + \frac{(5/2)z^2 + (19/6)z + (7/6)}{[z-(1/2)][z^2 + z + (1/3)]}$$

By partial fraction expansion, we can write

$$\frac{(5/2)z^2 + (19/6)z + (7/6)}{[z-(1/2)][z^2 + z + (1/3)]} = \frac{A}{z-(1/2)} + \frac{Bz + C}{z^2 + z + (1/3)} = \frac{81/26}{z-(1/2)} + \frac{-(8/13)z - (10/39)}{z^2 + z + (1/3)}$$

$$\therefore \qquad H(z) = 1 + \frac{81/26}{z-(1/2)} + \frac{-(8/13)z - (10/39)}{z^2 + z + (1/3)}$$

$$= 1 + \frac{(81/26)z^{-1}}{1-(1/2)z^{-1}} + \frac{-(8/13)z^{-1} - (10/39)z^{-2}}{1 + z^{-1} + (1/3)z^{-2}}$$

Let $$H(z) = \frac{Y(z)}{X(z)} = 1 + \frac{(81/26)z^{-1}}{1-(1/2)z^{-1}} + \frac{-(8/13)z^{-1} - (10/39)z^{-2}}{1 + z^{-1} + (1/3)z^{-2}}$$

$$Y(z) = X(z) + Y_1(z) + Y_2(z)$$

where $Y_1(z) = \dfrac{(81/26)z^{-1}}{1-(1/2)z^{-1}} X(z)$

and $$Y_2(z) = \frac{-(8/13)z^{-1} - (10/39)z^{-2}}{1 + z^{-1} + (1/3)z^{-2}} X(z)$$

Also $$\frac{Y_1(z)}{X(z)} = H_1(z) \text{ and } \frac{Y_2(z)}{X(z)} = H_2(z)$$

On cross multiplying the above equations for $Y_1(z)$ and $Y_2(z)$, we have

$$Y_1(z) - \frac{1}{2}z^{-1}Y_1(z) = \frac{81}{26}z^{-1}X(z)$$

i.e. $$Y_1(z) = \frac{81}{26}z^{-1}X(z) + \frac{1}{2}z^{-1}Y_1(z)$$

and $$Y_2(z) + z^{-1}Y_2(z) + \frac{1}{3}z^{-2}Y_2(z) = -\frac{8}{13}z^{-1}X(z) - \frac{10}{39}z^{-2}X(z)$$

$$\therefore \qquad Y_2(z) = -\frac{8}{13} z^{-1} X(z) - \frac{10}{39} z^{-2} X(z) - z^{-1} Y_2(z) - \frac{1}{3} z^{-2} Y_2(z)$$

Using the equation for $Y_1(z)$, the direct form-I structure of $H_1(z)$ can be drawn as shown in Figure 4.44(a).

Using equation for $Y_2(z)$ the direct form-I structure of $H_2(z)$ can be drawn as shown in Figure 4.44(b).

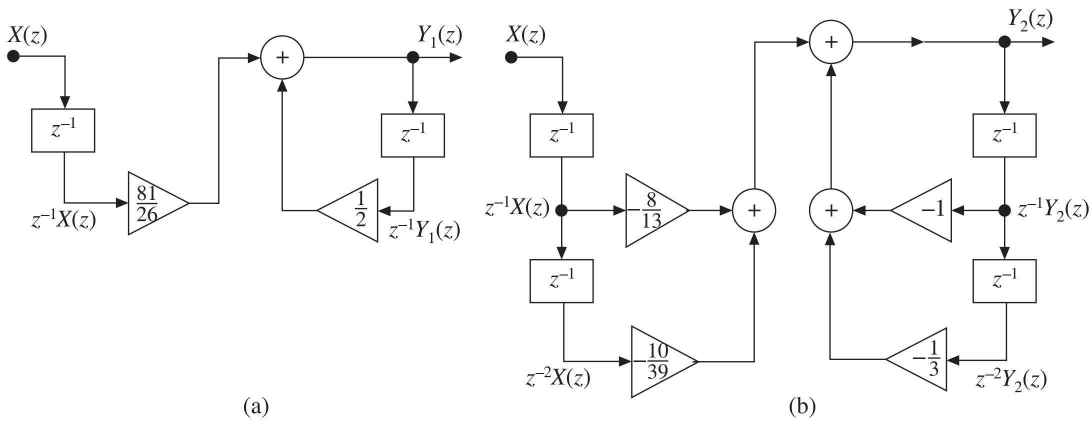


Figure 4.44 Direct form-I structure of (a) $H_1(z)$ and (b) $H_2(z)$ (Example 4.12).

The parallel structure of $H(z)$ is obtained by connecting the direct form-I structures of $H_1(z)$ and $H_2(z)$ as shown in Figure 4.45.

4.4 STRUCTURES FOR REALIZATION OF FIR SYSTEMS

FIR (Finite duration impulse response) systems are the systems whose impulse response consists of finite number of samples. They are designed by using only a finite number of samples of the infinite duration impulse response. The convolution formula for FIR system is given by

$$y(n) = \sum_{k=0}^{N-1} h(k)\, x(n-k)$$

where $h(n) = 0$ for $n < 0$ and $n \geq N$. This implies that, the impulse response selects only N samples of the input signal. In effect, the system acts as a window that views only the most recent N input signal samples in forming the output. It neglects or simply forgets all prior input samples. Thus, a FIR system has a finite memory of length N samples.

A system whose output depends only on the present and past inputs and not on past outputs is called a non-recursive system. Hence, for non-recursive systems, the output $y(n)$ is given by

$$y(n) = \mathrm{F}[x(n), x(n-1), \ldots, x(n-M)]$$

Hence, in general, an FIR system is of non-recursive type.

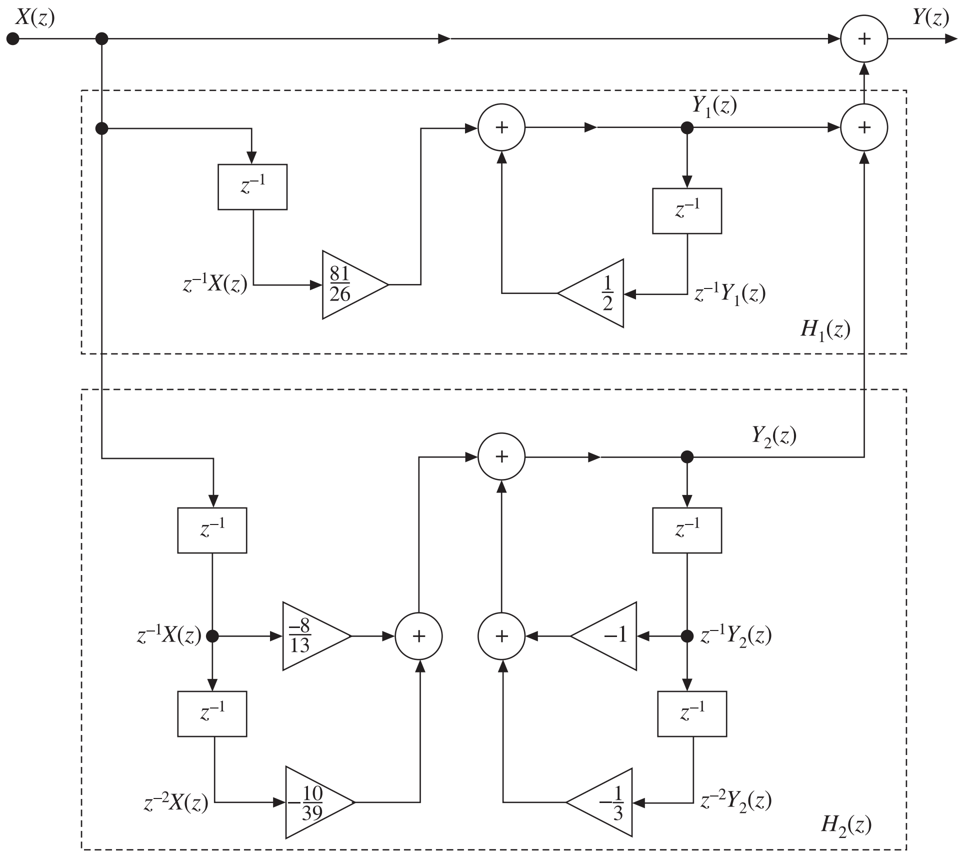


Figure 4.45 Parallel structure of $H(z)$ (Example 4.12).

In non-recursive system, $y(n_0)$ can be computed immediately without having $y(n_0 - 1)$, $y(n_0 - 2)$, ... Hence, the output of non-recursive system can be computed in any order.

[i.e. $y(50)$, $y(5)$, $y(2)$, $y(100)$, ...]

Transfer function of FIR system

In general, an FIR system is described by the difference equation

$$y(n) = \sum_{k=0}^{N-1} b_k x(n-k)$$

i.e. in general, in Finite Impulse Response (FIR) systems, the output at any instant depends only on the present and the past inputs. It does not depend on the past outputs.

Taking Z-transform on both sides of equation for $y(n)$, we get

$$y(z) = \sum_{k=0}^{N-1} b_k z^{-k} X(z)$$

The transfer function of the FIR system is:

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{N-1} b_k z^{-k} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_{N-1} z^{-(N-1)}$$

Also, we know that $H(z) = Z[h(n)]$, where $h(n)$ is the impulse response of the FIR system. Let us replace the index n by k.

$$\therefore \qquad H(z) = \sum_{k=0}^{N-1} h(k) z^{-k} = h(0) + h(1)z^{-1} + h(2)z^{-2} + \cdots + h(n-1) z^{-(N-1)}$$

On comparing the above two equations for $H(z)$, we can say that $b_k = h(k)$ for $k = 0, 1, 2, ..., (N-1)$.

The above equations for $Y(z)$ and $H(z)$ can be viewed as a computational procedure to determine the output sequence $y(n)$ from the input sequence $x(n)$. These equations can be used to construct the block diagram of the system using delays, adders and multipliers.

This block diagram is referred to as realization of the system or equivalently as a structure for realizing the system.

The different types of structures for realizing FIR systems are:

1. Direct form realization
2. Transposed form realization
3. Cascade realization
4. Lattice structure realization
5. Linear phase realization

4.4.1 Direct Form Realization of FIR System

Since there are no denominator components or poles in an FIR system, the direct form has only one structure which is called *direct form*. It realizes directly either the difference equation or the system function.

The direct form structure can be obtained from the general equation for $Y(z)$ governing the FIR system.

$$Y(z) = \sum_{k=0}^{N-1} b_k z^{-k} X(z)$$

$$= b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + \cdots + b_{N-2} z^{-(N-2)} X(z) + b_{N-1} z^{-(N-1)} X(z)$$

The direct form structure using the above equation for $Y(z)$ can be drawn as shown in Figure 4.46.

The direct form realization of Figure 4.46 is often called a *transversal* or *tapped-delay-line* filter because the output is just tapping the delayed inputs. This structure is also called a *canonical structure* because the number of delay elements used and the order of the filter is the same.

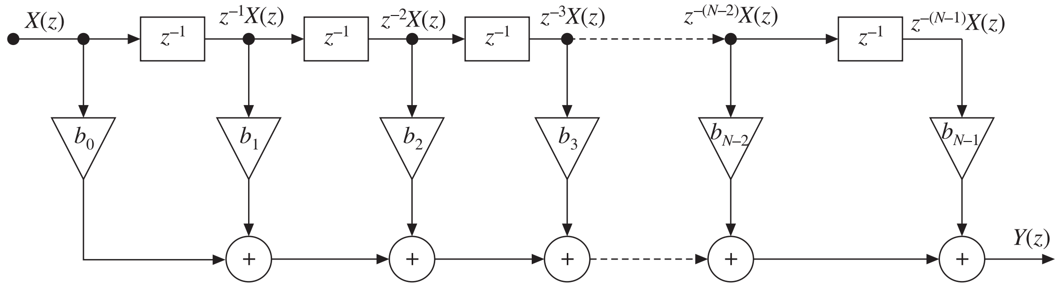


Figure 4.46 Direct form structure of FIR system.

4.4.2 Transposed Form Structure Realization of FIR System

The transposed form structure realization has already been discussed earlier with respect to IIR systems. The same holds good for FIR systems also.

Procedure to realize transposed form structure of FIR system

1. First realize the given difference equation or transfer function by using the direct form structure.
2. Reverse or transpose the direction of signal flow and interchange the input and output nodes.
3. Replace the junction points by adders and adders by junction points.
4. Fold the structure, which is the transposed form realization of an FIR system.

In general, the transposed structure realization of an FIR system has no advantages compared to the direct form structure. Whatever be the number of additions, multiplications and storage components needed for the direct form structure, the same number of elements are needed for the transposed structure too. The direct form structure and the transposed form structure of a general Nth order FIR system are shown in Figure 4.47.

EXAMPLE 4.13 Realize the second order FIR system

$$y(n) = 2x(n) + 4x(n-1) - 3x(n-2)$$

by using the transposed form structure.

Solution: The given system is:

$$y(n) = 2x(n) + 4x(n-1) - 3x(n-2)$$

Taking Z-transform on both sides, we have

$$Y(z) = 2X(z) + 4z^{-1}X(z) - 3z^{-2}X(z)$$

The direct form structure, the recovered realization structure and the transposed structure are shown in Figure 4.48 [(a), (b) and (c) respectively].

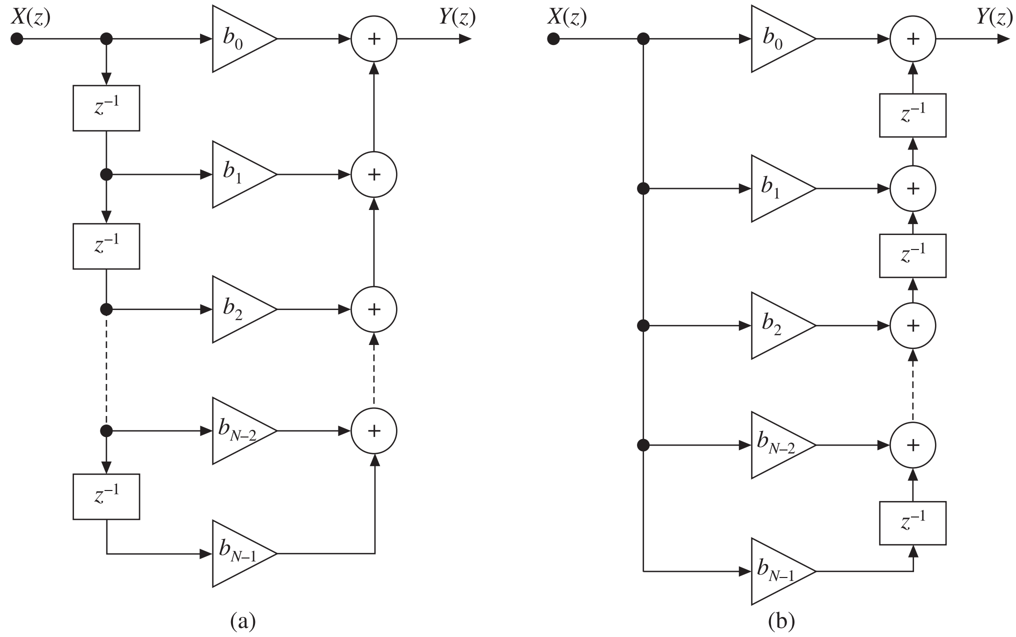


Figure 4.47 (a) A general direct form structure of FIR system (b) Transposed structure of FIR system.

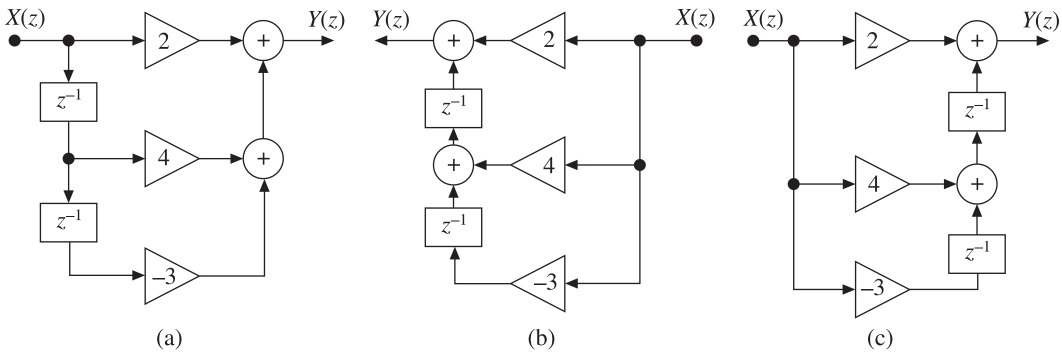


Figure 4.48 (a) Direct form structure (b) Recovered realization structure and (c) Transposed structure (Example 4.13).

4.4.3 Cascade Form Structure Realization of FIR System

The cascade form structure is nothing, but a cascade connection or a series connection of direct form structures. Hence, in cascade structure, the given transfer function $H(z)$ is broken into the product of many sub-transfer functions $H_1(z)$, $H_2(z)$, ..., $H_N(z)$ and each of these sub-transfer functions is realized separately in direct form structure and all these are connected in series or cascade.

The block diagram representation of a cascade form structure is shown in Figure 4.49.

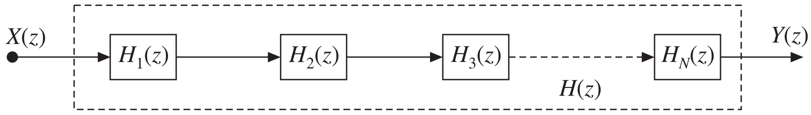


Figure 4.49 Block diagram of cascade structure.

The transfer function of FIR system, $H(z)$ is an $(N-1)$th order polynomial in z.

When N is odd, then $(N-1)$ will be even and so $H(z)$ will have $(N-1)/2$ second order factors. When N is odd, then

$$H(z) = \sum_{k=0}^{N-1} b_k z^{-k} = \prod_{i=1}^{\frac{N-1}{2}} (C_{0i} + C_{1i} z^{-1} + C_{2i} z^{-2}) = H_1 H_2 \ldots H_{\frac{N-1}{2}}$$

Each second order factor of the above equation for $H(z)$ can be realized in direct form and all the second order systems are connected in cascade to realize $H(z)$ as shown in Figure 4.50.

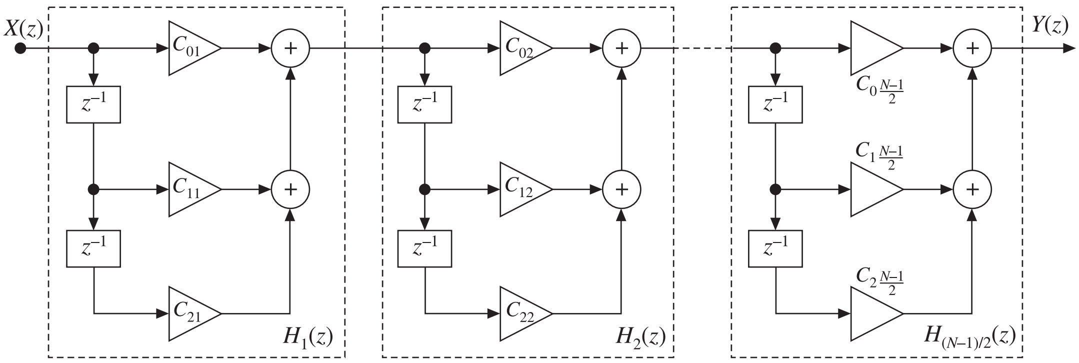


Figure 4.50 Cascade structure of FIR system when N is odd.

When N is even, then $(N-1)$ will be odd and so $H(z)$ will have one first order factor and $(N-2)/2$ second order factors.

When N is even, then

$$H(z) = \sum_{k=0}^{N-1} b_k z^{-k} = (C_{0i} + C_{1i} z^{-1}) \prod_{i=2}^{N/2} (C_{0i} + C_{1i} z^{-1} + C_{2i} z^{-2})$$

$$= H_1, H_2, \ldots, H_{N/2}$$

In this case, the cascade structure will have one first order section and $(N-2)/2$ second order sections.

Each one of them can be realized in direct form and all of them connected in cascade as shown in Figure 4.51.

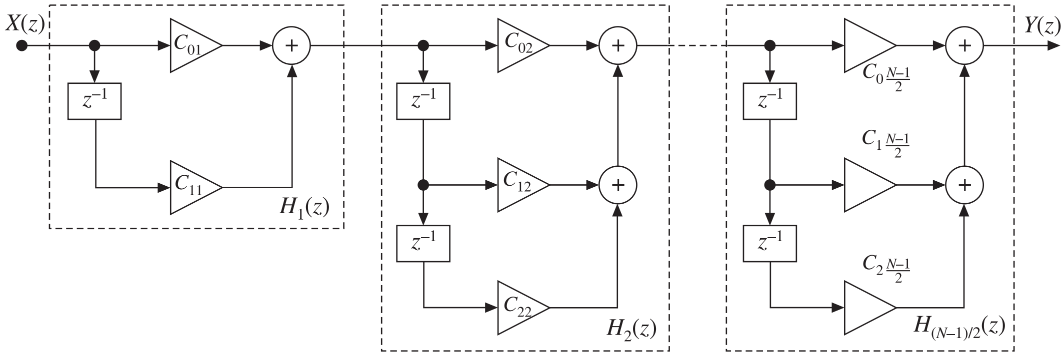


Figure 4.51 Cascade structure of FIR system when N is even.

4.4.4 Lattice Structure Realization of FIR Systems

The FIR structures discussed till now are used in general. However, the lattice structure is extensively used in digital speech processing.

The lattice structure consists of two different paths through which the input $x(n)$ is processed. Hence, the lattice structure has two different output set-ups: $y(n)$ and $y'(n)$. $y(n)$ is the real output and $y'(n)$ is the supporting output, which offers support for obtaining the output for the next stage. A single stage lattice structure is shown in Figure 4.52, wherein K is called the reflection coefficient. Therefore, the output from the first stage of the lattice structure, which is the first order FIR system, is given below.

$$y_1(n) = x(n) + K_1 x(n-1)$$

$$y_1'(n) = K_1 x(n) + x(n-1)$$

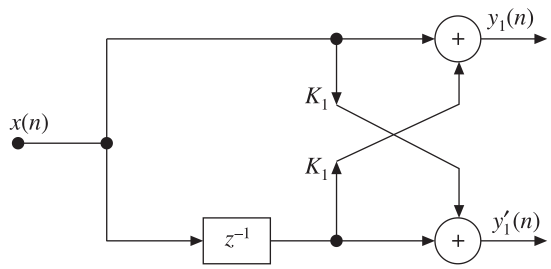


Figure 4.52 Single stage lattice structure.

Similarly, the two-stage lattice structure is shown in Figure 4.53. The output from the second stage of Figure 4.53, which is the second order FIR system is given as follows:

$$y_2(n) = y_1(n) + K_2 y_1'(n-1)$$

$$y_2'(n) = K_2 y_1(n) + y_1'(n-1)$$

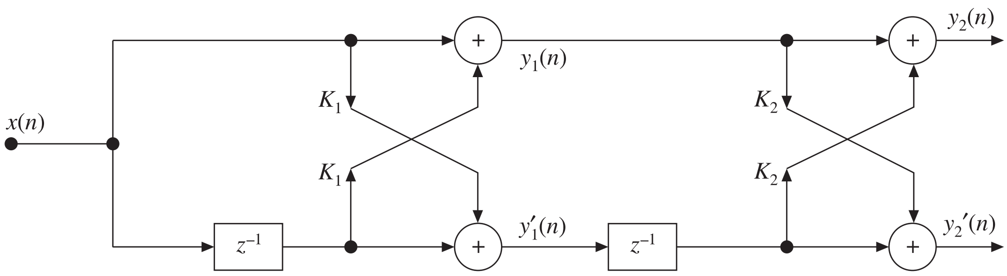


Figure 4.53 Two-stage lattice structure.

Substituting the values of $y_1(n)$ and $y'_1(n)$ in the above equations for $y_2(n)$ and $y'_2(n)$ we get

$$y_2(n) = [x(n) + K_1 x(n-1)] + K_2[K_1 x(n-1) + x(n-2)]$$

or $$y_2(n) = x(n) + K_1[1 + K_2]x(n-1) + K_2 x(n-2)$$

$$y'_2(n) = K_2[x(n) + K_1 x(n-1)] + K_1 x(n-1) + x(n-2)]$$

or $$y'_2(n) = K_2 x(n) + K_1[1 + K_2]x(n-1) + x(n-2)$$

Let $$\alpha_2(0) = 1,\ \alpha_2(1) = K_1(1 + K_2) \text{ and } \alpha_2(2) = K_2$$

Then the equation for $y_2(n)$ changes to

$$y_2(n) = \alpha_2(0)\,x(n) + \alpha_2(1)\,x(n-1) + \alpha_2(2)\,x(n-2)$$

or $$y_2(n) = \sum_{k=0}^{2} \alpha_2(k)\,x(n-k)$$

Therefore, in general, the output of the mth order FIR system by using the lattice structure can be written as:

$$y_m(n) = \sum_{k=0}^{m} \alpha_m(k)x(n-k)$$

The above equation is the convolution sum. Using the convolution property it can be re-written as:

$$Y_m(z) = \alpha_m(z)\,X(z)$$

Hence, the transfer function of the FIR system can be obtained as

$$\alpha_m(z) = \frac{Y_m(z)}{X(z)}$$

The FIR filter having a system function given above is called the *forward prediction.*

Procedure to realize the lattice structure of FIR system

1. If the coefficient of the present input $x(n)$ is not unity, convert it to unity by taking common of the coefficient of the present input.
2. Find the order of the difference equation and compare the coefficients of the given difference equation with the coefficients of the same order lattice structure output involving the reflection coefficients K_1, K_2, K_3, ...
3. Assign the calculated values of K_1, K_2, K_3, ... and construct the structure.

In order to realize by using lattice structure, it is enough to find the values of reflection coefficients. Therefore, it is easily programmable. But the number of components used, specially adders and multipliers will increase.

EXAMPLE 4.14 Realize a system with $H(z) = 5 + 3z^{-1}$ by using lattice structure.

Solution: Given

$$H(z) = \frac{Y(z)}{X(z)} = 5 + 3z^{-1}$$

or

$$Y(z) = 5X(z) + 3z^{-1}X(z)$$

Taking the inverse Z-transform on both sides, we get

$$y(n) = 5x(n) + 3x(n-1)$$

$$= 5[x(n) + \frac{3}{5}x(n-1)] = 5p(n)$$

This is of first order. So comparing $p(n) = x(n) + \frac{3}{5}x(n-1)$ with the standard equation for first order lattice structure, i.e. with $y(n) = x(n) + K_1x(n-1)$, we get

$$K_1 = \frac{3}{5}$$

Therefore, the lattice structure realization of the given first order FIR system is as shown in Figure 4.54.

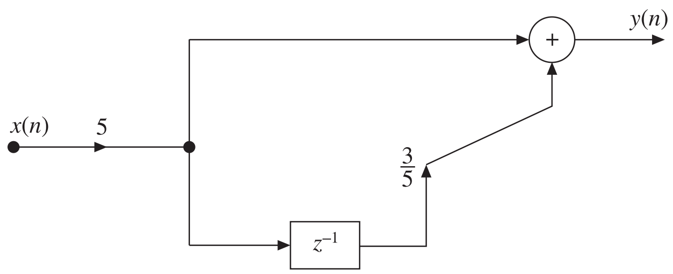


Figure 4.54 Lattice structure (Example 4.14).

EXAMPLE 4.15 Determine the lattice coefficients corresponding to the FIR system with the system function $H(z) = 1 + \frac{7}{9}z^{-1} + \frac{3}{5}z^{-2}$ and realize it.

Solution: Given
$$H(z) = \frac{Y(z)}{X(z)} = 1 + \frac{7}{9}z^{-1} + \frac{3}{5}z^{-2}$$

$\therefore$
$$Y(z) = X(z) + \frac{7}{9}z^{-1}X(z) + \frac{3}{5}z^{-2}X(z)$$

Taking inverse Z-transform on both sides, we get

$$y(n) = x(n) + \frac{7}{9}x(n-1) + \frac{3}{5}x(n-2)$$

This corresponds to a second order system. Comparing this with the standard equation for second order lattice structure, i.e. with $y(n) = x(n) + K_1(1+K_2)x(n-1) + K_2x(n-2)$, we get

$$K_2 = \frac{3}{5}$$

and
$$K_1(1+K_2) = \frac{7}{9}$$

or
$$K_1 = \frac{7/9}{1+(3/5)} = \frac{35}{72}$$

Therefore, the lattice structure realization of the given FIR system is shown in Figure 4.55.

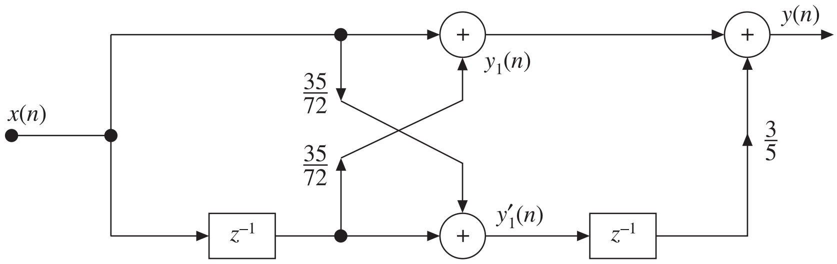


Figure 4.55 Lattice structure (Example 4.15).

4.4.5 Linear Phase Realizations

An FIR system is said to be linear phase, if it satisfies the condition

$$h(k) = \pm h(N-1-k)$$

An FIR system which satisfies the relation $h(k) = h(N-1-k)$ is called a *symmetric FIR system* and an FIR system which satisfies the relation $h(k) = -h(N-1-k)$ is called an

antisymmetric FIR system. Here we discuss symmetric FIR systems. The symmetric condition may be viewed as

1. Odd symmetry
2. Even symmetry

The impulse response is symmetrical, i.e. $h(k) = h(N-1-k)$ means

$$h(0) = h(N-1);\ h(1) = h(N-2);\ \ldots;\ h\left(\frac{N}{2}-1\right) = h\left(\frac{N}{2}\right)$$

By using this symmetry condition, it is possible to reduce the number of multipliers required for the realization of FIR system.

Odd symmetry (N = odd)

A function is said to be of odd symmetry if it has an odd number of samples and satisfies the condition $h(k) = h(N-1-k)$.

For example, if $N = 11$ (i.e. N is odd), then a linear phase FIR system will have

$$h(0) = h(9-1-0) = h(8),\ h(1) = h(9-1-1) = h(7),\ h(2) = h(9-1-2) = h(6)$$

$$h(3) = h(9-1-3) = h(5),\ h(4) = h(9-1-4) = h(4)$$

In case of odd symmetry, it is obvious that the central sample is the same for both LHS and RHS. Hence, in case of odd symmetry, the common or central sample will be lying at $(N-1)/2$ The graphical representation of the above impulse response for $N = 11$ is shown in Figure 4.56.

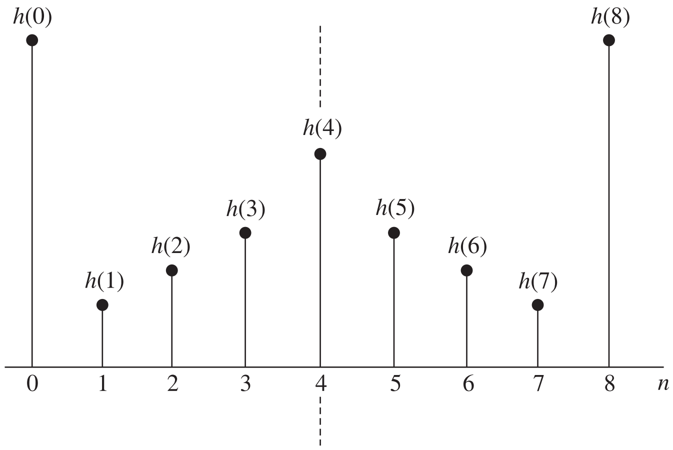


Figure 4.56 Linear phase impulse response sequence for $N = 11$.

The linear phase FIR system can be remodeled as

$$H(z) = \sum_{k=0}^{N-1} h(k)\, z^{-k} = \frac{Y(z)}{X(z)}$$

or $$Y(z) = [h(0) + h(1)z^{-1} + h(2)z^{-2} + \cdots + h(N-2)z^{-(N-2)} + h(N-1)z^{-(N-1)}]X(z)$$

For an odd symmetry linear phase FIR system, we have

$$h(0) = h(N-1),\ h(1) = h(N-2),\ \ldots,\ h\left(\frac{N-3}{2}\right) = h\left(\frac{N+1}{2}\right),\ h\left(\frac{N-1}{2}\right) = h\left(\frac{N-1}{2}\right)$$

The impulse $h\left(\frac{N-1}{2}\right)$ will remain single.

Substituting these in the expression for $Y(z)$, we have the direct form structure of linear phase FIR system as

$$Y(z) = h_0[X(z) + z^{-(N-1)}X(z)] + h_1[z^{-1}X(z) + z^{-(N-2)}X(z)] + \cdots +$$
$$h_{\frac{N-5}{2}}\left[z^{-\frac{(N-5)}{2}}X(z) + z^{-\frac{(N+3)}{2}}X(z)\right] + h_{\frac{N-3}{2}}\left[z^{-\frac{(N-3)}{2}}X(z) + z^{-\frac{(N+1)}{2}}X(z)\right] + h_{\frac{N-1}{2}}z^{-\frac{(N-1)}{2}}X(z)$$

which is constructed as shown in Figure 4.57.

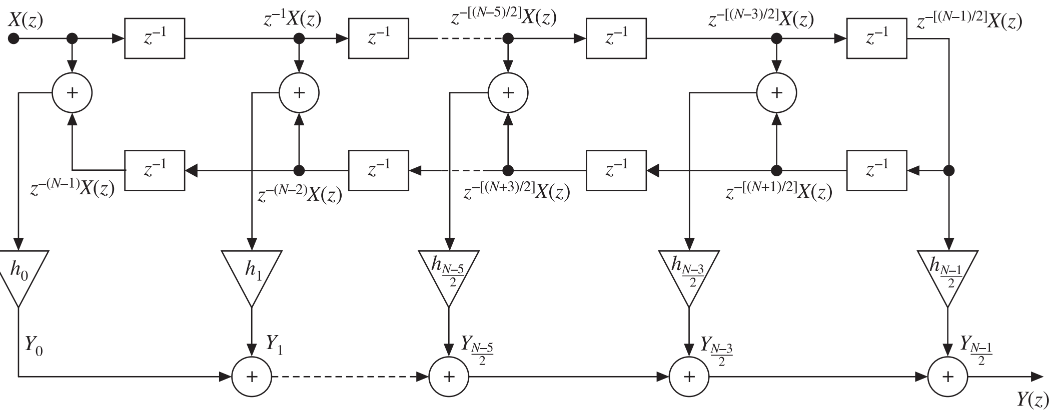


Figure 4.57 Direct form realization of FIR system when N is odd.

In Figure 4.57

$$Y_0 = h_0[X(z) + z^{-(N-1)}X(z)];\ Y_1 = h_1[z^{-1}X(z) + z^{-(N-2)}X(z)]$$

$$Y_{\frac{N-5}{2}} = h_{\frac{N-5}{2}}\left[z^{-\frac{(N-5)}{2}}X(z) + z^{-\frac{(N+3)}{2}}X(z)\right]$$

$$Y_{\frac{N-3}{2}} = h_{\frac{N-3}{2}}\left[z^{-\frac{(N-3)}{2}}X(z) + z^{-\frac{(N+1)}{2}}X(z)\right]$$

Even symmetry

A function is said to have even symmetry if it has even number of samples and satisfies the condition $h(k) = h(N - 1 - k)$.

For example, if $N = 8$ (i.e. N is even), then a linear phase symmetric FIR system will have

$$h(0) = h(8-1-0) = h(7),\ h(1) = h(8-1-1) = h(6),\ h(2) = h(8-1-2) = h(5)$$

and
$$h(3) = h(8-1-3) = h(4)$$

In the case of even symmetry, since there is no central sample, a virtual central sample is selected at $k = (N - 1)/2$. In this case virtual sample is at $k = (8–1)/2 = 3.5$.

The graphical representation of the above impulse responses for $N = 8$ is shown in Figure 4.58.

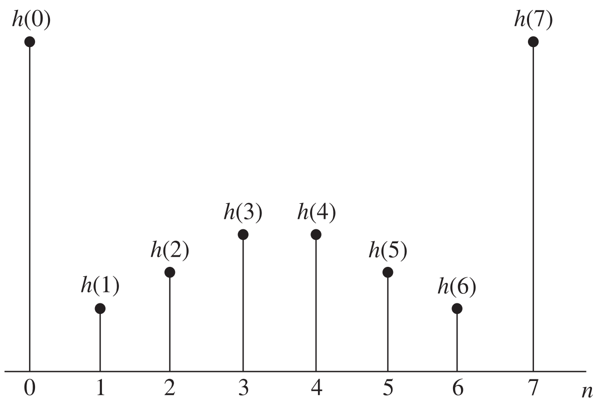


Figure 4.58 Linear phase impulse response sequence for $N = 8$.

For an even symmetry linear phase FIR system, we have

$$h(0) = h(N-1),\ h(1) = h(N-2),\ \ldots,\ h\left(\frac{N}{2}-1\right) = h\left(\frac{N}{2}\right)$$

Substituting these in the expression for $Y(z)$, we have the direct form structure of linear phase FIR system as:

$$Y(z) = h_0[X(z) + z^{-(N-1)}X(z)] + h_1[z^{-1}X(z) + z^{-(N-2)}X(z)] + \cdots + h_{\frac{N}{2}-1}[z^{-\left(\frac{N}{2}-1\right)}X(z) + z^{-\frac{N}{2}}X(z)]$$

as shown in Figure 4.59.

In Figure 4.59.

$$Y_0 = h_0[X(z) + z^{-(N-1)}X(z)];\quad Y_1 = h_1[z^{-1}X(z) + z^{-(N-2)}X(z)]$$

$$Y_{\frac{N}{2}-2} = h_{\frac{N}{2}-2}\left[z^{-\left(\frac{N}{2}-2\right)}X(z) + z^{-\left(\frac{N}{2}+1\right)}X(z)\right]$$

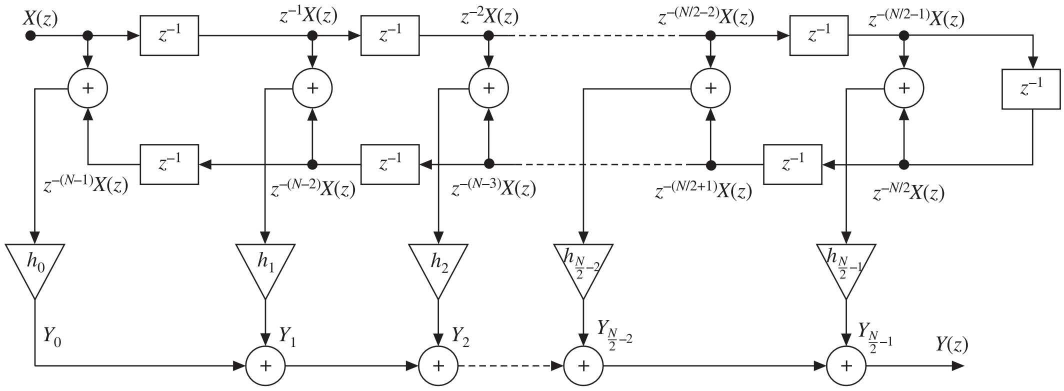


Figure 4.59 Direct form realization of FIR system when N is even.

$$Y_{\frac{N}{2}-1} = h_{\frac{N}{2}-1}\left[z^{-\left(\frac{N}{2}-1\right)}X(z) + z^{-\left(\frac{N}{2}\right)}X(z)\right]$$

$$Y_{\frac{N-1}{2}} = h_{\frac{N-1}{2}}\left[z^{-\left(\frac{N-1}{2}\right)}X(z)\right]$$

EXAMPLE 4.16 Draw the direct form structure of the FIR system described by the transfer function

$$H(z) = 1 + \frac{1}{5}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{3}z^{-3} + \frac{1}{7}z^{-4} + \frac{1}{6}z^{-5}$$

Solution: Let $H(z) = \dfrac{Y(z)}{X(z)} = 1 + \dfrac{1}{5}z^{-1} + \dfrac{3}{4}z^{-2} + \dfrac{1}{3}z^{-3} + \dfrac{1}{7}z^{-4} + \dfrac{1}{6}z^{-5}$

$$\therefore \qquad Y(z) = X(z) + \frac{1}{5}z^{-1}X(z) + \frac{3}{4}z^{-2}X(z) + \frac{1}{3}z^{-3}X(z) + \frac{1}{7}z^{-4}X(z) + \frac{1}{6}z^{-5}X(z)$$

The direct form structure of FIR system can be obtained directly from the above equation for $Y(z)$ as shown in Figure 4.60.

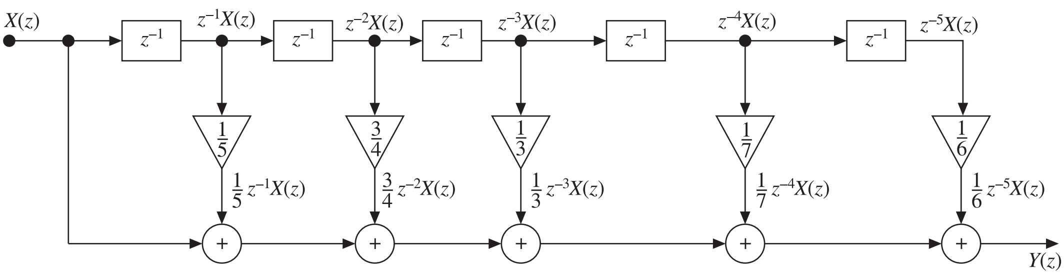


Figure 4.60 Direct form structure of $H(z)$ (Example 4.16).

EXAMPLE 4.17 Realize the filter transfer function given by the expression below using the direct form

$$H(z) = (1 - z^{-1})(1 + 2z^{-1} - 3z^{-2})$$

Solution: Since the given filter transfer function has only the numerator part or zeros, it is an FIR system. To realize in direct form, multiply the factors and obtain a single expression.

Given $$H(z) = \frac{Y(z)}{X(z)} = (1 - z^{-1})(1 + 2z^{-1} - 3z^{-2}) = 1 + z^{-1} - 5z^{-2} + 3z^{-3}$$

On cross multiplying, we get

$$Y(z) = X(z) + z^{-1}X(z) - 5z^{-2}X(z) + 3z^{-3}X(z)$$

The direct form realization of $Y(z)$ is shown in Figure 4.61.

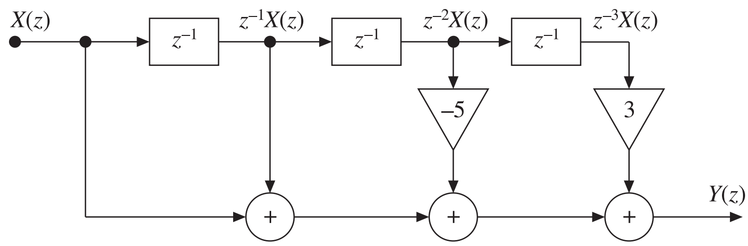


Figure 4.61 Direct form realization of $H(z)$ (Example 4.17).

EXAMPLE 4.18 Realize the following systems with minimum number of multipliers.

(a) $H(z) = \frac{1}{3} + \frac{1}{5}z^{-1} + \frac{2}{3}z^{-2} + \frac{1}{5}z^{-3} + \frac{1}{3}z^{-4}$

(b) $H(z) = \frac{1}{2} + \frac{1}{4}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{2}z^{-3}$

(c) $H(z) = \left(1 + \frac{1}{3}z^{-1} + z^{-2}\right)\left(1 + \frac{1}{5}z^{-1} + z^{-2}\right)$

Solution:

(a) Given $$H(z) = \frac{1}{3} + \frac{1}{5}z^{-1} + \frac{2}{3}z^{-2} + \frac{1}{5}z^{-3} + \frac{1}{3}z^{-4}$$

By the definition of Z-transform, we get

$$H(z) = \sum_{n=0}^{\infty} h(n)z^{-n} = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + \cdots$$

On comparing the above two equations for $H(z)$, we get

Impulse response $$h(n) = \left\{\frac{1}{3}, \frac{1}{5}, \frac{2}{3}, \frac{1}{5}, \frac{1}{3}\right\}$$

Here $h(n)$ satisfies the condition, $h(n) = H(N-1-n)$ and so the impulse response is symmetrical. Hence, the system has linear phase and can be realized with minimum number of multipliers.

Let $$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{3} + \frac{1}{5}z^{-1} + \frac{2}{3}z^{-2} + \frac{1}{5}z^{-3} + \frac{1}{3}z^{-4}$$

$$\therefore \quad Y(z) = \frac{1}{3}X(z) + \frac{1}{5}z^{-1}X(z) + \frac{2}{3}z^{-2}X(z) + \frac{1}{5}z^{-3}X(z) + \frac{1}{3}z^{-4}X(z)$$

$$= \frac{1}{3}[X(z) + z^{-4}X(z)] + \frac{1}{5}[z^{-1}X(z) + z^{-3}X(z)] + \frac{2}{3}z^{-2}X(z)$$

The direct form structure of linear phase FIR system is constructed using the above equation for $Y(z)$ as shown in Figure 4.62.

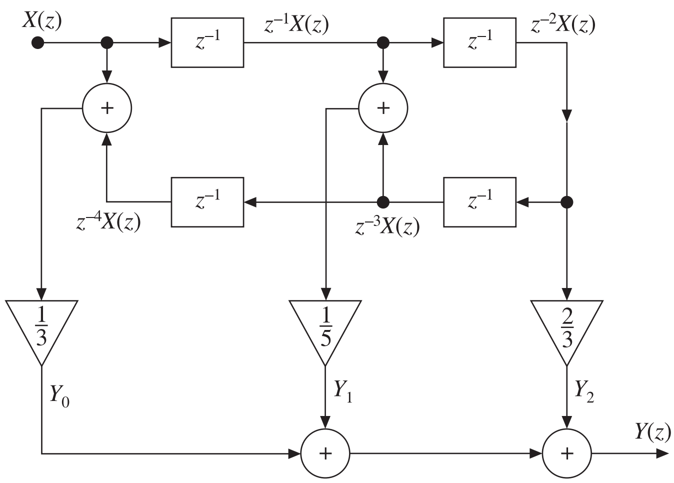


Figure 4.62 Linear phase realization of $H(z)$ (Example 4.18(a)).

(b) Given $$H(z) = \frac{1}{2} + \frac{1}{4}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{2}z^{-3}$$

Let $$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{2} + \frac{1}{4}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{2}z^{-3}$$

$$\therefore \qquad Y(z) = \frac{1}{2}X(z) + \frac{1}{4}z^{-1}X(z) + \frac{1}{4}z^{-2}X(z) + \frac{1}{2}z^{-3}X(z)$$

$$= \frac{1}{2}[X(z) + z^{-3}X(z)] + \frac{1}{4}[z^{-1}X(z) + z^{-2}X(z)]$$

The direct form realization of $H(z)$ with minimum number of multipliers (i.e. linear phase realization) is obtained using the above equation for $Y(z)$ as shown in Figure 4.63.

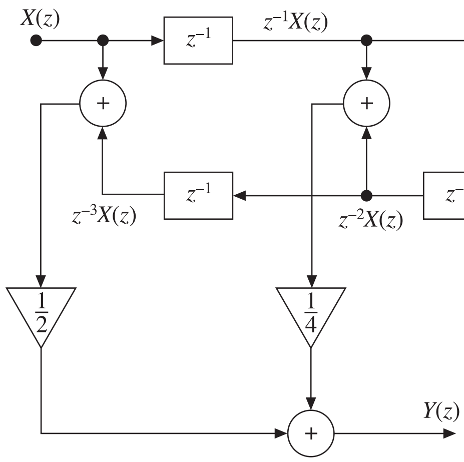


Figure 4.63 Linear phase realization of $H(z)$ [Example 4.18(b)].

(c) Given $$H(z) = \left(1 + \frac{1}{3}z^{-1} + z^{-2}\right)\left(1 + \frac{1}{5}z^{-1} + z^{-2}\right)$$

The given system can be realized as cascade of two second order systems. Each system can be realized with minimum number of multipliers (i.e. linear phase realization).

Let $$H(z) = H_1(z)H_2(z)$$

where $$H_1(z) = 1 + \frac{1}{3}z^{-1} + z^{-2} \text{ and } H_2(z) = 1 + \frac{1}{5}z^{-1} + z^{-2}$$

Let $$H_1(z) = \frac{Y_1(z)}{X(z)} = 1 + \frac{1}{3}z^{-1} + z^{-2}$$

$$\therefore \qquad Y_1(z) = X(z) + \frac{1}{3}z^{-1}X(z) + z^{-2}X(z)$$

$$= [X(z) + z^{-2}X(z)] + \frac{1}{3}z^{-1}X(z)$$

The linear phase realization structure of $H_1(z)$ using the above equation for $Y_1(z)$ is as shown in Figure 4.64(a).

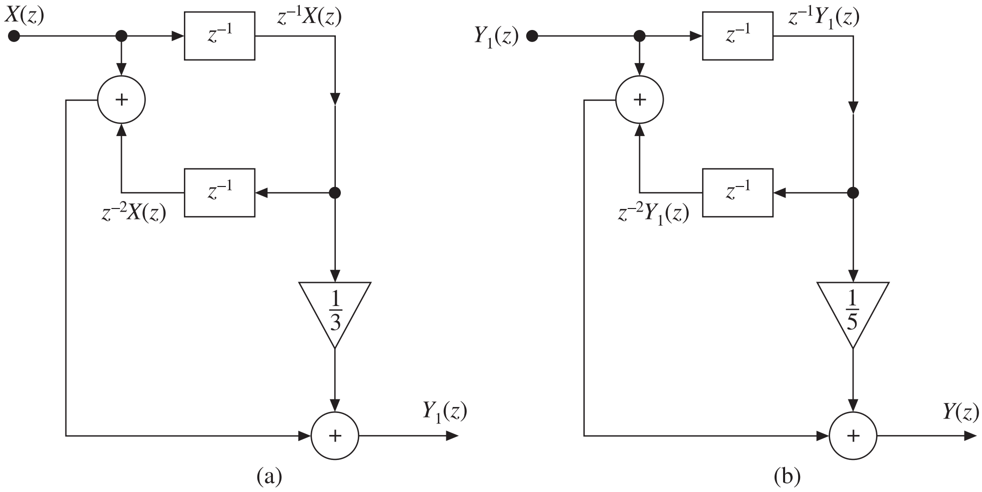


Figure 4.64 Linear phase realization of (a) $H_1(z)$ and (b) $H_2(z)$ [Example 4.18(c)].

Let
$$H_2(z) = \frac{Y(z)}{Y_1(z)} = 1 + \frac{1}{5}z^{-1} + z^{-2}$$

$\therefore$
$$Y(z) = Y_1(z) + \frac{1}{5}z^{-1}Y_1(z) + z^{-2}Y_1(z)$$
$$= [Y_1(z) + z^{-2}Y_1(z)] + \frac{1}{5}z^{-1}Y_1(z)$$

The linear phase realization structure of $H_2(z)$ using the above equation for $Y(z)$ is as shown in Figure 4.64(b).

The linear phase structure of $H(z)$ is obtained by connecting the systems $H_1(z)$ and $H_2(z)$ in cascade as shown in Figure 4.65.

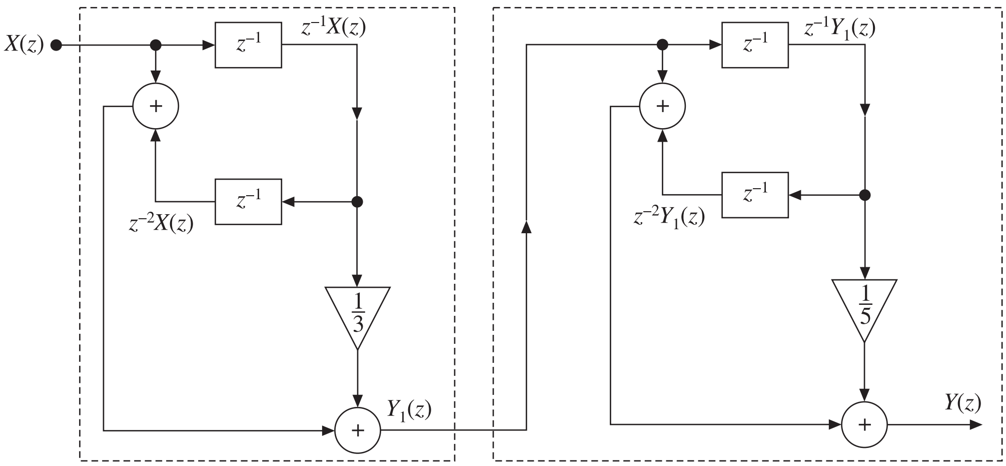


Figure 4.65 Cascade realization of $H(z)$ [Example 4.18(c)].

TABLE 4.2 Comparison between IIR and FIR systems

IIR system	*FIR system*
• Impulse response has infinite number of samples.	Impulse response has finite number of samples.
• Output depends upon present input sample, past input samples and past output samples.	Output depends upon present input sample and past input samples.
• Both poles and zeros are present.	Only zeros are present.
• It is a recursive system.	It is a non-recursive system.
• Not always stable. Finite word length effect can make it unstable.	Stability is guaranteed, because the system impulse response is of finite length and therefore bounded.
• It is a closed loop system.	It is an open loop system.
• More complex to implement.	Less complex to implement.
• Reliability is not always guaranteed.	Reliability is always guaranteed.
• Design is first done in analog domain and then transformed into a digital domain.	Design is done directly in digital domain.
• Magnitude response is better, sharp cut-off with lower order itself.	It requires higher order system to get sharp magnitude response.
• It does not have linear phase characteristics.	It has linear phase characteristics.

SHORT QUESTIONS WITH ANSWERS

1. What is cascade form realization?
Ans. Cascade form realization is one in which the given transfer function is expressed as a product of several transfer functions and each of these transfer functions is realized in direct form-II and then all of them are cascaded.

2. What is parallel form realization?
Ans. Parallel form realization is one in which the given transfer function is expressed in its partial fractions and each factor is realized in direct form-II and then all those realized structures are connected in parallel.

3. What is recursive and non-recursive system? Give an example.
Ans. If the output $y(n)$ at time n of a system depends on present and past inputs as well as on past outputs, then the system is called a recursive system.

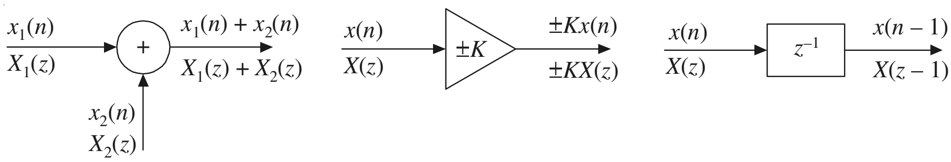
Example: $y(n) = y(n-1) + 0.5x(n) + x(n-1)$

If the output $y(n)$ at time n of a system does not depend on past output values, then the system is called a non-recursive system. In non-recursive systems, the outputs are functions of present and past outputs.

Example: $y(n) = x(n) + 2x(n-1) - 1.5x(n-2)$

4. What are the basic elements used to construct the block diagram of a discrete-time system? Draw their symbols.

Ans. The basic elements used to construct the block diagram of a discrete-time system are: Adder, Constant multiplier and Unit delay element. The symbols are given below.



(a) Adder (b) Multiplier (c) Unit delay element

5. What is an IIR system?

Ans. An IIR system is one which is designed by selecting all the infinite samples of impulse response.

6. Write the convolution sum formula for IIR system?

Ans. The convolution sum formula for IIR system is

$$y(n) = \sum_{k=0}^{\infty} h(k)\, x(n-k)$$

7. Write the general difference equation and transfer function of an IIR system.

Ans. In general, an IIR system is described by the difference equation

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

Its transfer function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

8. What are the factors that influence the choice of structure for realization of an LTI system?

Ans. The major factors that influence the choice of structure for realization of LTI system are: computational complexity, memory requirements and finite word length effects. Other factors that influence the choice of realization are: whether the structure lends itself to parallel processing or whether the computations can be pipelined.

9. List the different types of structures for realization of IIR systems.
Ans. The different types of structures for the realization of IIR systems are:

(a) Direct form-I structure
(b) Direct form-II structure
(c) Transposed form structure
(d) Cascade form structure
(e) Parallel form structure
(f) Lattice structure
(g) Ladder structure

10. What is the advantage of direct form-II structure when compared to direct form-I structure? How?
Ans. The advantage of direct form-II structure when compared to direct form-I structure is:

The direct form-II structure requires less amount of memory when compared to direct form-I structure. This is because in direct form-II structure, the number of delay elements required is exactly half of that for direct form-I structure when the number of poles and zeros are equal.

11. Why Direct form-I is called non-canonical structure?
Ans. Since the number of delay elements used in direct form-I is more than the order of the difference equation, direct form-I is called a non-canonical structure.

12. Why Direct form-II is called canonical structure?
Ans. Since the number of delay elements used in direct form-II is the same as that of the order of the difference equation, direct form-II is called a canonical structure.

13. What are the difficulties in cascade realization?
Ans. The difficulties in cascade realization are:
(a) Decision of pairing poles and zeros.
(b) Deciding the order of cascading the first and second order sections.
(c) Scaling multipliers should be provided between individual sections to prevent the filter variables from becoming too large or too small.

14. What is the advantage in cascade and parallel realization of IIR systems?
Ans. The advantage in cascade and parallel realization of IIR systems is that the sensitivity of frequency response characteristics to quantization of the coefficients is minimized.

15. What is an FIR system?
Ans. An FIR system is one which is designed by selecting only a finite number of samples of impulse response.

16. Write the convolution sum formula for FIR system?
Ans. The convolution sum formula for FIR system is

$$y(n) = \sum_{k=0}^{N-1} h(k)\, x(n-k)$$

17. Write the general difference equation and transfer function of an FIR system.
Ans. In general, an FIR system is described by the difference equation

$$y(n) = \sum_{k=0}^{N-1} b_k x(n-k)$$

Its transfer function is

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{N-1} b_k z^{-k} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_{N-1} z^{-(N-1)}$$

18. List the different types of structures for realizing FIR systems.
Ans. The different types of structures for realizing FIR systems are:
(a) Direct form realization
(b) Transposed form realization
(c) Cascade realization
(d) Lattice structure realization
(e) Linear phase realization

19. What is the advantage of linear phase realization of FIR systems?
Ans. The advantage of linear phase realization of FIR systems is that the system can be realized with minimum number of delay elements.

REVIEW QUESTIONS

1. Discuss the basic elements used to construct the block diagram of discrete-time systems.
2. Explain the factors that influence the choice of structure for realization of a LTI system.
3. Write the difference equations for FIR and IIR system and hence derive the transfer functions of FIR and IIR systems.
4. Discuss the different methods of realization of IIR systems and explain how conversion can be made from direct form-I structure to direct form-II structure.
5. Discuss the different methods of realization of FIR systems.
6. Compare FIR and IIR systems.
7. Compare cascade and parallel form realizations.

FILL IN THE BLANKS

1. A system whose output $y(n)$ at time n depends only on the present and past inputs is called a _______ system.
2. A system whose output $y(n)$ at time n depends on the past outputs is called a _______ system.
3. The basic elements used to construct the block diagram of a discrete-time system are _______, _______, _______.
4. _______ refers to the number of arithmetic operations required to compute an output value $y(n)$ for the system.

5. _______ refers to the number of memory locations required to store the system parameters, past inputs and outputs and any intermediate computed values.
6. _______ refer to the quantization effects that are inherent in any digital implementation of the system, either in hardware or in software.
7. In _______ systems, the impulse response consists of infinite number of samples.
8. The convolution sum formula for IIR system is _______.
9. The difference equation describing an IIR system is _______.
10. The transfer function of an IIR system is _______.
11. The different types of structures for realizing IIR systems are _______, _______, _______, _______.
12. _______ structure provides a direct relation between time domain and *z*-domain equations.
13. Direct form-II realization of discrete-time systems uses less number of _______ than direct form-I realization.
14. In _______ systems, the impulse response consists of finite number of samples.
15. The convolution sum formula for FIR systems is _______.
16. The difference equation describing an FIR system is _______.
17. The transfer function of an FIR system is _______.
18. The different types of structures for realizing FIR systems are _______, _______, _______.
19. In FIR systems, for linear phase response, the _______ should be symmetrical.
20. Linear phase results in reduction of the number of _______ required for the realization of FIR system.

OBJECTIVE TYPE QUESTIONS

1. A system whose output $y(n)$ at time n depends on any number of past output values is called a
   (a) recursive system (b) non-recursive system
   (c) causal system (d) non-causal system
2. A system whose output $y(n)$ at time n depends only on present and past input values is called a
   (a) recursive system (b) non-recursive system
   (c) causal system (d) non-causal system
3. The structure which uses less number of delay elements is
   (a) direct form-I (b) direct form-II
   (c) cascade form (d) parallel form

4. The number of multipliers required for the realization of FIR systems is reduced if we choose

(a) direct form (b) cascade form

(c) parallel form (d) linear phase realization

PROBLEMS

1. Construct the block diagram and signal flow graph of a discrete-time system whose input-output relations are described by the difference equation

(a) $y(n) = 0.5x(n) + 0.5x(n-1)$

(b) $y(n) = 0.25y(n-1) + 0.5x(n) + 0.75x(n-1)$

2. Find the digital network in direct and transposed form for the system described by the difference equation

$$y(n) = x(n) - 0.3x(n-1) - 0.7x(n-2) + 0.6y(n-1) + 0.8y(n-2)$$

3. Determine the direct form-I and direct form-II realizations of the following LTI systems.

(a) $y(n) = -0.5y(n-1) + 0.25y(n-2) + 0.125y(n-3) + x(n) + 0.5x(n-1) + 0.75x(n-2)$

(b) $y(n) = -\frac{3}{8}y(n-1) + \frac{3}{32}y(n-2) + \frac{1}{64}y(n-3) + x(n) + 3x(n-1) + 2x(n-2)$

4. An LTI system is described by the difference equation

$$y(n) = a_1 y(n-1) + x(n) + b_1 x(n-1)$$

Realize it in direct form-I structure and convert it to direct form-II structure.

5. Obtain the direct form-I, direct form-II, cascade and parallel realization of the following LTI systems.

(a) $y(n) = \frac{13}{12}y(n-1) - \frac{9}{24}y(n-2) + \frac{1}{24}y(n-3) + x(n) - 4x(n-1) + 3x(n-2)$

(b) $y(n) = \frac{1}{6}y(n-1) + \frac{1}{6}y(n-2) + x(n) + \frac{1}{6}x(n-1) - \frac{1}{6}x(n-2)$

6. Realize the system in cascade and parallel forms

(a) $$H(z) = \frac{1 + \frac{1}{3}z^{-1}}{\left(1 - 2z^{-1} + \frac{1}{3}z^{-2}\right)\left(1 - z^{-1} + \frac{1}{5}z^{-2}\right)}$$

(b) $$H(z) = \frac{1 + \frac{1}{2}z^{-1}}{\left(1 - z^{-1} + \frac{1}{4}z^{-2}\right)\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)}$$

7. Realize the following IIR system in cascade and parallel forms

$$y(n)+\frac{1}{4}y(n-1)-\frac{1}{8}y(n-2)=x(n)-2x(n-1)+x(n-2)$$

8. Determine the direct form-I, direct form-II, cascade and parallel realization of the following LTI systems.

(a) $H(z)=\dfrac{z^3-2z^2+z-1}{(z-0.4)(z^2+2z-0.4)}$ (b) $H(z)=\dfrac{1+0.5z^{-1}}{(1-z^{-1}+0.5z^{-2})(1-z^{-1}+0.8z^{-2})}$

9. Realize the IIR filter using ladder structure

(a) $H(z)=\dfrac{2z^2+3z+4}{z^2+5z+7}$ (b) $H(z)=\dfrac{z^3+3z^2+2z+5}{2z^2+z+4}$

10. Realize the second order IIR system using the transposed form structure

$$y(n)+\frac{1}{2}y(n-1)+\frac{1}{4}y(n-2)=2x(n)+\frac{1}{3}x(n-1)+\frac{1}{5}x(n-2)$$

11. Determine the lattice coefficients corresponding to the IIR filter described by

$y(n)-\dfrac{1}{2}y(n-1)+\dfrac{1}{7}y(n-2)=x(n)+\dfrac{1}{5}x(n-1)$ and realize it.

12. Draw the direct form structure of the FIR system described by the transfer function

(a) $H(z)=1+\dfrac{1}{3}z^{-1}+\dfrac{1}{2}z^{-2}+\dfrac{1}{4}z^{-3}+\dfrac{1}{5}z^{-4}+\dfrac{1}{7}z^{-5}$

(b) $H(z)=2+\dfrac{1}{2}z^{-1}+\dfrac{1}{4}z^{-2}+\dfrac{1}{3}z^{-3}+\dfrac{1}{6}z^{-4}$

(c) $H(z)=1+\dfrac{1}{2}z^{-1}+\dfrac{3}{4}z^{-2}+\dfrac{1}{4}z^{-3}+\dfrac{1}{2}z^{-4}+\dfrac{1}{8}z^{-5}$

13. Realize the following systems with minimum number of multipliers

(a) $H(z)=1+3z^{-1}+2z^{-2}+5z^{-3}+2z^{-4}+3z^{-5}+z^{-6}$

(b) $H(z)=0.2+0.6z^{-1}+0.7z^{-2}+0.8z^{-3}+0.9z^{-4}+0.8z^{-5}+0.7z^{-6}+0.6z^{-7}+0.2z^{-8}$

(c) $H(z)=0.5+0.75z^{-1}+0.8z^{-2}+0.9z^{-3}+2z^{-4}+0.9z^{-5}+0.8z^{-6}+0.75z^{-7}+0.5z^{-8}$

(d) $H(z)=\dfrac{1}{4}+\dfrac{1}{2}z^{-1}+\dfrac{3}{4}z^{-2}+\dfrac{1}{2}z^{-3}+\dfrac{1}{4}z^{-4}$

(e) $H(z)=\left(1+\dfrac{1}{2}z^{-1}+z^{-2}\right)\left(1+\dfrac{1}{4}z^{-1}+z^{-2}\right)$

14. Realize the following systems in cascade form

(a) $H(z) = \left(1 + \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{3}z^{-1} + z^{-2}\right)$

(b) $H(z) = \left(1 + \frac{1}{2}z^{-1} + z^{-2}\right)\left(1 + \frac{1}{5}z^{-1} + \frac{1}{3}z^{-2}\right)$

15. Realize the following second order FIR system using the transposed form structure

$$y(n) = 3x(n) + 5x(n-1) - 2x(n-2)$$

16. Determine the lattice coefficients corresponding to the FIR system with the system function $H(z) = 1 + \frac{5}{12}z^{-1} + \frac{2}{3}z^{-2}$ and realize it.

MATLAB PROGRAMS

Program 4.1

% Parallel form realization of IIR filters

```
clc; clear all; close all;
num=[2 10 23 34 31 16 4];
den=[36 78 87 59 26 7 1];
[r1 p1 k1]=residuez(num,den);
[r2 p2 k2]=residue(num,den);
disp('parallel form 1')
disp('residues are')
disp(r1)
disp('poles are at')
disp(p1)
disp('constant value')
disp(k1)
disp('parallel form II')
disp('residues are')
disp(r2)
disp('poles are at')
disp(p2)
disp('constant value')
disp(k2)
```

Output:

parallel form 1

```
residues are
  -0.5952 - 0.7561i
  -0.5952 + 0.7561i
  -0.5556 - 2.2785i
  -0.5556 + 2.2785i
  -0.8214 + 4.3920i
  -0.8214 - 4.3920i
```

```
poles are at
  -0.5000 + 0.2887i
  -0.5000 - 0.2887i
  -0.3333 + 0.4714i
  -0.3333 - 0.4714i
  -0.2500 + 0.4330i
  -0.2500 - 0.4330i
constant value
     4
```

parallel form II

```
residues are
   0.5159 + 0.2062i
   0.5159 - 0.2062i
   1.2593 + 0.4976i
   1.2593 - 0.4976i
  -1.6964 - 1.4537i
  -1.6964 + 1.4537i
poles are at
  - 0.5000 + 0.2887i
  - 0.5000 - 0.2887i
  - 0.3333 + 0.4714i
  - 0.3333 - 0.4714i
  - 0.2500 + 0.4330i
  - 0.2500 - 0.4330i
constant value
    0.0556
```

Program 4.2

% Direct form to cascade form conversion

```
clc; clear all; close all;
b=[4 5 6]; %numerator coefficients of direct form
a=[1 2 3]; %denominator coefficients of direct form
% compute gain coefficient
b0=b(1); a0=a(1);
b=b/b0; a=a/a0;
m=length(b);
n=length(a);
if n > m
    b= [b zeros(1,n-m)];
```

```
elseif m > n
   a=[a zeros(1,m–n)];
end
k=floor(n/2);
B=zeros(k,3);  A=zeros(k,3);
if k*2== n
    b=[b 0];
    a=[a 0];
end
broots=cplxpair(roots(b));
aroots=cplxpair(roots(a));
for i=1:2:2*k
  brow=broots(i:1:i+1,:);
  brow=real(poly(brow));
  B(fix((i+1)/2),:)=brow;
  arow=aroots(i:1:i+1,:);
  arow=real(poly(arow));
  A(fix((i+1)/2),:)=arow;
end
disp('numerator coefficients of cascade form')
disp(brow)
disp('denominator coefficients of cascade form')
disp(arow)
```

Output:

```
numerator coefficients of cascade form
    1.0000    1.2500    1.5000

denominator coefficients of cascade form
    1.0000    2.0000    3.0000
```

Program 4.3

% Cascade form realization of FIR & IIR filters

```
clc; clear all; close all;
b=[4 5 6]; %numerator coefficients of direct form
a=[1 2 3]; %denominator coefficients of direct form
% compute gain coefficient
b0=b(1);
x=[1 2 3 8 9 4 6 7 10];% input sequence
[k l]=size(b);
n=length(x);
w=zeros(k+1,n);
w(1,:)=x;
for i=1:1:k
    w(i+1,:)=filter(b(i,:),a(i,:),w(i,:));
end
y=b0*w(k+1,:);
disp('output of the final filter operation')

disp(y)
```

Output:

```
output of the final filter operation
    16    20    24   128    48   –44   336   –212    –140
```

Program 4.4

% Cascade form to direct form conversion

```
clc; clear all; close all;
B=[4 5 6]; %numerator coefficients of cascade form
A=[1 2 3]; %denominator coefficients of cascade form
% compute gain coefficient
b0=B(1);
[k l]=size(B);
```

```
b=[1];
a=[1];

for i=1:1:k
    b=conv(b,B(i,:));
    a=conv(a,A(i,:));
end

b=b*b0;
disp('numerator coefficients of direct form')
disp(b)
disp('denominator coefficients of direct form')
disp(a)
```

Output:

```
numerator coeffcients of direct form
    16    20    24

denominator coeffcients of direct form
     1     2     3
```

Program 4.5

%Direct form to parallel form conversion

```
clc; clear all;close all;
b=[4 5 6]; %numerator coefficients of direct form
a=[1 2 3]; %denominator coefficients of direct form
m=length(b);
n=length(a);
[r1 p1 c]=residuez(b,a);
p=cplxpair(p1,10000000*eps);
p2=cplxpair(p1);
I=[];
for j=1:1:length(p2)
```

```
    for i=1:1:length(p1)
        if(abs(p1(i)–p2(j)) < 0.0001)
            I=[I,i];
        end
    end
end
I=I';
r=r1(I);

K=floor(n/2);B=zeros(K,2);  A=zeros(K,3);

if K*2 ==n
    for i=1:2:n-2
        Brow=r(i:1:i+1,:);
        Arow=p(i:1:i+1,:);
        [Brow Arow]=residuez(Brow,Arow,[]);
        B(fix((i+1)/2),:)=real(Brow);
        A(fix((i+1)/2),:)=real(Arow);
    end

    [Brow Arow]=residuez(r(n–1),p(n–1),[]);
    B(K,:)=[real(Brow) 0]; A(K,:)=[real(Arow) 0];

else
    for i=1:2:n–1
        Brow=r(i:1:i+1,:);
        Arow=p(i:1:i+1,:);
        [Brow Arow]=residuez(Brow,Arow,[]);
        B(fix((i+1)/2),:)=real(Brow);
        A(fix((i+1)/2),:)=real(Arow);
    end
end

disp('numerator coefficients of parallel form')
disp(B)
disp('denominator coefficients of parallel form')
disp(A)
```

Output:

```
numerator coefficients of parallel form
    2.0000    1.0000

denominator coefficients of parallel form
    1.0000    2.0000    3.0000
```

Program 4.6

% Parallel form to direct form conversion

```
clc;clear all;close all;
C=[0];
A=[1 1 0.9; 1 0.4 –0.4];
B=[2 4; 3 1];

[K,L]=size(A);
R=[]; P=[];
for i=1:1:K
    [r p k]=residuez(B(i,:),A(i,:));
    R=[R;r];
    P=[P;p];
end
[b a]=residuez(R,P,C);
b=b(:)';
a=a(:)';

disp('numerator coefficients of direct form')
disp(b)
disp('denominator coefficients of direct form')
disp(a)
```

Output:

```
numerator coefficients of direct form

    5.0000    8.8000    4.5000   −0.7000         0

denominator coefficients of direct form

    1.0000    1.4000    0.9000   −0.0400   −0.3600
```

5

Discrete-time Fourier Transform

5.1 INTRODUCTION

A continuous-time signal can be represented in the frequency domain using Laplace transform or continuous-time Fourier transform (CTFT). Similarly, a discrete-time signal can be represented in the frequency domain using Z-transform or discrete-time Fourier transform. The Fourier transform of a discrete-time signal is called discrete-time Fourier transform (DTFT). DTFT is very popular for digital signal processing because of the fact that using this the complicated operation of convolution of two sequences in the time domain can be converted into a much simpler multiplicative operation in the frequency domain. In this chapter, we discuss about DTFT, its properties and its use in the analysis of signals.

5.2 DISCRETE-TIME FOURIER TRANSFORM (DTFT)

The Fourier transform of discrete-time signals is called the discrete-time Fourier transform (DTFT).

If $x(n)$ is the given discrete-time sequence, then $X(\omega)$ or $X(e^{j\omega})$ is the discrete-time Fourier transform of $x(n)$.

The DTFT of $x(n)$ is defined as:

$$\mathrm{F}[x(n)] = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

The inverse DTFT of $X(\omega)$ is defined as:

$$\mathrm{F}^{-1}[X(\omega)] = x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

We also refer to $x(n)$ and $X(\omega)$ as a Fourier transform pair and this relation is expressed as:

$$x(n) \xleftrightarrow{\text{FT}} X(\omega)$$

5.3 EXISTENCE OF DTFT

The Fourier transform exists for a discrete-time sequence $x(n)$ if and only if the sequence is absolutely summable, i.e. the sequence has to satisfy the condition:

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

The DTFT does not exist for the sequences that are growing exponentially (ex. $a^n\, u(n)$, $a > 1$) since they are not absolutely summable. Therefore, DTFT method of analyzing a system can be applied for a limited class of signals. Moreover this method can be applied only to asymptotically stable systems and it cannot be applied for unstable systems. That is, DTFT can be used only for the systems whose system function $H(z)$ has poles inside the unit circle.

The Fourier transform $X(\omega)$ of a signal $x(n)$ represents the frequency content of $x(n)$. We can say that, by taking Fourier transform, the signal $x(n)$ is decomposed into its frequency components. Hence $X(\omega)$ is called signal spectrum.

The difference between the Fourier transforms of a discrete-time signal and analog signal are as follows:

1. The Fourier transform of analog signals consists of a spectrum with a frequency range $-\infty$ to ∞. But the Fourier transform of discrete-time signals is unique in the frequency range $-\pi$ to π (or equivalently 0 to 2π). Also Fourier transforms of discrete-time signals are periodic with period 2π. Hence the frequency range for any discrete-time signal is limited to $-\pi$ to π (or 0 to 2π) and any frequency outside this interval has an equivalent frequency within this interval.
2. Since the analog signals are continuous, the Fourier transform of analog signals involves integration, but the Fourier transform of discrete-time signals involves summation because the signals are discrete.

5.4 RELATION BETWEEN Z-TRANSFORM AND FOURIER TRANSFORM

The Z-transform of a discrete sequence $x(n)$ is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

where z is a complex variable.

The Fourier transform of a discrete-time sequence $x(n)$ is defined as:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

The $X(z)$ can be viewed as a unique representation of the sequence $x(n)$ in the complex z-plane.

Let $z = re^{j\omega}$

$$\therefore \qquad X(z) = \sum_{n=-\infty}^{\infty} x(n)\,(re^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [x(n)\,r^{-n}]\,e^{-j\omega n}$$

The RHS is the Fourier transform of $x(n)\,r^{-n}$, i.e. the Z-transform of $x(n)$ is the Fourier transform of $x(n)\,r^{-n}$.

When $r = 1$,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\,e^{-j\omega n} = X(\omega)$$

The RHS is the Fourier transform of $x(n)$. So we can conclude that the Fourier transform of $x(n)$ is same as the Z-transform of $x(n)$ evaluated along the unit circle centred at the origin of the z-plane.

$$\therefore \qquad X(\omega) = X(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n)\,z^{-n}\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n)\,e^{-j\omega n}$$

For $X(\omega)$ to exist, the ROC must include the unit circle. Since ROC cannot contain any poles of $X(z)$ all the poles must lie inside the unit circle. Therefore, we can conclude that Fourier transform can be obtained for any sequence $x(n)$, from its Z-transform $X(z)$ if the poles of $X(z)$ are inside the unit circle.

EXAMPLE 5.1 Find the DTFT of the following sequences:

(a) $\delta(n)$ (b) $u(n)$

(c) $\delta(n-m)$ (d) $u(n-m)$

(e) $a^n u(n)$ (f) $-a^n u(-n-1)$

(g) $\delta(n+3) - \delta(n-3)$ (h) $u(n+3) - u(n-3)$

Solution:

(a) Given $x(n) = \delta(n)$

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

$$X(\omega) = \mathrm{F}\{\delta(n)\} = \sum_{n=-\infty}^{\infty} \delta(n)\,e^{-j\omega n}\Big|_{n=0} = 1$$

$$\therefore \qquad \mathrm{F}\{\delta(n)\} = 1$$

$$\boxed{\delta(n) \xleftrightarrow{\text{FT}} 1}$$

(b) Given $x(n) = u(n)$

$$u(n) = \begin{cases} 1 & \text{for } n \ge 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$X(\omega) = \text{F}\{u(n)\} = \sum_{n=-\infty}^{\infty} u(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (1)\, e^{-j\omega n} = \frac{1}{1 - e^{-j\omega}}$$

$\therefore$
$$\text{F}\{u(n)\} = \frac{1}{1 - e^{-j\omega}}$$

$$\boxed{u(n) \xleftrightarrow{\text{FT}} \frac{1}{1 - e^{-j\omega}}}$$

(c) Given $x(n) = \delta(n - m)$

$$\delta(n - m) = \begin{cases} 1 & \text{for } n = m \\ 0 & \text{for } n \ne m \end{cases}$$

$$X(\omega) = \text{F}\{\delta(n - m)\} = \sum_{n=-\infty}^{\infty} \delta(n - m)\, e^{-j\omega n} = e^{-j\omega n}\Big|_{n=m} = e^{-j\omega m}$$

$\therefore$
$$\text{F}\{\delta(n - m)\} = e^{-j\omega m}$$

$$\boxed{\delta(n - m) \xleftrightarrow{\text{FT}} e^{-j\omega m}}$$

(d) Given $x(n) = u(n - m)$

$$u(n - m) = \begin{cases} 1 & \text{for } n \ge m \\ 0 & \text{for } n < m \end{cases}$$

$$X(\omega) = \text{F}\{u(n - m)\} = \sum_{n=-\infty}^{\infty} u(n - m)\, e^{-j\omega n} = \sum_{n=m}^{\infty} (1)\, e^{-j\omega n}$$

$$= e^{-j\omega m} + e^{-j\omega(m+1)} + e^{-j\omega(m+2)} + \cdots$$

$$= e^{-j\omega m}(1 + e^{-j\omega} + e^{-j2\omega} + \cdots)$$

$$= \frac{e^{-j\omega m}}{1 - e^{-j\omega}}$$

$\therefore$
$$\text{F}\{u(n - m)\} = \frac{e^{-j\omega m}}{1 - e^{-j\omega}}$$

$$\boxed{u(n - m) \xleftrightarrow{\text{FT}} \frac{e^{-j\omega m}}{1 - e^{-j\omega}}}$$

(e) Given $x(n) = a^n u(n)$

$$X(\omega) = \text{F}\{a^n u(n)\} = \sum_{n=-\infty}^{\infty} a^n u(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1-ae^{-j\omega}}$$

$\therefore$ $$\text{F}\{a^n u(n)\} = \frac{1}{1-ae^{-j\omega}}$$

$$\boxed{a^n u(n) \xleftrightarrow{\text{FT}} \frac{1}{1-ae^{-j\omega}}}$$

(f) Given $x(n) = -a^n u(-n-1)$

$$X(\omega) = \text{F}\{-a^n u(-n-1)\}$$

$$= \sum_{n=-\infty}^{\infty} -a^n u(-n-1)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} -a^n e^{-j\omega n} = -\sum_{n=1}^{\infty} a^{-n} e^{j\omega n} = -\sum_{n=1}^{\infty} (a^{-1}e^{j\omega})^n$$

$$= -\left[a^{-1}e^{j\omega} + (a^{-1}e^{j\omega})^2 + (a^{-1}e^{j\omega})^3 + \cdots\right]$$

$$= -a^{-1}e^{j\omega}\left[1 + (a^{-1}e^{j\omega})^1 + (a^{-1}e^{j\omega})^2 + \cdots\right]$$

$$= \frac{-a^{-1}e^{j\omega}}{1-a^{-1}e^{j\omega}}$$

$$= \frac{1}{1-ae^{-j\omega}}$$

$\therefore$ $$\text{F}\{-a^n u(-n-1)\} = \frac{1}{1-ae^{-j\omega}}$$

$$\boxed{-a^n u(-n-1) \xleftrightarrow{\text{FT}} \frac{1}{1-ae^{-j\omega}}}$$

(g) Given $x(n) = \delta(n+3) - \delta(n-3)$

$$X(\omega) = \text{F}\{\delta(n+3) - \delta(n-3)\}$$

$$= \sum_{n=-\infty}^{\infty} \{\delta(n+3) - \delta(n-3)\}\, e^{-j\omega n}$$

$$= e^{-j\omega n}\Big|_{n=-3} - e^{-j\omega n}\Big|_{n=3}$$

$$= e^{j3\omega} - e^{-j3\omega} = 2j \sin 3\omega$$

(h) Given $x(n) = u(n+3) - u(n-3)$

$$X(\omega) = \text{F}\{u(n+3) - u(n-3)\}$$

$$= \sum_{n=-\infty}^{\infty} \{u(n+3) - u(n-3)\}\, e^{-j\omega n} = \sum_{n=-3}^{\infty} (1)\, e^{-j\omega n} - \sum_{n=3}^{\infty} (1)\, e^{-j\omega n}$$

$$= e^{j3\omega} + e^{j2\omega} + e^{j\omega} + 1 + e^{-j\omega} + e^{-j2\omega} + \cdots - e^{-j3\omega} - e^{-j4\omega} - \cdots$$

$$= e^{j3\omega} + e^{j2\omega} + e^{j\omega} + 1 + e^{-j\omega} + e^{-j2\omega}$$

EXAMPLE 5.2 Find the DTFT of:

(a) $x(n) = \{1, -2, 2, 3\}$

(b) $x(n) = 3^n u(n)$

(c) $x(n) = (0.5)^n\, u(n) + 2^n\, u(-n-1)$

(d) $x(n) = \left(\dfrac{1}{4}\right)^n u(n+1)$

(e) $x(n) = \begin{cases} n, & -4 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$

(f) $x(n) = \begin{cases} 1, & 0 \le n \le 3 \\ 0, & \text{otherwise} \end{cases}$

(g) $x(n) = a^{|n|}$

Solution:

(a) Given $x(n) = \{1, -2, 2, 3\}$

$$X(\omega) = \text{F}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= x(0) + x(1)\, e^{-j\omega} + x(2)\, e^{-j2\omega} + x(3)\, e^{-j3\omega}$$

$$= 1 - 2e^{-j\omega} + 2e^{-j2\omega} + 3e^{-j3\omega}$$

(b) Given $x(n) = 3^n u(n)$. The given sequence is not absolutely summable. Therefore, its DTFT does not exist.

(c) Given $x(n) = (0.5)^n u(n) + 2^n u(-n-1)$

$$X(\omega) = \text{F}\{x(n)\} = \sum_{n=-\infty}^{\infty} \{(0.5)^n\, u(n) + 2^n\, u(-n-1)\}\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \{(0.5)^n\, u(n)\}\, e^{-j\omega n} + \sum_{n=-\infty}^{\infty} \{2^n\, u(-n-1)\}\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (0.5)^n\, e^{-j\omega n} + \sum_{n=-\infty}^{-1} 2^n\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (0.5e^{-j\omega})^n + \sum_{n=1}^{\infty} (2^{-1}\, e^{j\omega})^n$$

$$= \frac{1}{1 - 0.5e^{-j\omega}} + \frac{2^{-1}\, e^{j\omega}}{1 - 2^{-1}\, e^{j\omega}}$$

$$= \frac{1}{1 - 0.5e^{-j\omega}} - \frac{1}{1 - 2e^{-j\omega}}$$

(d) Given $$x(n) = \left(\frac{1}{4}\right)^n u(n+1)$$

$$X(\omega) = \mathrm{F}\left\{\left(\frac{1}{4}\right)^n u(n+1)\right\} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{4}\right)^n u(n+1)\, e^{-j\omega n}$$

$$= \sum_{n=-1}^{\infty} \left(\frac{1}{4}\right)^n e^{-j\omega n} = \sum_{n=-1}^{\infty} \left(\frac{1}{4}e^{-j\omega}\right)^n$$

$$= \left(\frac{1}{4}e^{-j\omega}\right)^{-1} \left[\sum_{n=0}^{\infty} \frac{1}{4}e^{-j\omega}\right] = 4e^{j\omega}\left[\frac{1}{1-(1/4)\, e^{-j\omega}}\right]$$

$$= \frac{4e^{j\omega}}{1-(1/4)\, e^{-j\omega}}$$

(e) Given $$x(n) = \begin{cases} n, & -4 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$$

$$X(\omega) = \mathrm{F}\{x(n)\} = \sum_{n=-4}^{4} n e^{-j\omega n}$$

$$= -4e^{j4\omega} - 3e^{j3\omega} - 2e^{j2\omega} - e^{j\omega} + e^{-j\omega} + 2e^{-j2\omega} + 3e^{-j3\omega} + 4e^{-4j\omega}$$

$$= -2j\,\{4 \sin 4\omega + 3 \sin 3\omega + 2 \sin 2\omega + \sin \omega\}$$

(f) Given $$x(n) = \begin{cases} 1, & 0 \le n \le 3 \\ 0, & \text{otherwise} \end{cases}$$

By definition of Fourier transform,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} = \sum_{n=0}^{3} (1)\, e^{-j\omega n}$$

$$= \frac{1-e^{-j4\omega}}{1-e^{-j\omega}} = \frac{1-e^{-j2\omega}e^{-j2\omega}}{1-e^{(-j\omega/2)}e^{(-j\omega/2)}}$$

$$= \frac{e^{-j2\omega}\{e^{j2\omega} - e^{-j2\omega}\}}{e^{(-j\omega/2)}\{e^{(j\omega/2)} - e^{(-j\omega/2)}\}} = \left[\frac{2j \sin 2\omega}{2j \sin(\omega/2)}\right] e^{-j2\omega + j\omega/2}$$

$$= \frac{\sin 2\omega}{\sin(\omega/2)}\, e^{-j(3/2)\omega}$$

(g) Given $$x(n) = a^{|n|}$$

$$X(\omega) = \mathrm{F}\{a^{|n|}\} = \sum_{n=-\infty}^{\infty} a^{|n|}\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} a^{-n}\, e^{-j\omega n} + \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=1}^{\infty} (ae^{j\omega})^n + \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

$$= \frac{ae^{j\omega}}{1-ae^{j\omega}} + \frac{1}{1-ae^{-j\omega}} = \frac{1-a^2}{1-2a\cos\omega + a^2}$$

EXAMPLE 5.3 Find the DTFT of the following sequences:

(a) $\sin\left(\dfrac{n\pi}{2}\right)u(n)$ (b) $\cos\left(\dfrac{n\pi}{3}\right)u(n)$

(c) $\left(\dfrac{1}{2}\right)^n \sin\left(\dfrac{n\pi}{4}\right)u(n)$ (d) $\left(\dfrac{1}{2}\right)^{n-2} u(n-2)$

(e) $\cos(\omega_0 n)\ u(n)$ (f) $\sin(\omega_0 n)\ u(n)$

Solution:

(a) Given $$x(n) = \sin\left(\frac{n\pi}{2}\right)u(n)$$

$$X(\omega) = \mathrm{F}\left\{\sin\left(\frac{n\pi}{2}\right)u(n)\right\} = \sum_{n=-\infty}^{\infty}\left\{\sin\left(\frac{n\pi}{2}\right)u(n)\right\}e^{-j\omega n} = \sum_{n=0}^{\infty}\sin\left(\frac{n\pi}{2}\right)e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty}\frac{e^{j(n\pi/2)} - e^{-j(n\pi/2)}}{2j}e^{-j\omega n} = \frac{1}{2j}\left[\sum_{n=0}^{\infty}e^{j[(\pi/2)-\omega]n} - \sum_{n=0}^{\infty}e^{-j[(\pi/2)+\omega]n}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{1-e^{j[(\pi/2)-\omega]}} - \frac{1}{1-e^{-j[(\pi/2)+\omega]}}\right]$$

$$= \frac{1}{2j}\left[\frac{1-e^{-j(\pi/2)}e^{-j\omega} - 1 + e^{j(\pi/2)}e^{-j\omega}}{1+e^{-j2\omega} - e^{-j\omega}[e^{j(\pi/2)} + e^{-j(\pi/2)}]}\right]$$

$$= \frac{e^{-j\omega}\sin(\pi/2)}{1+e^{-j2\omega}} = \frac{e^{-j\omega}}{1+e^{-j2\omega}}$$

(b) Given $$x(n) = \cos\left(\frac{n\pi}{3}\right)u(n)$$

$$X(\omega) = \mathrm{F}\left\{\cos\left(\frac{n\pi}{3}\right)u(n)\right\} = \sum_{n=-\infty}^{\infty}\left\{\cos\left(\frac{n\pi}{3}\right)u(n)\right\}e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty}\left[\frac{e^{j(n\pi/3)} + e^{-j(n\pi/3)}}{2}u(n)\right]e^{-j\omega n} = \frac{1}{2}\left[\sum_{n=0}^{\infty}e^{j[(\pi/3)-\omega]n} + \sum_{n=0}^{\infty}e^{-j[(\pi/3)+\omega]n}\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{j[(\pi/3)-\omega]}} + \frac{1}{1-e^{-j[(\pi/3)+\omega]}}\right]$$

$$= \frac{1}{2}\left[\frac{1-e^{-j(\pi/3)}e^{-j\omega} + 1 - e^{j(\pi/3)}e^{-j\omega}}{1+e^{-j2\omega} - e^{-j\omega}[e^{j(\pi/3)} + e^{-j(\pi/3)}]}\right]$$

$$= \frac{1}{2}\left[\frac{2-e^{-j\omega}}{1-e^{-j\omega}+e^{-j2\omega}}\right]$$

(c) Given $$x(n) = \left(\frac{1}{2}\right)^n \sin\left(\frac{n\pi}{4}\right) u(n)$$

$$X(\omega) = \mathrm{F}\left\{\left(\frac{1}{2}\right)^n \sin\left(\frac{n\pi}{4}\right) u(n)\right\}$$

$$= \sum_{n=-\infty}^{\infty} \left\{\left(\frac{1}{2}\right)^n \sin\left(\frac{n\pi}{4}\right) u(n)\right\} e^{-j\omega n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \sin\left(\frac{n\pi}{4}\right) e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \left\{\frac{e^{j(n\pi/4)} - e^{-j(n\pi/4)}}{2j}\right\} e^{-j\omega n}$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{j[(\pi/4)-\omega]n} - \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j[(\pi/4)+\omega]n}\right]$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty} \left(\frac{1}{2} e^{j[(\pi/4)-\omega]}\right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{2} e^{-j[(\pi/4)+\omega]}\right)^n\right]$$

$$= \frac{1}{2j}\left[\frac{1}{1-(1/2)\,e^{j[(\pi/4)-\omega]}} - \frac{1}{1-(1/2)\,e^{-j[(\pi/4)+\omega]}}\right]$$

$$= \frac{(1/2)\,e^{-j\omega}\sin(\pi/4)}{1+(1/4)\,e^{-j2\omega} - e^{-j\omega}\cos(\pi/4)}$$

$$= \frac{(1/2\sqrt{2})\,e^{-j\omega}}{1-(1/\sqrt{2})\,e^{-j\omega} + (1/4)e^{-j2\omega}}$$

(d) Given $$x(n) = \left(\frac{1}{2}\right)^{n-2} u(n-2)$$

$$X(\omega) = \mathrm{F}\{x(n)\} = \sum_{n=-\infty}^{\infty} \left[\left(\frac{1}{2}\right)^{n-2} u(n-2)\right] e^{-j\omega n}$$

$$= \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^{n-2} e^{-j\omega n}$$

$$= e^{-j2\omega} + \frac{1}{2}e^{-j3\omega} + \left(\frac{1}{2}\right)^2 e^{-j4\omega} + \cdots$$

$$= e^{-j2\omega}\left[1 + \frac{1}{2}e^{-j\omega} + \left(\frac{1}{2}\right)^2 e^{-j2\omega} + \cdots\right]$$

$$= \frac{e^{-j2\omega}}{1-(1/2)\,e^{-j\omega}}$$

(e) Given $x(n) = \cos(\omega_0 n)\; u(n)$

$$X(\omega) = \text{F}\{x(n)\} = \sum_{n=-\infty}^{\infty} \{\cos(\omega_0 n)\, u(n)\}\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left[\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}\right] e^{-j\omega n}$$

$$= \frac{1}{2}\left\{\sum_{n=0}^{\infty} [e^{j(\omega_0-\omega)}]^n + \sum_{n=0}^{\infty} [e^{-j(\omega_0+\omega)}]^n\right\}$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{j(\omega_0-\omega)}} + \frac{1}{1-e^{-j(\omega_0+\omega)}}\right]$$

$$= \frac{1}{2}\left[\frac{1-e^{-j(\omega_0+\omega)} + 1 - e^{j(\omega_0-\omega)}}{1+e^{-j2\omega} - e^{-j\omega}(e^{j\omega_0} + e^{-j\omega_0})}\right]$$

$$= \frac{1-e^{-j\omega}\cos\omega_0}{1-2e^{-j\omega}\cos\omega_0 + e^{-j2\omega}}$$

(f) Given $x(n) = \sin(\omega_0 n)\; u(n)$

$$X(\omega) = \text{F}\{x(n)\} = \sum_{n=-\infty}^{\infty} \{\sin(\omega_0 n)\, u(n)\}\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left\{\frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j}\right\} e^{-j\omega n} = \sum_{n=0}^{\infty} \left\{\frac{e^{j(\omega_0-\omega)n} - e^{-j(\omega_0+\omega)n}}{2j}\right\}$$

$$= \frac{1}{2j}\left[\frac{1}{1-e^{j(\omega_0-\omega)}} - \frac{1}{1-e^{-j(\omega_0+\omega)}}\right]$$

$$= \frac{1}{2j}\left[\frac{1-e^{-j(\omega_0+\omega)} - 1 + e^{j(\omega_0-\omega)}}{1+e^{-j2\omega} - e^{-j\omega}(e^{j\omega_0} + e^{-j\omega_0})}\right]$$

$$= \frac{e^{-j\omega}\sin\omega_0}{1-2e^{-j\omega}\cos\omega_0 + e^{-j2\omega}}$$

EXAMPLE 5.4 Find the DTFT of the rectangular pulse sequence:

$$x(n) = \begin{cases} A, & |n| \le N \\ 0, & |n| > N \end{cases}$$

Solution: Given

$$x(n) = \begin{cases} A, & |n| \le N \\ 0, & |n| > N \end{cases}$$

$$X(\omega) = \sum_{n=-N}^{N} Ae^{-j\omega n} = \sum_{n=-N}^{-1} Ae^{-j\omega n} + \sum_{n=0}^{N} Ae^{-j\omega n}$$

$$= \sum_{n=1}^{N} Ae^{j\omega n} + \sum_{n=0}^{N} Ae^{-j\omega n} = Ae^{j\omega} \sum_{n=0}^{N-1} e^{j\omega n} + A \sum_{n=0}^{N} e^{-j\omega n}$$

$$= Ae^{j\omega} \left[\frac{1 - e^{j\omega N}}{1 - e^{j\omega}} \right] + A \left[\frac{1 - e^{-j\omega(N+1)}}{1 - e^{-j\omega}} \right]$$

$$= A \left[\frac{e^{j\omega} - e^{j\omega(N+1)}}{1 - e^{j\omega}} \right] + A \left[\frac{1 - e^{-j\omega(N+1)}}{1 - e^{-j\omega}} \right]$$

$$= A \left[\frac{e^{j\omega} - 1 - e^{j\omega(N+1)} + e^{j\omega N} + 1 - e^{j\omega} - e^{-j\omega(N+1)} + e^{-j\omega N}}{1 + 1 - e^{j\omega} - e^{-j\omega}} \right]$$

$$= A \left[\frac{(e^{j\omega N} + e^{-j\omega N}) - (e^{j\omega(N+1)} + e^{-j\omega(N+1)})}{2 - (e^{j\omega} + e^{-j\omega})} \right]$$

$$= A \left[\frac{2 \cos \omega N - 2 \cos \omega(N+1)}{2 - 2 \cos \omega} \right]$$

$$= A \left[\frac{2 \sin \omega[N + (1/2)] \sin(\omega/2)}{2 \sin^2(\omega/2)} \right] = \frac{A \sin \omega[N + (1/2)]}{\sin(\omega/2)}$$

5.5 INVERSE DISCRETE-TIME FOURIER TRANSFORM

The process of finding the discrete-time sequence $x(n)$ from its frequency response $X(\omega)$ is called the inverse discrete-time Fourier transform.

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

The integral solution of the above equation for $x(n)$, i.e, for the inverse Fourier transform is useful for analytic purpose, but it is usually very difficult to evaluate for typical functional forms of $X(\omega)$. An alternate and more useful method of determining the values of $x(n)$ follows directly from the definition of the Fourier transform.

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} = \cdots + x(-2)\, e^{j2\omega} + x(-1)\, e^{j\omega} + x(0) + x(1)\, e^{-j\omega} + x(2)\, e^{-j2\omega} + \cdots$$

From the defining equation of $X(\omega)$ we can say that, if $X(\omega)$ can be expressed as a series of complex exponentials as shown in the above equation for $X(\omega)$, then $x(n)$ is simply the coefficient of $e^{-j\omega n}$. Inverse Fourier transform can be obtained by using the partial fraction method or by using the convolution theorem.

EXAMPLE 5.5 Determine the signal $x(n)$ for the given Fourier transforms:

(a) $X(\omega) = e^{-j\omega}$ for $-\pi \le \omega \le \pi$ (b) $X(\omega) = e^{-j\omega}(1 + \cos \omega)$

Solution:

(a) Given $X(\omega) = e^{-j\omega}$

$$x(n) = \mathrm{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-1)}\, d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega(n-1)}}{j(n-1)}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\pi(n-1)} - e^{-j\pi(n-1)}}{j(n-1)}\right] = \frac{1}{\pi(n-1)}\left[\frac{e^{j\pi(n-1)} - e^{-j\pi(n-1)}}{2j}\right]$$

$$= \frac{\sin \pi(n-1)}{\pi(n-1)}$$

(b) Given $X(\omega) = e^{-j\omega}(1 + \cos\omega)$

$$= e^{-j\omega}\left(1 + \frac{e^{j\omega} + e^{-j\omega}}{2}\right)$$

$$= e^{-j\omega} + 0.5 + 0.5e^{-j2\omega}$$

$$= x(0) + x(1)\, e^{-j\omega} + x(2)e^{-j2\omega}$$

Therefore, $x(0) = 0.5,\ x(1) = 1,\ x(2) = 0.5$

$x(n) = 0$, otherwise

i.e. $x(n) = \{0.5,\ 1,\ 0.5\}$

EXAMPLE 5.6 Obtain the impulse response of the system described by

$$H(\omega) = \begin{cases} 1, & \text{for } |\omega| \le \omega_0 \\ 0, & \text{for } \omega_0 \le |\omega| \le \pi \end{cases}$$

Solution: Given
$$H(\omega) = \begin{cases} 1, & \text{for } |\omega| \le \omega_0 \\ 0, & \text{for } \omega_0 \le |\omega| \le \pi \end{cases}$$

The impulse response $h(n)$ is given by

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} (1)\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\left(\frac{e^{j\omega n}}{jn}\right)_{-\omega_0}^{\omega_0}$$

$$= \frac{1}{2\pi}\left(\frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{jn}\right) = \frac{\sin(\omega_0 n)}{\pi n}$$

EXAMPLE 5.7 Find the inverse Fourier transform of the following:

$$X(\omega)=\begin{cases}1, & \dfrac{\pi}{3}\le|\omega|\le\dfrac{2\pi}{3}\\[2mm] 0, & \dfrac{2\pi}{3}\le|\omega|\le\pi\end{cases}$$

Solution: The Fourier transform $X(\omega)$ is given. Then, the inverse Fourier transform of $X(\omega)$ is given by

$$x(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(\omega)\,e^{j\omega n}\,d\omega$$

$$=\frac{1}{2\pi}\left[\int_{-2\pi/3}^{-\pi/3}1\,e^{j\omega n}\,d\omega+\int_{\pi/3}^{2\pi/3}1\,e^{j\omega n}\,d\omega\right]=\frac{1}{2\pi}\left[\left(\frac{e^{j\omega n}}{jn}\right)_{-2\pi/3}^{-\pi/3}+\left(\frac{e^{j\omega n}}{jn}\right)_{\pi/3}^{2\pi/3}\right]$$

$$=\frac{1}{n\pi}\left[\frac{e^{-jn(\pi/3)}-e^{-jn(2\pi/3)}+e^{jn(2\pi/3)}-e^{jn(\pi/3)}}{2j}\right]$$

$$=\frac{1}{n\pi}\left[\frac{e^{jn(2\pi/3)}-e^{-jn(2\pi/3)}}{2j}-\frac{e^{jn(\pi/3)}-e^{-jn(\pi/3)}}{2j}\right]=\frac{1}{n\pi}\left[\sin n\frac{2\pi}{3}-\sin n\frac{\pi}{3}\right]$$

EXAMPLE 5.8 Find the inverse Fourier transform of

$$X(\omega)=2+e^{-j\omega}+3e^{-j3\omega}+4e^{-j4\omega}$$

Solution: Given $\quad X(\omega)=2+e^{-j\omega}+3e^{-j3\omega}+4e^{-j4\omega}$

We know that $\quad X(\omega)=\sum_{n=-\infty}^{\infty}x(n)\,e^{-j\omega n}$

$$=\cdots+x(-2)\,e^{j2\omega}+x(-1)\,e^{j\omega}+x(0)+x(1)\,e^{-j\omega}+x(2)\,e^{-j2\omega}$$
$$+\,x(3)\,e^{-j3\omega}+x(4)\,e^{-j4\omega}+\cdots$$

Comparing the above two values of $X(\omega)$, we get

i.e. $\quad x(0)=2,\ x(1)=1,\ x(2)=0,\ x(3)=3,\ x(4)=4$

$$x(n)=\{2, 1, 0, 3, 4\}$$

5.6 PROPERTIES OF DISCRETE-TIME FOURIER TRANSFORM

5.6.1 Linearity Property

The linearity property of DTFT states that

If $\quad \mathrm{F}[x_1(n)]=X_1(\omega)$ and $\mathrm{F}[x_2(n)]=X_2(\omega)$

Then $\quad \mathrm{F}[ax_1(n)+bx_2(n)]=aX_1(\omega)+bX_2(\omega)$

Proof:
$$F\{ax_1(n)+bx_2(n)\} = \sum_{n=-\infty}^{\infty} [ax_1(n)+bx_2(n)]\, e^{-j\omega n}$$
$$= \sum_{n=-\infty}^{\infty} ax_1(n)\, e^{-j\omega n} + \sum_{n=-\infty}^{\infty} bx_2(n)\, e^{-j\omega n} = aX_1(\omega)+bX_2(\omega)$$

5.6.2 Periodicity Property

The periodicity property of DTFT states that the DTFT $X(\omega)$ is periodic in ω with period 2π, i.e.

$$X(\omega + 2n\pi) = X(\omega)$$

Implication: We need only one period of $X(\omega)$ {i.e., $\omega \in (0, 2\pi)$ or $(-\pi, \pi)$} for analysis and not the whole range $-\infty < \omega < \infty$.

5.6.3 Time Shifting Property

The time shifting property of DTFT states that

If $$F[x(n)] = X(\omega)$$

Then $F[x(n-m)] = e^{-j\omega m} X(\omega)$ where m is an integer.

Proof:
$$F\{x(n-m)\} = \sum_{n=-\infty}^{\infty} x(n-m)\, e^{-j\omega n}$$

Let $n - m = p$

$\therefore$ $$n = p + m$$

$\therefore$ $$F\{x(n-m)\} = \sum_{p=-\infty}^{\infty} x(p)\, e^{-j\omega(p+m)}$$
$$= e^{-j\omega m} \sum_{p=-\infty}^{\infty} x(p)\, e^{-j\omega p} = e^{-j\omega m}\, X(\omega)$$

This result shows that the time shifting of a signal by m units does not change its amplitude spectrum but the phase spectrum is changed by $-\omega m$.

5.6.4 Frequency Shifting Property

The frequency shifting property of DTFT states that

If $$F\{x(n)\} = X(\omega)$$

Then $$F\{x(n)\, e^{j\omega_0 n}\} = X(\omega - \omega_0)$$

Proof:
$$F\{x(n)\, e^{j\omega_0 n}\} = \sum_{n=-\infty}^{\infty} \{x(n)\, e^{j\omega_0 n}\}\, e^{-j\omega n}$$
$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\omega-\omega_0)n} = X(\omega - \omega_0)$$

This property is the dual of the time shifting property.

5.6.5 Time Reversal Property

The time reversal property of DTFT states that

If $$F\{x(n)\} = X(\omega)$$

Then $$F\{x(-n)\} = X(-\omega)$$

Proof:

$$F\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(-\omega)n}$$

$$= X(-\omega)$$

That is, folding in the time domain corresponds to the folding in the frequency domain.

5.6.6 Differentiation in the Frequency Domain Property

The differentiation in the frequency domain property of DTFT states that

If $$F\{x(n)\} = X(\omega)$$

Then $$F\{n\,x(n)\} = j\frac{d}{d\omega}[X(\omega)]$$

Proof:

$$F\{x(n)\} = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

Differentiating both sides w.r.t. ω, we get

$$\frac{d}{d\omega}\{X(\omega)\} = \frac{d}{d\omega}\left\{\sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}\right\}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\frac{d}{d\omega} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\,(-jn)\, e^{-j\omega n}$$

$$= -j\left\{\sum_{n=-\infty}^{\infty} n\,x(n)\, e^{-j\omega n}\right\}$$

$\therefore$ $$\sum_{n=-\infty}^{\infty} nx(n)\, e^{-j\omega n} = F\{n\,x(n)\} = j\frac{d}{d\omega}[X(\omega)]$$

5.6.7 Time Convolution Property

The time convolution property of DTFT states that

If $$F\{x_1(n)\} = X_1(\omega) \text{ and } F\{x_2(n)\} = X_2(\omega)$$

Then $$F\{x_1(n) * x_2(n)\} = X_1(\omega)\, X_2(\omega)$$

Proof: $$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

$\therefore$ $$F\{x_1(n) * x_2(n)\} = \sum_{n=-\infty}^{\infty} [x_1(n) * x_2(n)]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} [x_1(k)\, x_2(n-k)] \right\} e^{-j\omega n}$$

Interchanging the order of summations, we get

$$F\{x_1(n) * x_2(n)\} = \sum_{k=-\infty}^{\infty} x_1(k) \sum_{n=-\infty}^{\infty} x_2(n-k)\, e^{-j\omega n}$$

Put $n - k = p$ in the second summation.

$\therefore$ $$n = p + k$$

$$F\{x_1(n) * x_2(n)\} = \sum_{k=-\infty}^{\infty} x_1(k) \sum_{p=-\infty}^{\infty} x_2(p)\, e^{-j\omega(p+k)}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)\, e^{-j\omega k} \sum_{p=-\infty}^{\infty} x_2(p)\, e^{-j\omega p}$$

$$= X_1(\omega)\, X_2(\omega)$$

That is, the convolution of the signals in the time domain is equal to multiplying their spectra in the frequency domain.

5.6.8 Frequency Convolution Property

The frequency convolution property of DTFT states that

If $$F\{x_1(n)\} = X_1(\omega) \text{ and } F\{x_2(n)\} = X_2(\omega)$$

Then $$F\{x_1(n)\, x_2(n)\} = X_1(\omega) * X_2(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\theta)\, X_2(\omega - \theta)\, d\theta$$

Proof: $$F[x_1(n)\, x_2(n)] = \sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\theta)\, e^{j\theta n}\, d\theta \right] e^{-j\omega n}\, x_2(n)$$

Interchanging the order of summation and integration, we get

$$F\{x_1(n)\, x_2(n)\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\theta) \left[\sum_{n=-\infty}^{\infty} x_2(n)\, e^{-j(\omega-\theta)n} \right] d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\theta)\, X_2(\omega-\theta)\, d\theta$$

This operation is known as periodic convolution because it is the convolution of two periodic functions $X_1(\omega)$ and $X_2(\omega)$.

5.6.9 The Correlation Theorem

The correlation theorem of DTFT states that

If $\quad F\{x_1(n)\} = X_1(\omega)$ and $F\{x_2(n)\} = X_2(\omega)$

Then $\quad F\{R_{x_1x_2}(l)\} = X_1(\omega)\, X_2(-\omega) = \Gamma_{x_1x_2}(\omega)$

The function $\Gamma_{x_1x_2}(\omega)$ is called the cross energy spectrum of the signals $x_1(n)$ $x_2(n)$.

5.6.10 The Modulation Theorem

If $\quad F\{x(n)\} = X(\omega)$

Then $\quad F\{x(n)\cos\omega_0 n\} = \frac{1}{2}\{X(\omega+\omega_0) + X(\omega-\omega_0)\}$

Proof:

$$F\{x(n)\cos\omega_0 n\} = \sum_{n=-\infty}^{\infty} x(n) \frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2} e^{-j\omega n}$$

$$= \frac{1}{2}\left\{ \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\omega-\omega_0)n} + \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\omega+\omega_0)n} \right\}$$

$$= \frac{1}{2}\{X(\omega-\omega_0) + X(\omega+\omega_0)\}$$

5.6.11 Parseval's Theorem

If $\quad F\{x(n)\} = X(\omega)$

Then

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2\, d\omega$$

Proof:

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} x(n)\, x^*(n)$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega \right\}^*$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega)\, e^{-j\omega n}\, d\omega \right\}$$

Interchanging the order of summation and integration, we get

$$E = \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega) \left\{ \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \right\} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega)\, X(\omega)\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2\, d\omega$$

5.6.12 Symmetry Properties

The DTFT $X(\omega)$ is a complex function of ω and can be expressed as:

$$X(\omega) = X_R(\omega) + jX_I(\omega)$$

where $X_R(\omega)$ is real part and $X_I(\omega)$ is imaginary part of $X(\omega)$ respectively. We have

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cos \omega n - j \sum_{n=-\infty}^{\infty} x(n) \sin \omega n$$

i.e.

$$X_R(\omega) + jX_I(\omega) = \sum_{n=-\infty}^{\infty} x(n) \cos \omega n - j \sum_{n=-\infty}^{\infty} x(n) \sin \omega n$$

Comparing LHS and RHS, we have

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} x(n) \cos \omega n$$

$$X_I(\omega) = -\sum_{n=-\infty}^{\infty} x(n) \sin \omega n$$

Since $\cos(-\omega)n = \cos\omega n$, and $\sin(-\omega)n = -\sin\omega n$

$$X_R(-\omega) = \sum_{n=-\infty}^{\infty} x(n)\cos(-\omega)n = \sum_{n=-\infty}^{\infty} x(n)\cos\omega n = X_R(\omega)$$

i.e. $$X_R(-\omega) = X_R(\omega) \qquad \text{(Even symmetry)}$$

$$X_I(-\omega) = -\sum_{n=-\infty}^{\infty} x(n)\sin(-\omega)n = \sum_{n=-\infty}^{\infty} x(n)\sin\omega n = -X_I(\omega)$$

i.e. $$X_I(-\omega) = -X_I(\omega) \qquad \text{(Odd symmetry)}$$

Therefore, $X_R(\omega)$ is an even function of ω and $X_I(\omega)$ is an odd function of ω. We can write $X_R(\omega)$ in the polar form as:

$$X(\omega) = |X(\omega)|e^{j\theta(\omega)}$$

where $|X(\omega)|$ is the magnitude and $\theta(\omega)$ is the phase of $X(\omega)$.

Expanding $$X(\omega) = |X(\omega)|\{\cos\theta(\omega) + j\sin\theta(\omega)\}$$

i.e. $$X_R(\omega) + jX_I(\omega) = |X(\omega)|\cos\theta(\omega) + j|X(\omega)|\sin\theta(\omega)$$

Comparing LHS and RHS, we get

$$X_R(\omega) = |X(\omega)|\cos\theta(\omega)$$

$$X_I(\omega) = |X(\omega)|\sin\theta(\omega)$$

$$|X(\omega)|^2 = \{X_R(\omega)\}^2 + \{X_I(\omega)\}^2$$

i.e. $$X(\omega) = \sqrt{\{X_R(\omega)\}^2 + \{X_I(\omega)\}^2}$$

and $$\tan\theta(\omega) = \frac{X_I(\omega)}{X_R(\omega)}$$

or $$\theta(\omega) = \tan^{-1}\frac{X_I(\omega)}{X_R(\omega)}$$

Similarly, $$\begin{aligned}|X(-\omega)| &= \sqrt{\{X_R(-\omega)\}^2 + \{X_I(-\omega)\}^2}\\ &= \sqrt{|X_R(\omega)|^2 + |X_I(\omega)|^2}\\ &= |X(\omega)|\end{aligned}$$

Therefore, $X(\omega)$ is an even function of ω,

$$\begin{aligned}\theta(-\omega) &= \tan^{-1}\frac{X_I(-\omega)}{X_R(-\omega)}\\ &= \tan^{-1}\frac{-X_I(\omega)}{X_R(\omega)}\\ &= -\tan^{-1}\frac{X_I(\omega)}{X_R(\omega)} = -\theta(\omega)\end{aligned}$$

$\therefore \qquad \theta(-\omega) = -\theta(\omega)$

That is $\qquad \underline{|X(-\omega)} = -\underline{|X(\omega)}$

Therefore $\underline{|X(\omega)}$ is an odd function of ω.

TABLE 5.1 Properties of DTFT

Property	*Sequence*	DTFT
	$x(n)$	$X(\omega)$
	$x_1(n)$	$X_1(\omega)$
	$x_2(n)$	$X_2(\omega)$
Linearity	$ax_1(n) + bx_2(n)$	$aX_1(\omega) + bX_2(\omega)$
Time shifting	$x(n - m)$	$e^{-j\omega m} X(\omega)$
Time reversal	$x(-n)$	$X(-\omega_0)$
Frequency shifting	$x(n)e^{j\omega 0n}$	$X(\omega - \omega)$
Differentiation in frequency domain	$nx(n)$	$j\dfrac{d}{d\omega}[X(\omega)]$
Convolution	$x_1(n) * x_2(n)$	$X_1(\omega)\, X_2(\omega)$
Multiplication	$x_1(n)\, x_2(n)$	$X_1(\omega) * X_2(\omega)$
Correlation	$R_{x_1x_2}(l)$	$X_1(\omega)\, X_2(-\omega)$
Modulation theorem	$x(n) \cos \omega_0 n$	$\dfrac{1}{2}\{X(\omega+\omega_0) + X(\omega-\omega_0)\}$
Parseval's theorem	$\sum_{n=-\infty}^{\infty} \lvert x(n)\rvert^2$	$\dfrac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(\omega)\rvert^2\, d\omega$
Symmetry property	$x^*(n)$	$X(-\omega)$
	$x^*(-n)$	$X^*(\omega)$
	$x_R(n)$	$X_e(\omega)$
	$jx_I(n)$	$X_0(\omega)$
	$x_e(n)$	$X_R(\omega)$
	$x_0(n)$	$jX_I(\omega)$

EXAMPLE 5.9 Using properties of DTFT, find the DTFT of the following:

(a) $\left(\dfrac{1}{4}\right)^{|n-2|}$

(b) $\left(\dfrac{1}{3}\right)^{|n-3|} u(n-3)$

(c) $\delta(n-2) - \delta(n+2)$

(d) $u(n+1) - u(n+2)$

(e) $n2^n\, u(n)$

(f) $u(-n)$

(g) $n3^{-n}\, u(-n)$

(h) $e^{3n}\, u(n)$

Solution:

(a) Using the time shifting property, we have

$$\mathrm{F}\left\{\left(\frac{1}{4}\right)^{|n-2|}\right\} = e^{-j2\omega}\ \mathrm{F}\left\{\left(\frac{1}{4}\right)^{|n|}\right\}$$

$$\mathrm{F}\left\{\left(\frac{1}{4}\right)^{|n|}\right\} = \sum_{n=-\infty}^{\infty}\left(\frac{1}{4}\right)^{|n|} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1}\left(\frac{1}{4}\right)^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^{n} e^{-j\omega n}$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^{n} e^{j\omega n} + \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^{n} e^{-j\omega n}$$

$$= \frac{1}{4}e^{j\omega}\frac{1}{1-(1/4)\,e^{j\omega}} + \frac{1}{1-(1/4)\,e^{-j\omega}} = \frac{15/16}{(17/16)-(1/2)\cos\omega}$$

$$\therefore\quad \mathrm{F}\left\{\left(\frac{1}{4}\right)^{|n-2|}\right\} = e^{-j2\omega}\frac{15/16}{(17/16)-(1/2)\cos\omega}$$

(b) Using the time shifting property, we have

$$\mathrm{F}\left\{\left(\frac{1}{3}\right)^{|n-3|} u(n-3)\right\} = e^{-j3\omega}\ \mathrm{F}\left\{\left(\frac{1}{3}\right)^{n} u(n)\right\} = e^{-j3\omega}\left\{\frac{1}{1-(1/3)\,e^{-j\omega}}\right\}$$

(c) Using the time shifting property, we have

$$\mathrm{F}\{\delta(n-2)-\delta(n+2)\} = \mathrm{F}\{\delta(n-2)\} - \mathrm{F}\{\delta(n+2)\}$$

$$= e^{-j2\omega}\ \mathrm{F}\{\delta(n)\} - e^{j2\omega}\ \mathrm{F}\{\delta(n)\}$$

$$= e^{-j2\omega} - e^{j2\omega} = -2j\sin 2\omega$$

(d) Using the time shifting property, we have

$$\mathrm{F}\{u(n+1)-u(n+2)\} = \mathrm{F}\{u(n+1)\} - \mathrm{F}\{u(n+2)\}$$

$$= e^{j\omega}\ \mathrm{F}\{u(n)\} - e^{j2\omega}\ \mathrm{F}\{u(n)\}$$

$$= \frac{e^{j\omega}}{1-e^{-j\omega}} - \frac{e^{j2\omega}}{1-e^{-j\omega}}$$

(e) Using differentiation in the frequency domain property, we have

$$\mathrm{F}\left\{n\left(\frac{1}{2}\right)^{n} u(n)\right\} = j\frac{d}{d\omega}\left[\mathrm{F}\left\{\left(\frac{1}{2}\right)^{n} u(n)\right\}\right]$$

$$= j\frac{d}{d\omega}\left[\frac{1}{1-(1/2)\,e^{-j\omega}}\right]$$

$$= j\frac{\{-[-(1/2)\,e^{-j\omega}(-j)]\}}{\{1-(1/2)\,e^{-j\omega}\}^2} = \frac{(1/2)\,e^{-j\omega}}{\{1-(1/2)\,e^{-j\omega}\}^2}$$

(f) Using the time reversal property, we have

$$\text{F}\{u(-n)\} = \text{F}\{u(n)\}\big|_{\omega=-\omega} = \left\{\frac{1}{1-e^{-j\omega}}\right\}_{\omega=-\omega} = \frac{1}{1-e^{j\omega}}$$

(g) Using differentiation in frequency domain and time reversal properties, we have

$$\text{F}\{n3^{-n}\,u(-n)\} = j\frac{d}{d\omega}[\text{F}\{3^{-n}\,u(-n)\}]$$

$$= j\frac{d}{d\omega}[\text{F}\{3^{n}\,u(n)\}]_{\omega=-\omega} = j\frac{d}{d\omega}\left[\frac{1}{1-3e^{-j\omega}}\right]_{\omega=-\omega}$$

$$= j\frac{d}{d\omega}\left[\frac{1}{1-3e^{j\omega}}\right] = j\frac{-[-3e^{j\omega}(j)]}{[1-3e^{j\omega}]^2} = \frac{-3e^{j\omega}}{\{1-3e^{j\omega}\}^2}$$

(h) Using the frequency shifting property, we have

$$\text{F}\{e^{3n}\,u(n)\} = \text{F}\{u(n)\}\big|_{\omega=\omega-3}$$

$$= \left\{\frac{1}{1-e^{-j\omega}}\right\}_{\omega=\omega-3} = \frac{1}{1-e^{-j(\omega-3)}}$$

EXAMPLE 5.10 Find the inverse Fourier transform for the first order recursive filter

$$H(\omega) = (1 - ae^{-j\omega})^{-1}$$

Solution: Given $H(\omega) = (1-ae^{-j\omega})^{-1} = \dfrac{1}{1-ae^{-j\omega}}$

$$= 1 + ae^{-j\omega} + a^2e^{-j2\omega} + a^3e^{-j3\omega} + \cdots$$

Let $h(n)$ be the inverse Fourier transform of $H(\omega)$.

$$\therefore\ H(\omega) = \sum_{n=-\infty}^{\infty} h(n)\,e^{-j\omega n} = \cdots + h(-2)\,e^{j2\omega} + h(-1)\,e^{j\omega} + h(0) + h(1)\,e^{-j\omega} + h(2)\,e^{-j2\omega} + \cdots$$

On comparing the two expressions for $H(\omega)$, we can say that the samples of $h(n)$ are the coefficients of $e^{-j\omega n}$.

$$\therefore\qquad h(n) = \{1, a, a^2, \ldots, a^k, \ldots\}$$

i.e.
$$h(n) = \begin{cases} a^n, & n \ge 0 \\ 0, & n < 0 \end{cases} \quad \text{or } h(n) = a^n\,u(n)$$

EXAMPLE 5.11 Determine the output sequence from the output spectrum:

$$Y(\omega) = \frac{1}{4}\frac{e^{j2\omega} + 1 + e^{-j2\omega}}{1 - ae^{-j\omega}}$$

Solution: Given $$Y(\omega) = \frac{1}{4}\frac{e^{j2\omega} + 1 + e^{-j2\omega}}{1 - ae^{-j\omega}}$$

The output sequence $y(n)$ is the inverse Fourier transform of $Y(\omega)$.

$$\therefore \qquad y(n) = \text{F}^{-1}\left\{\frac{1}{4}\frac{e^{j2\omega} + 1 + e^{-j2\omega}}{1 - ae^{-j\omega}}\right\}$$

$$= \frac{1}{4}\left[\text{F}^{-1}\left\{\frac{e^{j2\omega}}{1 - ae^{-j\omega}}\right\} + \text{F}^{-1}\left\{\frac{1}{1 - ae^{-j\omega}}\right\} + \text{F}^{-1}\left\{\frac{e^{-j2\omega}}{1 - ae^{-j\omega}}\right\}\right]$$

Using the time shifting property, we have

$$\text{F}^{-1}\left\{\frac{e^{j2\omega}}{1 - ae^{-j\omega}}\right\} = \text{F}^{-1}\left\{\frac{1}{1 - ae^{-j\omega}}\right\}\Bigg|_{n=n+2} = a^n u(n)\Big|_{n=n+2} = a^{n+2}\, u(n+2)$$

Also, we know that

$$\text{F}^{-1}\left\{\frac{1}{1 - ae^{-j\omega}}\right\} = a^n u(n)$$

Using the time shifting property, we have

$$\text{F}^{-1}\left\{\frac{e^{-j2\omega}}{1 - ae^{-j\omega}}\right\} = \text{F}^{-1}\left\{\frac{1}{1 - ae^{-j\omega}}\right\}\Bigg|_{n=n-2} = a^{n-2}\, u(n-2)$$

$$\therefore \qquad y(n) = \frac{1}{4}\{a^{n+2}\, u(n+2) + a^n u(n) + a^{n-2}\, u(n-2)\}$$

EXAMPLE 5.12 The impulse response of a LTI system is $h(n) = \{1, 2, 1, -2\}$. Find the response of the system for the input $x(n) = \{1, 3, 2, 1\}$.

Solution: The response of the system $y(n)$ for an input $x(n)$ and impulse response $h(n)$ is given by

$$y(n) = x(n) * h(n)$$

Using the convolution property of Fourier transform, we get

$$Y(\omega) = X(\omega)\, H(\omega)$$

$$\therefore \qquad y(n) = \text{F}^{-1}\{X(\omega)\, H(\omega)\}$$

Given $$x(n) = \{1, 3, 2, 1\}$$

$$\therefore \qquad X(\omega) = 1 + 3e^{-j\omega} + 2e^{-j2\omega} + e^{-j3\omega}$$

Given $$h(n) = \{1, 2, 1, -2\}$$

$$\therefore \qquad H(\omega) = 1 + 2e^{-j\omega} + e^{-j2\omega} - 2e^{-j3\omega}$$

$$Y(\omega) = X(\omega)\, H(\omega) = (1 + 3e^{-j\omega} + 2e^{-j2\omega} + e^{-j3\omega})(1 + 2e^{-j\omega} + e^{-j2\omega} - 2e^{-j3\omega})$$
$$= 1 + 5e^{-j\omega} + 9e^{-j2\omega} + 6e^{-j3\omega} - 2e^{-j4\omega} - 3e^{-j5\omega} - 2e^{-j6\omega}$$

Taking inverse Fourier transform on both sides, we get

$$y(n) = 1 + 5\delta(n-1) + 9\delta(n-2) + 6\delta(n-3) - 2\delta(n-4) - 3\delta(n-5) - 2\delta(n-6)$$

or $$y(n) = \{1, 5, 9, 6, -2, -3, -2\}$$

EXAMPLE 5.13 Find the convolution of the signals given below using Fourier transform:

$$x_1(n) = \left(\frac{1}{2}\right)^n u(n)$$

$$x_2(n) = \left(\frac{1}{3}\right)^n u(n)$$

Solution: Given $$x_1(n) = \left(\frac{1}{2}\right)^n u(n)$$

$$\therefore \qquad X_1(\omega) = \frac{1}{1 - (1/2)\, e^{-j\omega}}$$

$$x_2(n) = \left(\frac{1}{3}\right)^n u(n)$$

$$\therefore \qquad X_2(\omega) = \frac{1}{1 - (1/3)\, e^{-j\omega}}$$

Using the convolution property of Fourier transform, we get

$$\mathrm{F}[x_1(n) * x_2(n)] = X_1(\omega)\, X_2(\omega)$$
$$= \left[\frac{1}{1 - (1/2)\, e^{-j\omega}}\right]\left[\frac{1}{1 - (1/3)\, e^{-j\omega}}\right]$$

$$\therefore \qquad x_1(n) * x_2(n) = \mathrm{F}^{-1}\left\{\left[\frac{1}{1 - (1/2)e^{-j\omega}}\right]\left[\frac{1}{1 - (1/3)e^{-j\omega}}\right]\right\}$$

Let $$X(\omega) = \left(\frac{1}{1 - (1/2)e^{-j\omega}}\right)\left(\frac{1}{1 - (1/3)e^{-j\omega}}\right) = \left[\frac{e^{j\omega}}{e^{j\omega} - (1/2)}\right]\left[\frac{e^{j\omega}}{e^{j\omega} - (1/3)}\right]$$

$$\therefore \qquad \frac{X(\omega)}{e^{j\omega}} = \frac{e^{j\omega}}{[e^{j\omega} - (1/2)][e^{j\omega} - (1/3)]}$$

$$= \frac{A}{e^{j\omega} - (1/2)} + \frac{B}{e^{j\omega} - (1/3)} = \frac{3}{e^{j\omega} - (1/2)} + \frac{-2}{e^{j\omega} - (1/3)}$$

$$\therefore \qquad X(\omega) = \frac{3e^{j\omega}}{e^{j\omega} - (1/2)} - \frac{2e^{j\omega}}{e^{j\omega} - (1/3)}$$

$$= 3\frac{1}{1 - (1/2)e^{-j\omega}} - 2\frac{1}{1 - (1/3)e^{-j\omega}}$$

Taking inverse Fourier transform on both sides, we have

$$x(n) = x_1(n) * x_2(n) = 3\left(\frac{1}{2}\right)^n u(n) - 2\left(\frac{1}{3}\right)^n u(n)$$

EXAMPLE 5.14 Consider a discrete-time LTI system with impulse response $h(n) = \left(\frac{1}{2}\right)^n u(n)$. Use Fourier transform to determine the response to the signal $x(n) = \left(\frac{3}{4}\right)^n u(n)$.

Solution: Given the impulse response $h(n)$ and the input $x(n)$ to the system, the response $y(n)$ is given by

$$y(n) = x(n) * h(n)$$

Using the convolution property of Fourier transform, we have

$$y(n) = x(n) * h(n) = \mathrm{F}^{-1}[X(\omega)\, H(\omega)]$$

where $X(\omega)$ and $H(\omega)$ are the Fourier transforms of $x(n)$ and $h(n)$, respectively.

$$x(n) = \left(\frac{3}{4}\right)^n u(n)$$

$$\therefore \qquad X(\omega) = \frac{1}{1 - (3/4)\, e^{-j\omega}}$$

$$h(n) = \left(\frac{1}{2}\right)^n u(n)$$

$$\therefore \qquad H(\omega) = \frac{1}{1 - (1/2)\, e^{-j\omega}}$$

$$\therefore \qquad Y(\omega) = X(\omega)\,H(\omega) = \left[\frac{1}{1-(3/4)\,e^{-j\omega}}\right]\left[\frac{1}{1-(1/2)\,e^{-j\omega}}\right]$$

$$= \left[\frac{e^{j\omega}}{e^{j\omega}-(3/4)}\right]\left[\frac{e^{j\omega}}{e^{j\omega}-(1/2)}\right]$$

$$\therefore \qquad \frac{Y(\omega)}{e^{j\omega}} = \frac{e^{j\omega}}{[e^{j\omega}-(3/4)]\,[e^{j\omega}-(1/2)]}$$

$$= \frac{A}{e^{j\omega}-(3/4)} + \frac{B}{e^{j\omega}-(1/2)} = \frac{3}{e^{j\omega}-(3/4)} + \frac{-2}{e^{j\omega}-(1/2)}$$

$$\therefore \qquad Y(\omega) = \frac{3e^{j\omega}}{e^{j\omega}-(3/4)} - \frac{2e^{j\omega}}{e^{j\omega}-(1/2)} = \frac{3}{1-(3/4)\,e^{-j\omega}} - \frac{2}{1-(1/2)\,e^{-j\omega}}$$

Taking inverse Fourier transform on both sides, we get the response

$$y(n) = 3\left(\frac{3}{4}\right)^n u(n) - 2\left(\frac{1}{2}\right)^n u(n)$$

5.7 TRANSFER FUNCTION

If $X(\omega)$ is the Fourier transform of the input signal $x(n)$, and $Y(\omega)$ is the Fourier transform of the output signal $y(n)$, then $H(\omega)$, the transfer function of the system is given by

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

i.e. the transfer function of an LTI system is defined as the ratio of the Fourier transform of the output to the Fourier transform of the input. It is also defined as the Fourier transform of the impulse response $h(n)$ of the system. It can be obtained as follows:

If the input to the system is of the form $e^{j\omega n}$, then the output $y(n)$ is given by

$$y(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(k)\, e^{j\omega(n-k)} = e^{j\omega n}\, H(\omega)$$

i.e. $\qquad y(n) = x(n)\; H(\omega)$

We have
$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

$$y(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega\; H(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, H(\omega)\, e^{j\omega n}\, d\omega$$

$y(n)$ in terms of $Y(\omega)$ is given by

$$y(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(\omega)\, e^{j\omega n}\, d\omega$$

Comparing the above two expressions for $y(n)$, we get

$$Y(\omega) = X(\omega)\ H(\omega)$$

$$\therefore \qquad H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

where $H(\omega)$ is known as the transfer function of the system.

5.8 FREQUENCY RESPONSE OF DISCRETE-TIME SYSTEMS

The frequency response of a linear time-invariant discrete-time system can be obtained by applying a spectrum of input sinusoids to the system. The frequency response gives the gain and phase response of the system to the input sinusoids at all frequencies.

Let $h(n)$ be the impulse response of an LTI discrete system, and let the input $x(n)$ to the system be a complex exponential $e^{j\omega n}$.

The output of the system $y(n)$ can be obtained by using convolution sum.

$$y(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

For $x(n) = e^{j\omega n}$,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, e^{j\omega(n-k)}$$

$$= e^{j\omega n} \sum_{k=-\infty}^{\infty} h(k)\, e^{-j\omega k}$$

$$= \underbrace{e^{j\omega n}}_{\text{Input}} \qquad \underbrace{H(\omega)}_{\text{Frequency response}}$$

That is, if we force the system with a complex exponential $e^{j\omega n}$, then the output is of the form $H(\omega)e^{j\omega n}$. Therefore, the output of the system is identical to the input modified in amplitude and phase by $H(\omega)$. The quantity $H(\omega)$ is the frequency response of the system.

It is same as the transfer function $H(\omega)$ in the frequency domain. The frequency response $H(\omega)$ is complex and can be expressed in the polar form as:

$$H(\omega) = |H(\omega)|\, e^{j\angle H(\omega)}$$

where the magnitude of $H(\omega)$, i.e. $|H(\omega)|$ is called the magnitude response and phase angle of $H(\omega)$, i.e. $\angle H(\omega)$ is called the phase response. The plot of $|H(\omega)|$ versus ω is called the magnitude response plot and the plot of $\angle H(\omega)$ versus ω is called the phase response plot.

Properties of frequency response

Frequency response is a complex function that describes the magnitude and phase shift of a filter over a range of frequencies. If $h(n)$ is a real sequence, the frequency response $H(\omega)$ has the following properties:

1. $H(\omega)$ takes on values for all ω, i.e. on a continuum of ω.
2. $H(\omega)$ is periodic in ω, with period of 2π.
3. The magnitude response $|H(\omega)|$ is an even function of ω and symmetrical about π.
4. The phase response $\angle H(\omega)$ is an odd function of ω and antisymmetrical about π.

EXAMPLE 5.15 Write a difference equation that characterizes a system whose frequency response is:

$$H(\omega) = \frac{1 - e^{-j\omega} - 3e^{-j2\omega}}{1 + (1/3)\, e^{-j\omega} + (1/6)\, e^{-j2\omega}}$$

Solution: Given $$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1 - e^{-j\omega} - 3e^{-j2\omega}}{1 + (1/3)\, e^{-j\omega} + (1/6)\, e^{-j2\omega}}$$

On cross multiplication, we get

$$Y(\omega) + \frac{1}{3} e^{-j\omega} Y(\omega) + \frac{1}{6} e^{-j2\omega} Y(\omega) = X(\omega) - e^{-j\omega} X(\omega) - 3e^{-j2\omega} X(\omega)$$

Taking inverse Fourier transform on both sides, we get the difference equation:

$$y(n) + \frac{1}{3} y(n-1) + \frac{1}{6} y(n-2) = x(n) - x(n-1) - 3x(n-2)$$

EXAMPLE 5.16 Find the frequency response of the following causal systems:

(a) $y(n) - y(n-1) + \dfrac{3}{16} y(n-2) = x(n) - \dfrac{1}{2} x(n-1)$

(b) $y(n) - \dfrac{1}{4} y(n-1) - \dfrac{3}{8} y(n-2) = x(n) + x(n-1)$

Solution: The frequency response of a system is given by $H(\omega) = [Y(\omega)/X(\omega)]$, where $Y(\omega)$ and $X(\omega)$ are the Fourier transforms of output and input signals respectively.

(a) Given $$y(n) - y(n-1) + \frac{3}{16} y(n-2) = x(n) - \frac{1}{2} x(n-1)$$

Taking Fourier transform on both sides, we have

$$Y(\omega) - e^{-j\omega}\, Y(\omega) + \frac{3}{16} e^{-j2\omega}\, Y(\omega) = X(\omega) - \frac{1}{2} e^{-j\omega}\, X(\omega)$$

i.e. $$Y(\omega)\left(1 - e^{-j\omega} + \frac{3}{16} e^{-j2\omega}\right) = X(\omega)\left(1 - \frac{1}{2} e^{-j\omega}\right)$$

$$\therefore \quad H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1 - (1/2)\, e^{-j\omega}}{1 - e^{-j\omega} + (3/16)\, e^{-j2\omega}} = \frac{e^{j\omega}\, [e^{j\omega} - (1/2)]}{e^{j2\omega} - e^{j\omega} + (3/16)}$$

(b) Given $$y(n) - \frac{1}{4}y(n-1) - \frac{3}{8}y(n-2) = x(n) + x(n-1)$$

Taking Fourier transform on both sides, we have

$$Y(\omega) - \frac{1}{4}e^{-j\omega}\,Y(\omega) - \frac{3}{8}e^{-j2\omega}\,Y(\omega) = X(\omega) + e^{-j\omega}\,X(\omega)$$

i.e. $$Y(\omega)\left(1 - \frac{1}{4}e^{-j\omega} - \frac{3}{8}e^{-j2\omega}\right) = X(\omega)\,(1 + e^{-j\omega})$$

$$\therefore H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1 + e^{-j\omega}}{1 - (1/4)\,e^{-j\omega} - (3/8)\,e^{-j2\omega}} = \frac{e^{j\omega}(e^{j\omega} + 1)}{e^{j2\omega} - (1/4)\,e^{j\omega} - (3/8)}$$

EXAMPLE 5.17 A discrete system is given by the following difference equation:

$$y(n) - 5y(n-1) = x(n) + 4x(n-1)$$

where $x(n)$ is the input and $y(n)$ is the output. Determine its magnitude and phase response.

Solution: Given $$y(n) - 5y(n-1) = x(n) + 4x(n-1)$$

Taking Fourier transform on both sides, we have

$$Y(\omega) - 5e^{-j\omega}\,Y(\omega) = X(\omega) + 4e^{-j\omega}\,X(\omega)$$

i.e. $$Y(\omega)\,\{1 - 5e^{-j\omega}\} = X(\omega)\,\{1 + 4e^{-j\omega}\}$$

The transfer function of the system, i.e. the frequency response of the system is:

$$\frac{Y(\omega)}{X(\omega)} = H(\omega) = \frac{1 + 4\,e^{-j\omega}}{1 - 5e^{-j\omega}} = \frac{e^{j\omega} + 4}{e^{j\omega} - 5}$$

$$= \frac{\cos\omega + j\sin\omega + 4}{\cos\omega + j\sin\omega - 5}$$

The magnitude response of the system is:

$$|H(\omega)| = \frac{\sqrt{(4 + \cos\omega)^2 + (\sin\omega)^2}}{\sqrt{(\cos\omega - 5)^2 + (\sin\omega)^2}} = \frac{\sqrt{17 + 8\cos\omega}}{\sqrt{26 - 10\cos\omega}}$$

The phase response of the system is:

$$\underline{|H(\omega)} = \tan^{-1}\left\{\frac{\sin\omega}{4 + \cos\omega}\right\} - \tan^{-1}\left\{\frac{\sin\omega}{\cos\omega - 5}\right\}$$

EXAMPLE 5.18 The output $y(n)$ for a linear shift-invariant system, with input $x(n)$ is given by

$$y(n) = x(n) - 2x(n-1) + x(n-2)$$

Determine the magnitude and phase response of the system.

Solution: Given $$y(n) = x(n) - 2x(n-1) + x(n-2)$$

Taking Fourier transform on both sides, we have

$$Y(\omega) = X(\omega) - 2e^{-j\omega} X(\omega) + e^{-j2\omega} X(\omega)$$

The transfer function of the system, i.e. the frequency response of the system is:

$$\begin{aligned} H(\omega) &= \frac{Y(\omega)}{X(\omega)} = 1 - 2e^{-j\omega} + e^{-j2\omega} \\ &= 1 - 2[\cos\omega - j\sin\omega] + \cos 2\omega - j\sin 2\omega \\ &= [1 - 2\cos\omega + \cos 2\omega] + j[2\sin\omega - \sin 2\omega] \end{aligned}$$

The magnitude response of the system is:

$$|H(\omega)| = \sqrt{[1 - 2\cos\omega + \cos 2\omega]^2 + [2\sin\omega - \sin 2\omega]^2}$$

The phase response of the system is given as:

$$\underline{|H(\omega)} = \tan^{-1}\left\{\frac{2\sin\omega - \sin 2\omega}{1 - 2\cos\omega + \cos 2\omega}\right\}$$

EXAMPLE 5.19 The impulse response of a system is:

$$h(n) = \begin{cases} 1 & \text{for } 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

Find the transfer function, frequency response, magnitude response and phase response.

Solution: The transfer function $H(\omega)$ is obtained by taking the Fourier transform of $h(n)$.

$$\therefore \qquad \begin{aligned} H(\omega) &= \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n} = \sum_{n=0}^{N-1} (1)\, e^{-j\omega n} \\ &= \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} \end{aligned}$$

The frequency response of the system is same as the transfer function, i.e.

$$H(\omega) = \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}$$

The magnitude function is given as:

$$\begin{aligned} |H(\omega)| &= \{H(\omega)\, H^*(\omega)\}^{1/2} \\ &= \left\{\left[\frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}\right]\left[\frac{1 - e^{j\omega N}}{1 - e^{j\omega}}\right]\right\}^{1/2} \end{aligned}$$

$$= \left\{ \frac{1 + 1 - e^{j\omega N} - e^{-j\omega N}}{1 + 1 - e^{j\omega} - e^{-j\omega}} \right\}^{1/2}$$

$$= \left\{ \frac{2 - (e^{j\omega N} + e^{-j\omega N})}{2 - (e^{j\omega} + e^{-j\omega})} \right\}^{1/2} = \left\{ \frac{1 - \cos \omega N}{1 - \cos \omega} \right\}^{1/2}$$

In order to determine the phase function, the real and imaginary parts of $H(\omega)$ have to be separated.

$$\therefore \qquad H(\omega) = \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} \times \frac{1 - e^{j\omega}}{1 - e^{j\omega}} = \frac{1 - e^{j\omega} - e^{-j\omega N} + e^{-j\omega(N-1)}}{2 - (e^{j\omega} + e^{-j\omega})}$$

$$= \frac{1 - (\cos \omega + j \sin \omega) - (\cos \omega N - j \sin \omega N) + [\cos \omega(N-1) - j \sin \omega(N-1)]}{2 - 2 \cos \omega}$$

Now,
$$H_R(\omega) = \frac{1 - \cos \omega - \cos \omega N + \cos \omega(N-1)}{2 - 2\cos \omega}$$

$$H_I(\omega) = \frac{-\sin \omega + \sin \omega N - \sin \omega(N-1)}{2 - 2 \cos \omega}$$

$$\therefore \qquad \underline{|H(\omega)} = \tan^{-1} \frac{H_I(\omega)}{H_R(\omega)}$$

$$= \tan^{-1} \left\{ \frac{-\sin \omega + \sin \omega N - \sin \omega(N-1)}{1 - \cos \omega - \cos \omega N + \cos \omega(N-1)} \right\}$$

EXAMPLE 5.20 The impulse response of a LTI system is given by $h(n)$. Find the frequency response, magnitude and phase response.

Solution: Given $\qquad h(n) = 0.6^n\, u(n)$

The frequency response of the system is:

$$H(\omega) = \text{F}\{0.6^n\, u(n)\} = \frac{1}{1 - 0.6e^{-j\omega}}$$

Here $H(\omega)$ is a complex function of frequency. To separate the real and imaginary parts of $H(\omega)$, multiply the numerator and denominator by the complex conjugate of the denominator. Thus,

$$H(\omega) = \frac{1}{1 - 0.6e^{-j\omega}} \times \frac{1 - 0.6e^{j\omega}}{1 - 0.6e^{j\omega}} = \frac{1 - 0.6e^{j\omega}}{1 - 0.6e^{j\omega} - 0.6e^{-j\omega} + 0.36}$$

$$= \frac{1 - 0.6 \cos \omega}{1.36 - 1.2 \cos \omega} - j \frac{0.6 \sin \omega}{1.36 - 1.2 \cos \omega}$$

$$\therefore \qquad H_R(\omega) = \frac{1 - 0.6 \cos \omega}{1.36 - 1.2 \cos \omega} \quad \text{and} \quad H_I(\omega) = \frac{-0.6 \sin \omega}{1.36 - 1.2 \cos \omega}$$

The magnitude function of $H(\omega)$ is:

$$|H(\omega)| = \{H_R(\omega)^2 + H_I(\omega)^2\}^{1/2}$$

$$= \left\{\left[\frac{1-0.6\cos\omega}{1.36-1.2\cos\omega}\right]^2 + \left[\frac{-0.6\sin\omega}{1.36-1.2\cos\omega}\right]^2\right\}^{1/2}$$

$$= \frac{1}{(1.36-1.2\cos\omega)^{1/2}}$$

The phase function of $H(\omega)$ is:

$$\underline{|H(\omega)} = \tan^{-1}\frac{H_I(\omega)}{H_R(\omega)} = \tan^{-1}\left\{\frac{-0.6\sin\omega}{1-0.6\cos\omega}\right\}$$

The magnitude function $|H(\omega)|$ can be obtained as follows:

$$|H(\omega)| = \{H(\omega)\,H^*(\omega)\}^{1/2} = \{H(\omega)\,H(-\omega)\}^{1/2}$$

$$= \left(\frac{1}{1-0.6e^{-j\omega}} \times \frac{1}{1-0.6e^{j\omega}}\right)^{1/2} = \left[\frac{1}{1+0.36-0.6\,(e^{j\omega}+e^{-j\omega})}\right]^{1/2}$$

$$= \frac{1}{(1.36-1.2\cos\omega)^{1/2}}$$

EXAMPLE 5.21 A causal LTI system is described by the difference equation:

$$y(n) - ay(n-1) = bx(n) + x(n-1)$$

where a is real and less than 1 in magnitude. Find a value of $b(b \neq a)$ such that the frequency response of the system satisfies $|H(\omega)| = 1$ for all ω (an all pass system, the magnitude of the frequency response is constant independent of frequency).

Solution: Given $\quad y(n) - ay(n-1) = bx(n) + x(n-1)$

Taking Fourier transform on both sides, we have

$$Y(\omega)\,[1-ae^{-j\omega}] = X(\omega)\,[b+e^{-j\omega}]$$

$$\therefore \qquad H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{b+e^{-j\omega}}{1-ae^{-j\omega}} = \frac{b+\cos\omega - j\sin\omega}{1-a\cos\omega + ja\sin\omega}$$

$$|H(\omega)|^2 = \frac{(b+\cos\omega)^2 + \sin^2\omega}{(1-a\cos\omega)^2 + a^2\sin^2\omega} = \frac{1+b^2+2b\cos\omega}{1+a^2-2a\cos\omega}$$

For magnitude to be independent of frequency,

$$\frac{d}{d\omega}|H(\omega)|^2 = 0$$

i.e.
$$\frac{d}{d\omega}\left(\frac{1+b^2+2b\cos\omega}{1+a^2-2a\cos\omega}\right)=0$$

Simplifying, we get $b = 1/a$.

EXAMPLE 5.22 A causal and stable LTI system has the property that

$$\left(\frac{4}{5}\right)^n u(n) \longrightarrow n\left(\frac{4}{5}\right)^n u(n)$$

(a) Determine the frequency response $H(\omega)$ for the system.
(b) Determine a difference equation relating any input $x(n)$ and the corresponding output $y(n)$.

Solution: Given
$$x(n)=\left(\frac{4}{5}\right)^n u(n)$$

$\therefore$
$$X(\omega)=\frac{1}{1-(4/5)e^{-j\omega}}$$

$$y(n)=n\left(\frac{4}{5}\right)^n u(n)$$

$\therefore$
$$Y(\omega)=j\frac{d}{d\omega}[X(\omega)]=\frac{(4/5)\,e^{-j\omega}}{[1-(4/5)\,e^{-j\omega}]^2}$$

$\therefore$
$$Y(\omega)=\frac{(4/5)\,e^{-j\omega}}{[1-(4/5)\,e^{-j\omega}]}\,\frac{1}{[1-(4/5)\,e^{-j\omega}]}=\frac{(4/5)\,e^{-j\omega}}{[1-(4/5)\,e^{-j\omega}]}\,X(\omega)$$

Therefore, the frequency response is:

$$H(\omega)=\frac{Y(\omega)}{X(\omega)}=\frac{(4/5)\,e^{-j\omega}}{[1-(4/5)\,e^{-j\omega}]}$$

Cross multiplying, we get

$$Y(\omega)-\frac{4}{5}e^{-j\omega}Y(\omega)=\frac{4}{5}e^{-j\omega}X(\omega)$$

Taking inverse Fourier transform, we get the difference equation:

$$y(n)-\frac{4}{5}y(n-1)=\frac{4}{5}x(n-1)$$

EXAMPLE 5.23 Determine the impulse response of all the four types of ideal filters shown in Figure 5.1.

Solution:

(a) For an ideal low-pass filter shown in Figure 5.1(a),

$$H(\omega)=\begin{cases}1, & \text{for } 0\le|\omega|\le\omega_c\\ 0, & \text{otherwise}\end{cases}$$

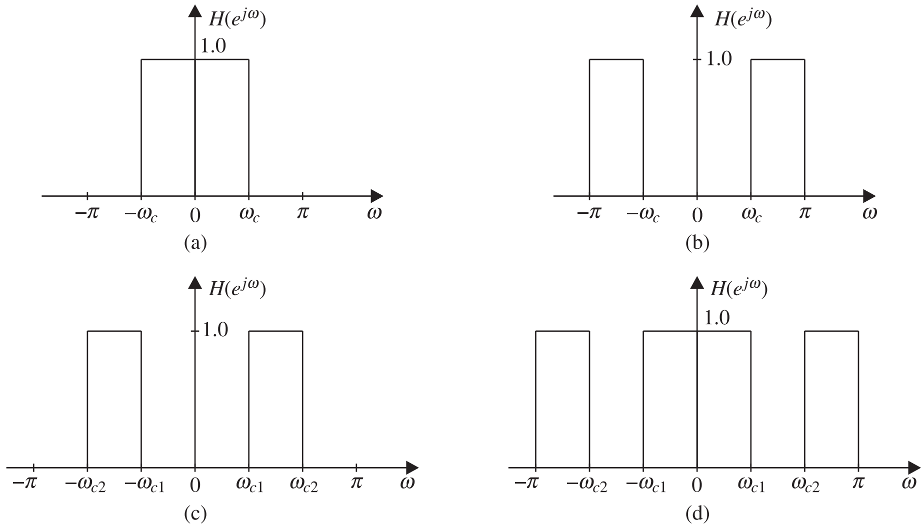


Figure 5.1 Frequency response of ideal filters, (a) low-pass filter, (b) high-pass filter, (c) band pass filter, and (d) band stop filter.

We have
$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)\, e^{j\omega n}\, d\omega$$
$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega = \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{\pi n(2j)} = \frac{\sin \omega_c n}{\pi n}$$

i.e.
$$h(n) = \frac{\sin \omega_c n}{\pi n} \qquad -\infty \le n \le \infty$$

(b) For an ideal high-pass filter shown in Figure 5.1(b),
$$H(\omega) = \begin{cases} 0, & \text{for } 0 \le |\omega| \le \omega_c \\ 1, & \text{for } \omega_c \le |\omega| \le \pi \end{cases}$$

We have
$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)\, e^{j\omega n}\, d\omega$$
$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-\omega_c} e^{j\omega n}\, d\omega + \int_{\omega_c}^{\pi} e^{j\omega n}\, d\omega\right]$$
$$= \frac{1}{2\pi jn}[e^{-j\omega_c n} - e^{-j\pi n} + e^{j\pi n} - e^{j\omega_c n}]$$
$$= \frac{1}{\pi n}\left[\left(\frac{e^{j\pi n} - e^{-j\pi n}}{2j}\right) - \left(\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right)\right]$$

$$= \frac{1}{\pi n}[\sin \pi n - \sin \omega_c n]$$

$$= -\frac{\sin \omega_c n}{\pi n}$$

i.e. $$h(n) = -\frac{\sin \omega_c n}{\pi n} \quad -\infty \le n \le \infty$$

(c) For an ideal band pass filter shown in Figure 5.1(c),

$$H(\omega) = \begin{cases} 1, & \text{for } \omega_{c1} \le |\omega| \le \omega_{c2} \\ 0, & \text{otherwise} \end{cases}$$

We have $$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left(\int_{-\omega_{c2}}^{-\omega_{c1}} e^{j\omega n}\, d\omega + \int_{\omega_{c1}}^{\omega_{c2}} e^{j\omega n}\, d\omega \right)$$

$$= \frac{1}{2\pi jn}[e^{-j\omega_{c1}n} - e^{-j\omega_{c2}n} + e^{j\omega_{c2}n} - e^{j\omega_{c1}n}]$$

$$= \frac{1}{\pi n}\left[\left(\frac{e^{j\omega_{c2}n} - e^{-j\omega_{c2}n}}{2j}\right) - \left(\frac{e^{j\omega_{c1}n} - e^{-j\omega_{c1}n}}{2j}\right)\right]$$

$$= \frac{1}{\pi n}[\sin \omega_{c2}n - \sin \omega_{c1}n]$$

i.e. $$h(n) = \frac{\sin \omega_{c2}n - \sin \omega_{c1}n}{\pi n} \quad -\infty \le n \le \infty$$

(d) For an ideal band stop filter shown in Figure 5.1(d),

$$H(\omega) = \begin{cases} 1, & \text{for } |\omega| \le \omega_{c1} \text{ and } \omega_{c2} \le |\omega| \le \pi \\ 0, & \text{for } \omega_{c1} \le |\omega| \le \omega_{c2} \end{cases}$$

We have $$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega) e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left(\int_{-\pi}^{-\omega_{c2}} e^{j\omega n}\, d\omega + \int_{-\omega_{c1}}^{\omega_{c1}} e^{j\omega n}\, d\omega + \int_{\omega_{c2}}^{\pi} e^{j\omega n}\, d\omega \right)$$

$$= \frac{1}{2\pi jn}[e^{-j\omega_{c2}n} - e^{-j\pi n} + e^{j\omega_{c1}n} - e^{-j\omega_{c1}n} + e^{j\pi n} - e^{j\omega_{c2}n}]$$

$$= \frac{1}{\pi n}\left[-\left(\frac{e^{j\omega_{c2}n} - e^{-j\omega_{c2}n}}{2j}\right) + \left(\frac{e^{j\omega_{c1}n} - e^{-j\omega_{c1}n}}{2j}\right) + \left(\frac{e^{j\pi n} - e^{-j\pi n}}{2j}\right)\right]$$

$$= \frac{1}{\pi n}[\sin \omega_{c1}n - \sin \omega_{c2}n]$$

i.e. $$h(n) = \frac{\sin \omega_{c1}n - \sin \omega_{c2}n}{\pi n} \qquad -\infty \le n \le \infty$$

SHORT QUESTIONS WITH ANSWERS

1. Define Fourier transform of a discrete-time signal.

Ans. The Fourier transform of a discrete-time signal $x(n)$ is defined as:

$$\mathrm{F}\{x(n)\} = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

The Fourier transform exists only if $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$, i.e. only if the sequence is absolutely summable.

2. Define inverse discrete-time Fourier transform.

Ans. The inverse discrete-time Fourier transform of $X(\omega)$ is defined as:

$$\mathrm{F}^{-1}\{X(\omega)\} = x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

3. Why the Fourier transform of a discrete-time signal is called signal spectrum?

Ans. By taking Fourier transform of a discrete-time signal $x(n)$ it is decomposed into its frequency components. Hence the Fourier transform is called signal spectrum.

4. List the differences between Fourier transform of discrete-time signal and analog signal.

Ans.

1. The Fourier transform of analog signal consists of a spectrum with frequency range $-\infty$ to ∞, but the Fourier transform of a discrete-time signal is unique in the range $-\pi$ to π (or 0 to 2π), and also it is periodic with periodicity of 2π.
2. The Fourier transform of analog signal involves integration, but Fourier transform of discrete-time signal involves summation.

5. Give some applications of discrete-time Fourier transform.

Ans. Some applications of discrete-time Fourier transform are:

1. The frequency response of LTI system is given by the Fourier transform of the impulse response of the system.

2. The ratio of the Fourier transform of output to the Fourier transform of input is called the transfer function of the system in the frequency domain.
3. The response of an LTI system can be easily computed using the convolution property of Fourier transform.

6. What is the relation between discrete-time Fourier transform and Z-transform?

Ans. The Fourier transform of the discrete-time sequence $x(n)$ is the Z-transform of the sequence $x(n)$ evaluated along the unit circle centred at the origin of the z-plane.

$$X(\omega) = X(z)\big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

7. Write the properties of frequency response of LTI system.

Ans. The properties of frequency response of LTI system are:

1. The frequency response is periodic function of ω with a period of 2π.
2. If $h(n)$ is real, then $|H(\omega)|$ is symmetric and $\underline{|H(\omega)}$ is antisymmetric.
3. If $h(n)$ is complex, then the real part of $H(\omega)$ is symmetric and the imaginary part of $H(\omega)$ is antisymmetric over the interval $0 \le \omega \le 2\pi$.
4. The frequency response is a continuous function of ω.

8. What is frequency response of LTI systems?

Ans. The Fourier transform of the impulse response $h(n)$ of the system is called frequency response of the system. It is denoted by $H(\omega)$. $\{F[h(n)] = H(\omega)\}$. The frequency response has two components: magnitude function $|H(\omega)|$ and phase function $\underline{|H(\omega)}$.

9. What is the sufficient condition for the existence of DTFT?

Ans. The sufficient condition for the existence of DTFT for a sequence $x(n)$ is:

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

i.e. the sequence $x(n)$ must be absolutely summable.

10. State Parseval's energy theorem for discrete-time aperiodic signals.

Ans. The Parseval's energy theorem for discrete-time aperiodic signals is given by

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2\, d\omega$$

11. Define DTFT pair.

Ans. The Fourier transform pair of a discrete-time signal is:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

and

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

12. If $F\{x(n)\} = X(\omega)$, what is the $F\{x(n-m)\}$?

Ans. $$F\{x(n-m)\} = e^{-j\omega m} X(\omega)$$

13. If $F\{x(n)\} = X(\omega)$, what is the $F\{e^{j\omega_0 n} x(n)\}$?

Ans. $$F\{e^{j\omega_0 n} x(n)\} = X(\omega - \omega_0)$$

14. If $F\{x(n)\} = X(\omega)$, what is the $F\{x(-n)\}$?

Ans. $$F\{x(-n)\} = X(-\omega)$$

15. If $F\{x(n)\} = X(\omega)$, what is the $F\{nx(n)\}$?

Ans. $$F\{nx(n)\} = j\frac{d}{d\omega}[X(\omega)]$$

16. If $F\{x_1(n)\} = X_1(\omega)$ and $F\{x_2(n)\} = X_2(\omega)$, what is the $F\{x_1(n) * x_2(n)\}$?

Ans. $$F\{x_1(n) * x_2(n)\} = X_1(\omega)\, X_2(\omega)$$

17. If $F\{x_1(n)\} = X_1(\omega)$ and $F\{x_2(n)\} = X_2(\omega)$, what is the $F\{x_1(n)\, x_2(n)\}$?

Ans. $$F\{x_1(n)\, x_2(n)\} = X_1(\omega) * X_2(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\theta)\, X_2(\omega - \theta)\, d\theta$$

18. If $F\{x(n)\} = X(\omega)$, what is the $F\{x(n)\cos\omega_0 n\}$?

Ans. $$F\{x(n)\cos\omega_0 n\} = \frac{1}{2}\{X(\omega - \omega_0) + X(\omega + \omega_0)\}$$

REVIEW QUESTIONS

1. State and prove the time shifting and frequency shifting properties of DTFT.
2. State and prove the time reversal and differentiation in frequency domain properties of DTFT.
3. State and prove the time convolution and frequency convolution properties of DTFT.
4. State and prove the modulation theorem and Parseval's theorem.
5. Compare the Fourier transforms of discrete-time signal and analog signal.
6. What are the applications of DTFT?

FILL IN THE BLANKS

1. The DTFT of $x(n)$ is defined as ______.
2. The DTFT exists only if ______.
3. The FT of a discrete-time signal is called ______.
4. The FT of a discrete-time signal is periodic with period ______.

5. The FT of an analog signal involves ______, but the FT of discrete-time signal involves ______.

6. The FT of analog signals consists of a spectrum with a frequency range ______, but the FT of a discrete-time signal is unique in the range ______.

7. The inverse Fourier transform of $X(\omega)$ is defined as ______.

8. The FT of $x(n)$ is nothing but the Z-transform of $x(n)$ evaluated along the ______ centred at the origin of z-plane.

9. The relation between DTFT $X(\omega)$ and Z-transform $X(z)$ is ______.

10. The FT of a discrete and aperiodic sequence is ______.

11. The frequency response of LTI system is given by the FT of the ______ of the system.

12. The impulse response is the inverse Fourier transform of the ______ of the system.

13. The ratio of the FT of the output to the FT of the input is called the ______ or ______ of the system.

14. The frequency response is a ______ function of ω.

15. The frequency response has two components: 1. ______ 2. ______.

16. If $h(n)$ is real, then $H(\omega)$ is ______ $\underline{|H(\omega)}$ ______.

17. If $h(n)$ is complex, then ______ part of $H(\omega)$ is symmetric and the ______ part of $H(\omega)$ is antisymmetric over the interval $0 \le \omega \le 2\pi$.

18. The Fourier transform of ______ is equal to the product $X(\omega)\,H(\omega)$.

OBJECTIVE TYPE QUESTIONS

1. The DTFT of a sequence $x(n)$ is defined as $X(\omega)$ =

(a) $\sum_{n=-\infty}^{\infty} x(n)\, e^{j\omega n}$ (b) $\sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$

(c) $\sum_{n=0}^{\infty} x(n)\, e^{j\omega n}$ (d) $\sum_{n=0}^{\infty} x(n)\, e^{-j\omega n}$

2. The inverse DTFT of $X(\omega)$ is defined as $x(n)$ =

(a) $\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{-j\omega n}\, d\omega$ (b) $\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$

(c) $\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega n}\, d\omega$ (d) $\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega n}\, d\omega$

3. The FT of a discrete-time signal is periodic with period

(a) 2π (b) π (c) ∞ (d) finite

4. The relation between DTFT and Z-transform is $X(\omega)$ =

(a) $X(z)\big|_{z=e^{j\omega}}$ (b) $X(z)\big|_{z=e^{-j\omega}}$ (c) $X(z)\big|_{z=j\omega}$ (d) $X(z)\big|_{z=\omega}$

5. The frequency response of LTI system is given by the FT of the ________ of the system.

(a) transfer function (b) output
(c) impulse response (d) input

6. The Fourier transform of $x(n) * h(n)$ is equal to

(a) $X(\omega)\, H(\omega)$ (b) $X(\omega) * H(\omega)$ (c) $X(\omega)\, H(-\omega)$ (d) $X(\omega) * H(-\omega)$

7. The FT of analog signal consists of a spectrum with frequency range

(a) $-\pi$ to π (b) 0 to 2π (c) 0 to ∞ (d) $-\infty$ to ∞

8. The FT of a discrete-time signal is unique in the range

(a) $-\infty$ to ∞ (b) 0 to ∞ (c) $-\pi$ to π (d) 0 to π

9. The FT of $\delta(n)$ is

(a) 0 (b) 1 (c) ∞ (d) not defined

10. The FT of $u(n)$ is

(a) $\dfrac{1}{1-e^{-j\omega}}$ (b) $\dfrac{1}{1-e^{j\omega}}$ (c) $\dfrac{1}{1-\omega}$ (d) $\dfrac{1}{1-j\omega}$

11. The FT of $a^n u(n)$ is

(a) $\dfrac{1}{1-ae^{j\omega}}$ (b) $\dfrac{1}{1-ae^{-j\omega}}$ (c) $\dfrac{1}{1-ja\omega}$ (d) $\dfrac{1}{1+ja\omega}$

12. The FT of $-a^n u(-n-1)$ is

(a) $\dfrac{1}{1-ae^{j\omega}}$ (b) $\dfrac{1}{1-ae^{-j\omega}}$ (c) $\dfrac{1}{1-ja\omega}$ (d) $\dfrac{1}{1+aj\omega}$

13. The FT of $\left(\dfrac{1}{2}\right)^{n-1} u(n-1)$ is

(a) $\dfrac{e^{-j\omega}}{1-(1/2)e^{-j\omega}}$ (b) $\dfrac{e^{j\omega}}{1-(1/2)e^{j\omega}}$ (c) $\dfrac{e^{-j\omega}}{1-(1/2)e^{j\omega}}$ (d) $\dfrac{e^{j\omega}}{1-(1/2)e^{-j\omega}}$

14. The FT of $2^n u(n)$ is

(a) $\dfrac{1}{1-2e^{-j\omega}}$ (b) $\dfrac{1}{1-2e^{j\omega}}$ (c) $\dfrac{1}{1+2e^{j\omega}}$ (d) does not exist

15. The FT of $\delta(n+2) - \delta(n-2)$ is

(a) $2j \sin 2\omega$ (b) $2 \cos 2\omega$
(c) $\sin 2\omega$ (d) $\cos 2\omega$

PROBLEMS

1. Find the DTFT of

   (a) $x(n) = \{2, -1, 3, 2\}$ (b) $\left(\frac{1}{4}\right)^n u(n+2)$

   (c) $x(n) = (0.2)^n u(n) - 2^n u(-n-1)$ (d) $x(n) = a^n \cos \omega_0 n$

2. Using properties of DTFT, find the FT of the following:

   (a) $\left(\frac{1}{2}\right)^{|n-3|}$ (b) $\left(\frac{1}{2}\right)^{n-4} u(n-4)$

   (c) $nu(-n)$ (d) $e^{j2n}\, u(n)$

   (e) $n3^{-n}\, u(-n)$

3. The impulse response of an LTI system is $h(n) = \{1, 2, 1, -1\}$. Find the response of the system for the input $x(n) = \{1, 3, 2, 1\}$.

4. Find the convolution of the sequences $x_1(n) = x_2(n) = \{1, \underset{\uparrow}{1}, 1\}$

5. Find the frequency response of $x(n) = \{2, 1, 2\}$.

6. Determine the output sequence from the spectrum $Y(\omega)$.

$$Y(\omega) = \frac{1}{3} \frac{e^{j\omega} + 1 + e^{-j\omega}}{1 - ae^{-j\omega}}$$

7. A system has unit sample response $h(n)$ given by

$$h(n) = -\frac{1}{4}\delta(n+1) + \frac{1}{2}\delta(n) - \frac{1}{4}\delta(n-1)$$

   Find the frequency response.

8. Determine the frequency response, magnitude response and phase response of the LTI system governed by the difference equation:

$$y(n) = x(n) + 0.81x(n-1) + 0.81x(n-2) - 0.45y(n-2)$$

MATLAB PROGRAMS

Program 5.1

% Fourier transform and Inverse Fourier transform of a given sequence

```
clc; clear all; close all;
syms x;
f = exp(–x^2);
disp('The input equation is')
disp(f)
a=fourier(f);
disp('The fourier transform of the input equation is')
disp(a)
b=ifourier(a);
disp('The Inverse fourier transform is')
disp(b)
```

Output:

```
The input equation is
exp(–x^2)

The fourier transform of the input equation is
pi^(1/2)*exp(–w^2/4)

The Inverse fourier transform is
exp(–x^2)
```

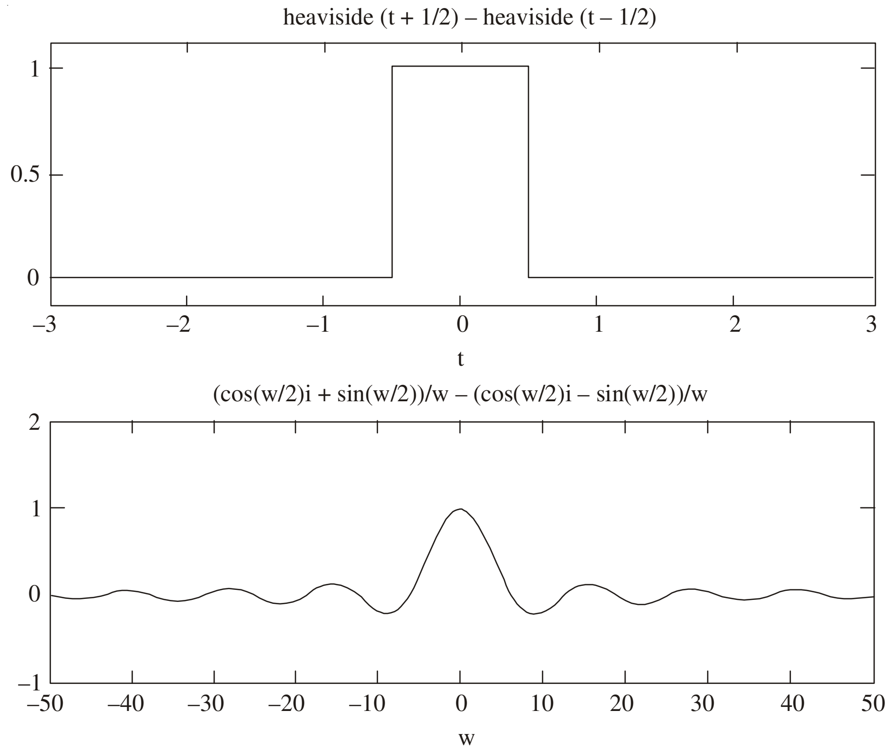
Program 5.2

% Fourier transform of a signal
% u(t+0.5)–u(t–0.5)

```
clc; close all; clear all;
syms t w
a = heaviside(t + 0.5) – heaviside(t – 0.5);
subplot(2,1,1),ezplot(a, [–3 3]);
```

```
b = fourier(a)
subplot(2,1,2);
ezplot(b, [–50 50]);
axis([–50 50 –1 2])

b =
(cos(w/2)*i + sin(w/2))/w – (cos(w/2)*i – sin(w/2))/w
```

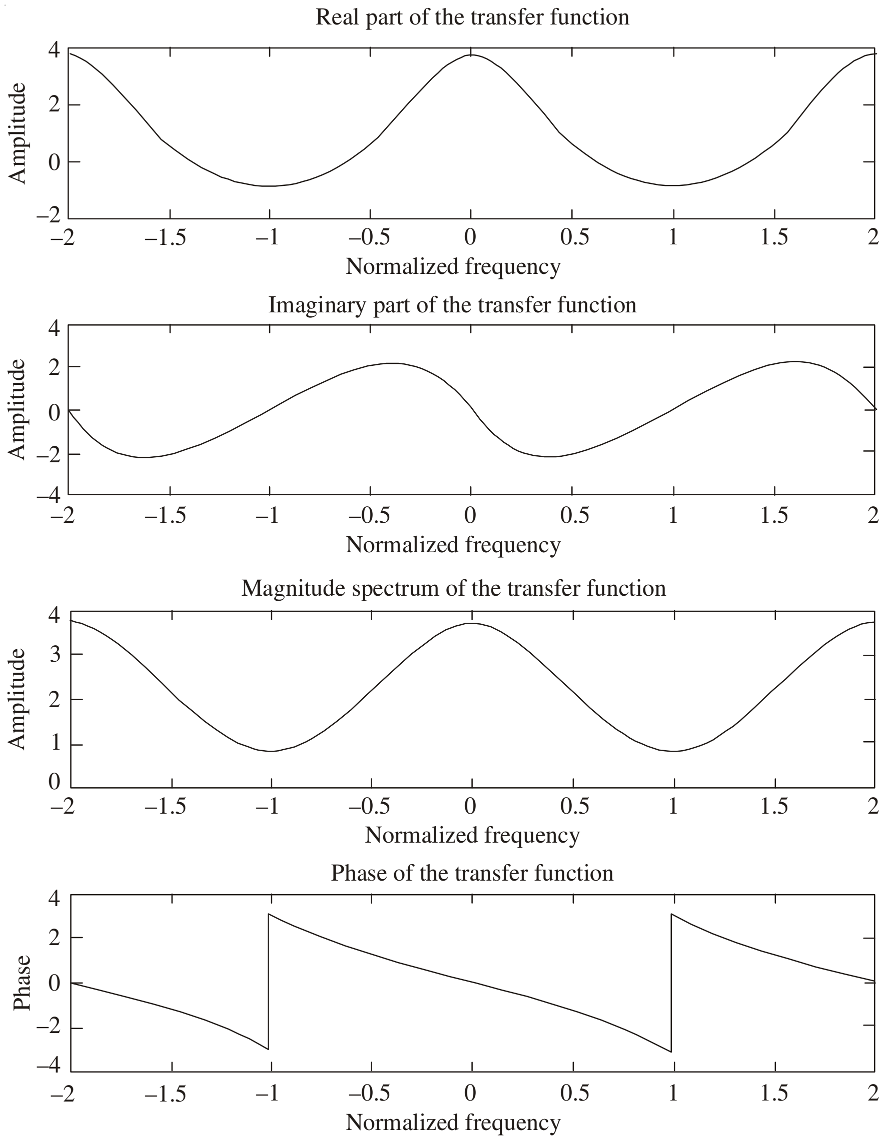
Output:



Program 5.3

```
% Evaluation and plotting of DTFT of the transfer function of the form a=e^(–jw)
% h(e)=[1+2*a^(–1)]/[1–0.2*a^(–1)]

clc; clear all; close all;
w=–2*pi:8*pi/511:2*pi;
num=[1 2];den=[1 –0.2];
h=freqz(num,den,w);
subplot(2,1,1);plot(w/pi,real(h));
xlabel(‘Normalized frequency’)
ylabel(‘Amplitude’)
title(‘Real part of the transfer function’)
subplot(2,1,2);plot(w/pi,imag(h));
xlabel(‘Normalized frequency’)
ylabel(‘Amplitude’)
title(‘Imaginary part of the transfer function’)
figure;
subplot(2,1,1);plot(w/pi,abs(h));
xlabel(‘Normalized frequency’)
ylabel(‘Amplitude’)
title(‘Magnitude spectrum of the transfer function’)
subplot(2,1,2);plot(w/pi,angle(h));
xlabel(‘Normalized frequency’)
ylabel(‘phase’)
title(‘phase of the transfer function’)
```

Output:

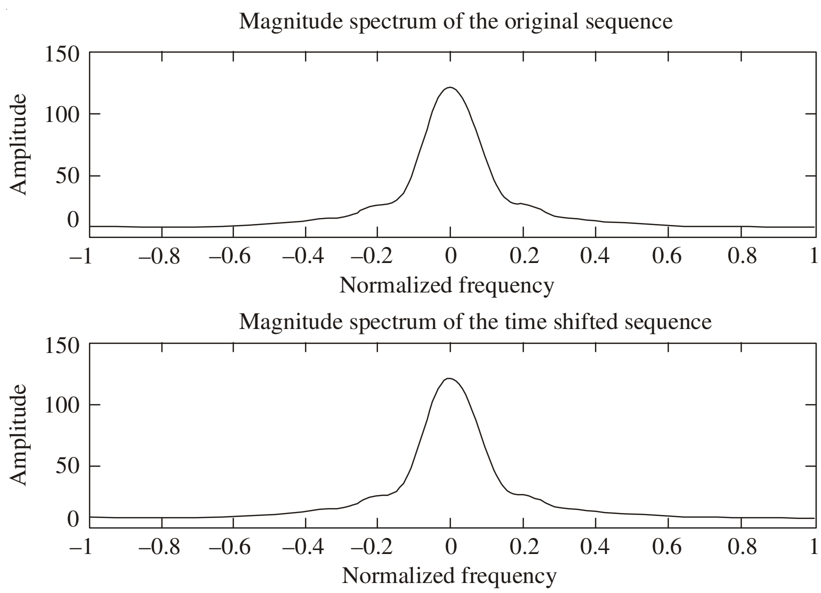


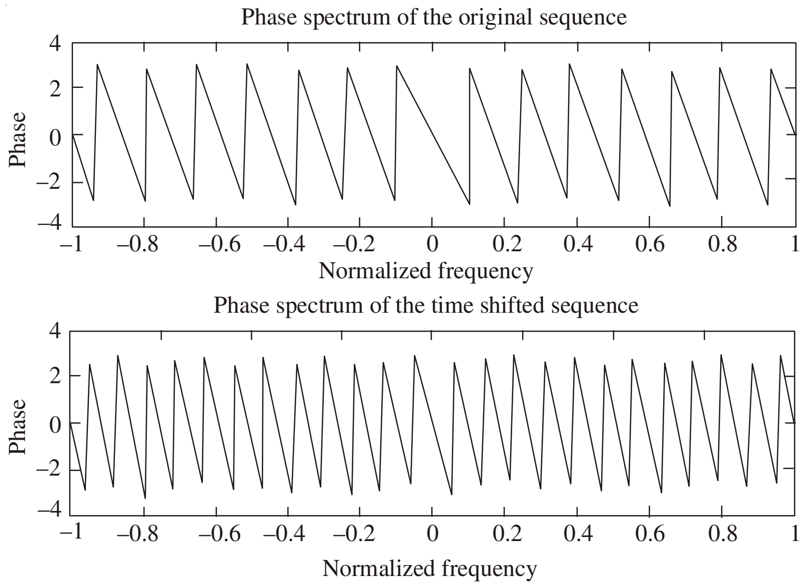
Program 5.4

```
% Time shifting property of DTFT
clc; clear all; close all
w=–pi:2*pi/255:pi;
d=10;num=1:15;
h1=freqz(num,1,w);
a=[zeros(1,d)  num];%shifting
```

```
h2=freqz(a,1,w);
subplot(2,1,1);plot(w/pi,abs(h1));
xlabel('Normalized frequency')
ylabel('Amplitude')
title('magnitude spectrum of the original sequence')
subplot(2,1,2); plot(w/pi,abs(h2));
xlabel('Normalized frequency')
ylabel('Amplitude')
title('magnitude spectrum of the time shifted sequence')
figure;
subplot(2,1,1);plot(w/pi,angle(h1));
xlabel('Normalized frequency')
ylabel('phase')
title('phase spectrum of the original sequence')
subplot(2,1,2),;plot(w/pi,angle(h2));
xlabel('Normalized frequency')
ylabel('phase')
title('phase spectrum of the time shifted sequence')
```

Output:



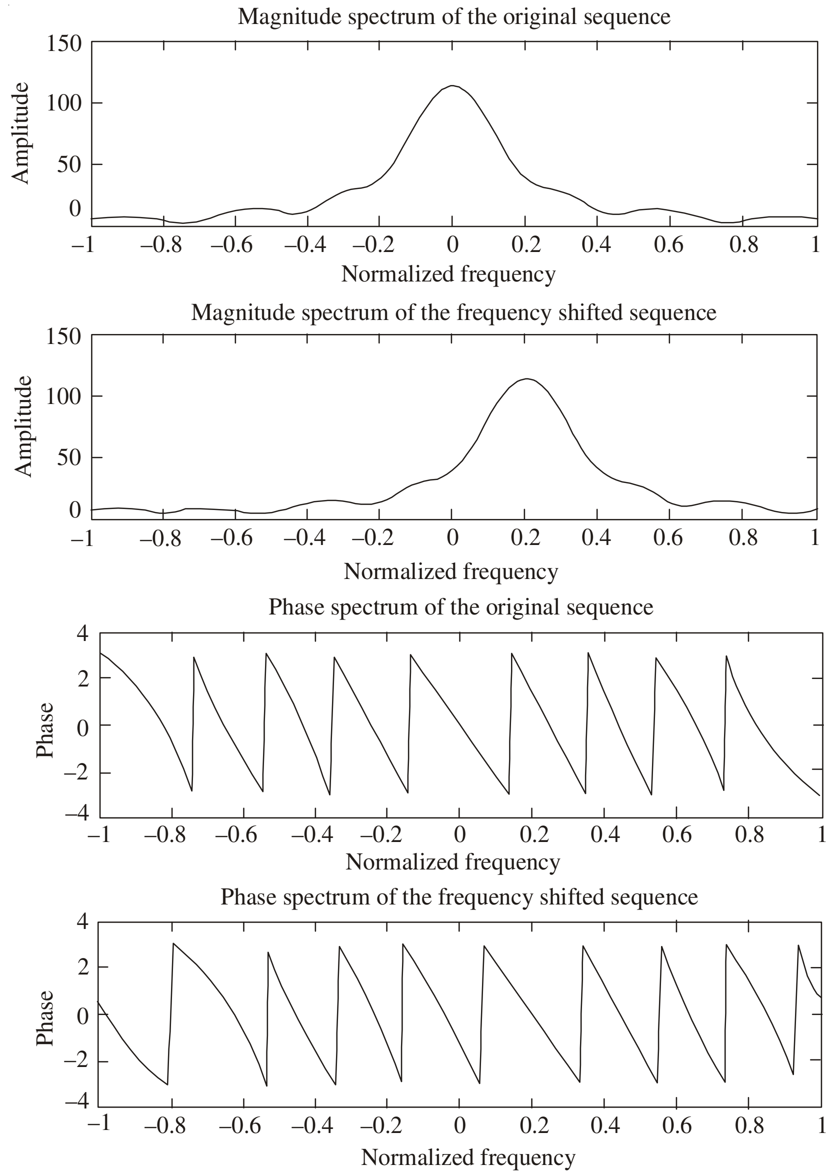


Program 5.5

```
% Frequency shifting property of DTFT
clc; clear all; close all;
w=–pi:2*pi/255:pi;
wo=0.2*pi;
num1=[1 3 5 7 5 11 13 17 18 21 12];
l=length(num1);
h1=freqz(num1,1,w);
n=0:l–1;
num2=exp(wo*i*n).*num1;
h2=freqz(num2,1,w);
subplot(2,1,1);plot(w/pi,abs(h1));
xlabel('Normalized frequency')
ylabel('Amplitude')
title('magnitude spectrum of the original sequence')
subplot(2,1,2);plot(w/pi,abs(h2));
xlabel('Normalized frequency')
ylabel('Amplitude')
title('magnitude spectrum of the Frequency shifted sequence')
figure;
subplot(2,1,1);plot(w/pi,angle(h1));
```

```
xlabel('Normalized frequency')
ylabel('phase')
title('phase spectrum of the original sequence')
subplot(2,1,2);plot(w/pi,angle(h2));
xlabel('Normalized frequency')
ylabel('phase')
title('phase spectrum of the Frequency shifted sequence')
```

Output:

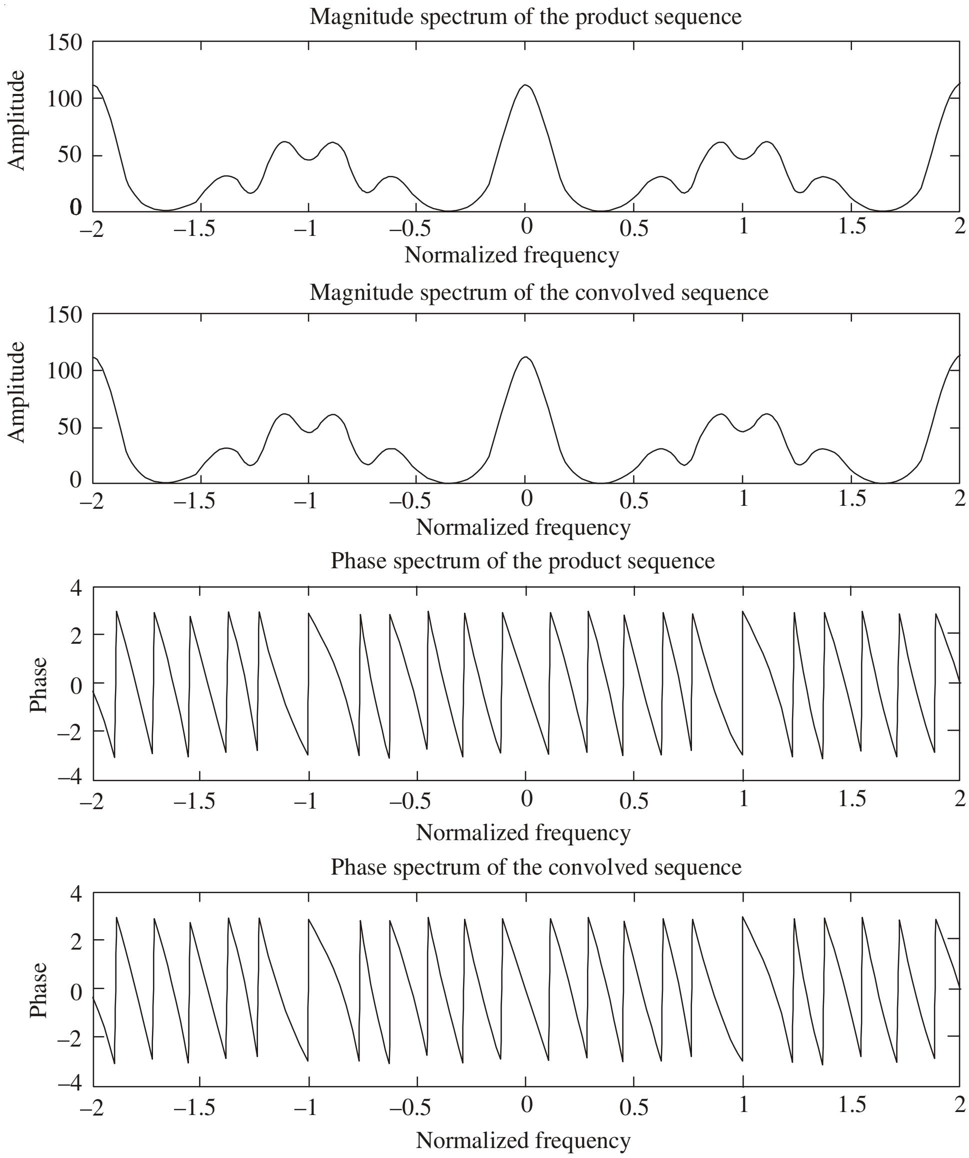


Program 5.6

% Time convolution property of DTFT

```
clc; clear all; close all;
w=–2*pi:2*pi/255:2*pi;
x1=[1 3 5 7 5 11 13 17 18 21 12];
x2=[1 –2 3 –2 1];
y=conv(x1,x2);
h1=freqz(x1,1,w);
h2=freqz(x2,1,w);
h=h1.*h2;
h3=freqz(y,1,w);
subplot(2,1,1); plot(w/pi,abs(h));
xlabel('Normalized frequency')
ylabel('Amplitude')
title('magnitude spectrum of the product sequence')
subplot(2,1,2); plot(w/pi,abs(h3));
xlabel('Normalized frequency')
ylabel('Amplitude')
title('magnitude spectrum of the convolved sequence')
figure
subplot(2,1,1),;plot(w/pi,angle(h));
xlabel('Normalized frequency')
ylabel('phase')
title('phase spectrum of the product sequence')
subplot(2,1,2);plot(w/pi,angle(h3));
xlabel('Normalized frequency')
ylabel('phase')
title('phase spectrum of the convolved sequence')
```

Output:

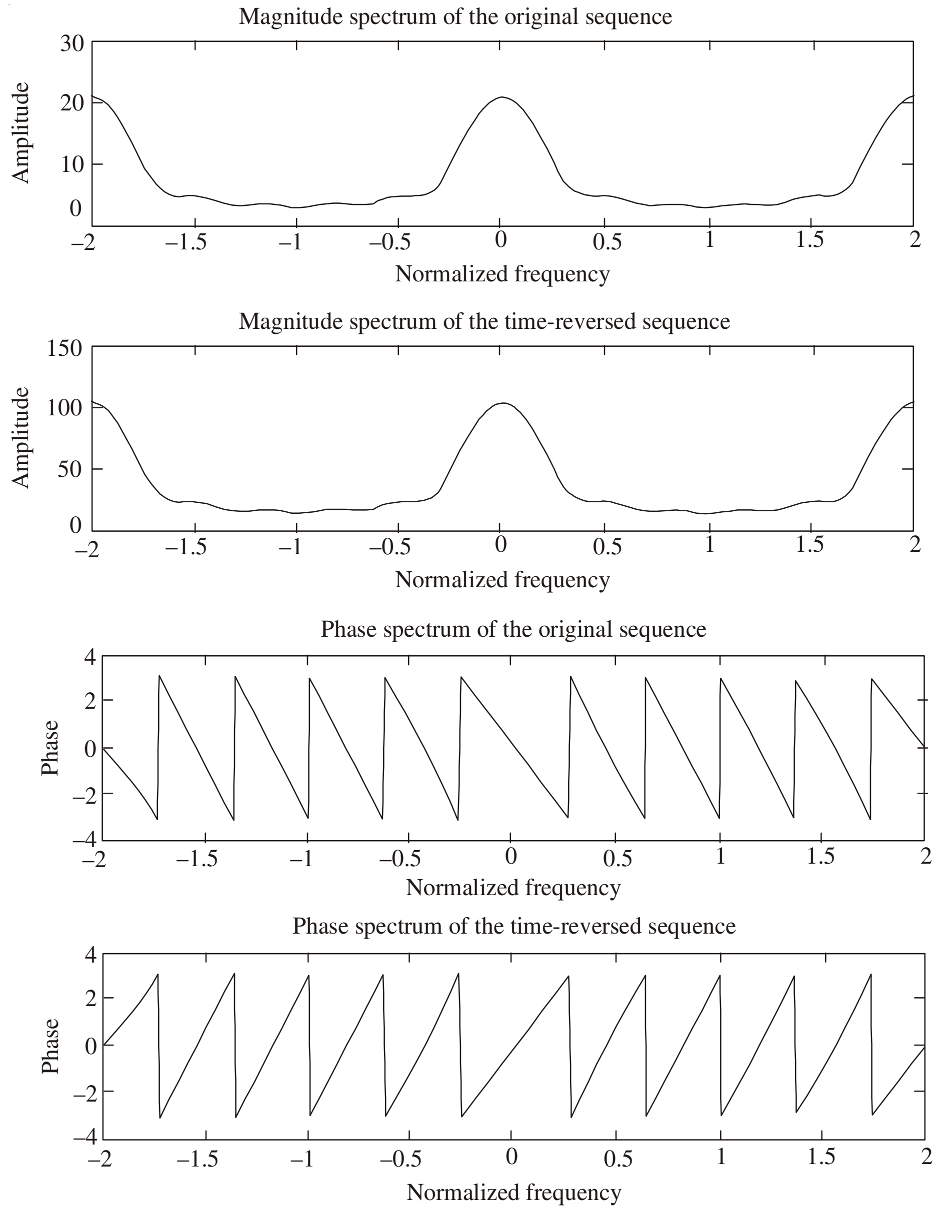

Magnitude spectrum of the product sequence
Amplitude
150
100
50
0
–2
–1.5
–1
–0.5
0
0.5
1
1.5
2
Normalized frequency
Magnitude spectrum of the convolved sequence
Amplitude
150
100
50
0
–2
–1.5
–1
–0.5
0
0.5
1
1.5
2
Normalized frequency
Phase spectrum of the product sequence
Phase
4
2
0
–2
–4
–2
–1.5
–1
–0.5
0
0.5
1
1.5
2
Normalized frequency
Phase spectrum of the convolved sequence
Phase
4
2
0
–2
–4
–2
–1.5
–1
–0.5
0
0.5
1
1.5
2
Normalized frequency

Program 5.7

% Time reversal property of DTFT

```
clc; clear all; close all
w=–2*pi:2*pi/255:2*pi;
num=[1 2 3 4 5 6];
l=length(num)–1;
h1=freqz(num,1,w);
h2=freqz(fliplr(num),1,w);
h3=exp(w*l*i).*h2;
subplot(2,1,1); plot(w/pi,abs(h1));
xlabel('Normalized frequency')
ylabel('Amplitude')
title('magnitude spectrum of the original sequence')
subplot(2,1,2);plot(w/pi,abs(h3));
xlabel('Normalized frequency')
ylabel('Amplitude')
title('magnitude spectrum of the time–reversed sequence')
figure
subplot(2,1,1); plot(w/pi,angle(h1));
xlabel('Normalized frequency')
ylabel('phase')
title('phase spectrum of the original sequence')
subplot(2,1,2); plot(w/pi,angle(h3));
xlabel('Normalized frequency')
ylabel('phase')
title('phase spectrum of the time–reversed sequence')
```

Output:

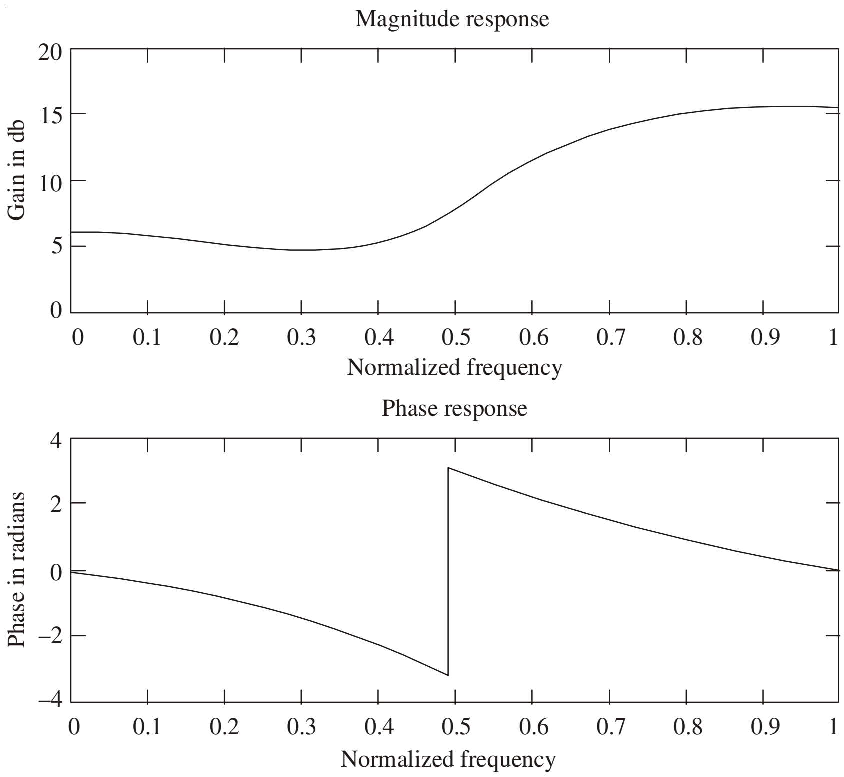

Magnitude spectrum of the original sequence
Amplitude
Normalized frequency
Magnitude spectrum of the time-reversed sequence
Amplitude
Normalized frequency
Phase spectrum of the original sequence
Phase
Normalized frequency
Phase spectrum of the time-reversed sequence
Phase
Normalized frequency

Program 5.8

Frequency response of the given system

```
clc; clear all; close all;
num=[1 –1 3];
den=[1 1/3 1/6];
[h,om]=freqz(num,den);
subplot(2,1,1);plot(om/pi,20*log10(abs(h)));
xlabel('normalized frequency')
ylabel('gain in db')
title('magnitude response')
subplot(2,1,2);plot(om/pi,angle(h));
xlabel('Normalized Frequency')
ylabel('phase in radians')
title('phase response')
```

Output:

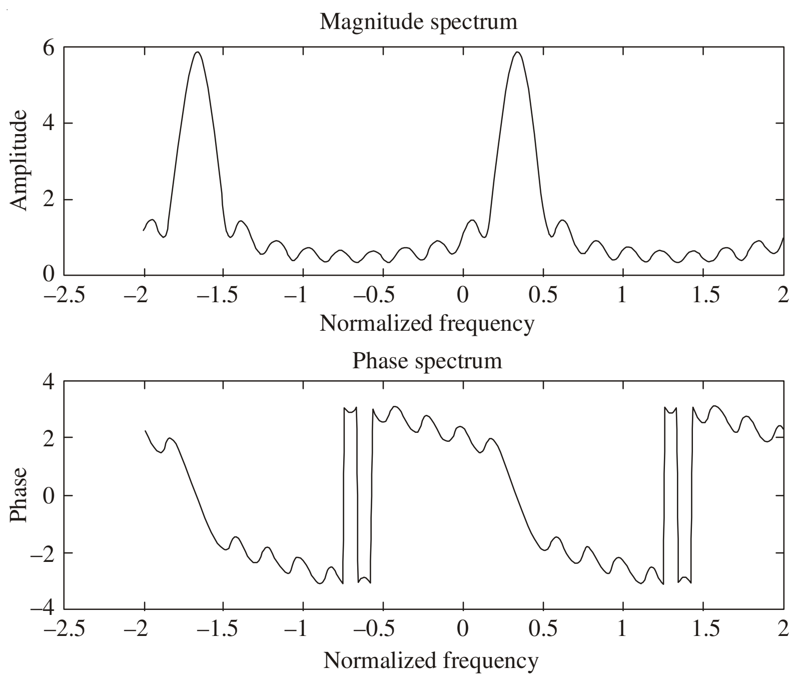


Program 5.9

% Periodicity property of DTFT

```
clc; clear all; close all;
n=1:10;
x=(0.9*exp(i*pi/3)).^n;
k=–200:200;
w=(pi/100)*k;
x1=x*exp(–i*pi/100).^(n'*k);
subplot(2,1,1); plot(w/pi,abs(x1));
xlabel('Normalized frequency')
ylabel('Amplitude')
title('magnitude spectrum ')
subplot(2,1,2); plot(w/pi,angle(x1));
xlabel('Normalized frequency')
ylabel('phase')
title('phase spectrum ')
```

Output:



6

Discrete Fourier Series (DFS) and Discrete Fourier Transform (DFT)

6.1 INTRODUCTION

Any periodic function can be expressed in a Fourier series representation. The discrete-time Fourier transform (DTFT) $X(\omega)$ of a discrete-time sequence $x(n)$ is a periodic continuous function of ω with a period of 2π. So it cannot be processed by a digital system. For processing by a digital system it should be converted into discrete form. The DFT converts the continuous function of ω to a discrete function of ω. Thus, DFT allows us to perform frequency analysis on a digital computer.

The DFT of a discrete-time signal $x(n)$ is a finite duration discrete frequency sequence. The DFT sequence is denoted by $X(k)$. The DFT is obtained by sampling one period of the Fourier transform $X(\omega)$ of the signal $x(n)$ at a finite number of frequency points. This sampling is conventionally performed at N equally spaced points in the period $0 \le \omega \le 2\pi$ or at $\omega_k = 2\pi k/N$; $0 \le k \le N - 1$. We can say that DFT is used for transforming discrete-time sequence $x(n)$ of finite length into discrete frequency sequence $X(k)$ of finite length.

The DFT is important for two reasons. First it allows us to determine the frequency content of a signal, that is to perform spectral analysis. The second application of the DFT is to perform filtering operation in the frequency domain.

Let $x(n)$ be a discrete-time sequence with Fourier transform $X(\omega)$, then the DFT of $x(n)$ denoted by $X(k)$ is defined as:

$$X(k) = X(\omega)\big|_{\omega = (2\pi k/N)}; \quad \text{for } k = 0, 1, 2, \ldots, N-1$$

The DFT of $x(n)$ is a sequence consisting of N samples of $X(k)$. The DFT sequence starts at $k = 0$, corresponding to $\omega = 0$, but does not include $k = N$ corresponding to $\omega = 2\pi$ (since the sample at $\omega = 0$ is same as the sample at $\omega = 2\pi$). Generally, the DFT is defined along

with number of samples and is called N-point DFT. The number of samples N for a finite duration sequence $x(n)$ of length L should be such that $N \geq L$.

The DTFT is nothing but the Z-transform evaluated along the unit circle centred at the origin of the z-plane. The DFT is nothing but the Z-transform evaluated at a finite number of equally spaced points on the unit circle centred at the origin of the z-plane.

To calculate the DFT of a sequence, it is not necessary to compute its Fourier transform, since the DFT can be directly computed.

DFT The N-point DFT of a finite duration sequence $x(n)$ of length L, where $N \geq L$ is defined as:

$$\text{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} = \sum_{n=0}^{N-1} x(n) W_N^{nk}; \quad \text{for } k = 0, 1, 2, \ldots, N-1$$

IDFT The Inverse Discrete Fourier transform (IDFT) of the sequence $X(k)$ of length N is defined as:

$$\text{IDFT}\{X(k)\} = x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi nk/N} = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-nk}; \quad \text{for } n = 0, 1, 2, \ldots, N-1$$

where $W_N = e^{-j(2\pi/N)}$ is called the twiddle factor.

The N-point DFT pair $x(n)$ and $X(k)$ is denoted as:

$$x(n) \xleftrightarrow[N]{\text{DFT}} X(k)$$

In this chapter, we discuss about discrete-time Fourier series and discrete Fourier transform.

6.2 DISCRETE FOURIER SERIES

The Fourier series representation of a periodic discrete-time sequence is called discrete Fourier series (DFS). Consider a discrete-time signal $x(n)$, that is periodic with period N defined by $x(n) = x(n + kN)$ for any integer value of k. The periodic function $x(n)$ can be synthesized as the sum of sine and cosine sequences (Trigonometric form of Fourier series) or equivalently as a linear combination of complex exponentials (Exponential form of Fourier series) whose frequencies are multiples of the fundamental frequency $\omega_0 = 2\pi/N$. This is done by constructing a periodic sequence for which each period is identical to the finite length sequence.

6.2.1 Exponential Form of Discrete Fourier Series

A real periodic discrete-time signal $x(n)$ of period N can be expressed as a weighted sum of complex exponential sequences. As the sinusoidal sequences are unique only for digital frequencies from 0 to 2π, the expansion has only a finite number of complex exponentials.

The exponential form of the Fourier series for a periodic discrete-time signal is given by

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j(2\pi/N)nk} \quad \text{for all } n$$

where the coefficients $X(k)$ are expressed as:

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j(2\pi/N)nk} \quad \text{for all } k$$

These equations for $x(n)$ and $X(k)$ are called DFS synthesis and analysis pair. Hence, $X(k)$ and $x(n)$ are periodic sequences.

The equivalent form for $X(k)$ is:

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk}$$

where W_N is defined as $W_N = e^{-j(2\pi/N)}$.

6.2.2 Trigonometric Form of Discrete Fourier Series

The trigonometric Fourier series representation of a continuous-time periodic signal $x(t)$ is given by

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

where the constants a_0, a_n and b_n can be determined as:

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\omega_0 t\,dt, \quad n = 1, 2, \ldots$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin n\omega_0 t\,dt, \quad n = 1, 2, \ldots$$

An alternative form of the discrete Fourier series comparable to the above equation for $x(t)$ for the continuous-time periodic signal can be easily found. However, it has different expressions for odd and even N.

For even N,

$$x(n) = A(0) + \sum_{k=1}^{(N/2)-1} A(k)\cos\left(k\frac{2\pi}{N}n\right) + \sum_{k=1}^{(N/2)-1} B(k)\sin\left(k\frac{2\pi}{N}n\right) + A\left(\frac{N}{2}\right)\cos \pi n$$

Here the last term contains $\cos \pi n$, which is equal to $(-1)^n$, the highest frequency sequence possible.

The constants $A(0)$, $A(k)$ and $B(k)$ will be as follows:

$$A(0) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)$$

$$A(k) = \frac{2}{N}\sum_{n=0}^{N-1} x(n)\cos\left(k\frac{2\pi}{N}n\right), \quad k = 1, 2, \ldots, \frac{N}{2} - 1$$

$$B(k) = \frac{2}{N}\sum_{n=0}^{N-1} x(n)\sin\left(k\frac{2\pi}{N}n\right), \quad k = 1, 2, \ldots, \frac{N}{2} - 1$$

$$A\left(\frac{N}{2}\right) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\cos \pi n$$

If N is odd, $A(0)$ remains the same as given above; however, $A(k)$ and $B(k)$ given above are good for k = 0, 1, 2,…, $(N – 1)/2$, and there will be no $A(N/2)$ coefficient.

So for odd N, we have

$$x(n) = A(0) + \sum_{k=1}^{(N-1)/2} A(k)\cos\left(k\frac{2\pi}{N}n\right) + \sum_{k=1}^{(N-1)/2} B(k)\sin\left(k\frac{2\pi}{N}n\right)$$

The constants $A(0)$, $A(k)$ and $B(k)$ will be

$$A(0) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)$$

$$A(k) = \frac{2}{N}\sum_{n=0}^{N-1} x(n)\cos\left(k\frac{2\pi}{N}n\right), \quad k = 1, 2, \ldots, \frac{N-1}{2}$$

$$B(k) = \frac{2}{N}\sum_{n=0}^{N-1} x(n)\sin\left(k\frac{2\pi}{N}n\right), \quad k = 1, 2, \ldots, \frac{N-1}{2}$$

6.2.3 Relationships between the Exponential and Trigonometric Forms of Discrete Fourier Series

For even N, the relationships between $X(k)$ of the exponential form and $A(k)$ and $B(k)$ of the trigonometric form for a real $x(n)$ are expressed by

$$A(0) = \frac{X(0)}{N}$$

$$A(k) = \frac{X(k) + X(N-k)}{N}, \qquad k = 1, 2, \ldots, \frac{N}{2} - 1$$

$$B(k) = \frac{j[X(k) - X(N-k)]}{N}, \quad k = 1, 2, \ldots, \frac{N}{2} - 1$$

Again if N is odd, $A(0)$ remains the same as given below and $A(k)$ and $B(k)$ will be for k = 0, 1, 2,…, $(N – 1)/2$ and there will be no $A(N/2)$.

So for odd N, the relationships between $A(k)$ and $B(k)$ of the trigonometric form and $X(k)$ of the exponential form for a real $x(n)$ are expressed by

$$A(0) = \frac{X(0)}{N}$$

$$A(k) = \frac{X(k) + X(N-k)}{N}, \qquad k = 1, 2, \ldots, \frac{N-1}{2}$$

$$B(k) = \frac{j[X(k) - X(N-k)]}{N}, \quad k = 1, 2, \ldots, \frac{N-1}{2}$$

EXAMPLE 6.1 Find both the exponential and trigonometric forms of the DFS representation of $x(n)$ shown in Figure 6.1.

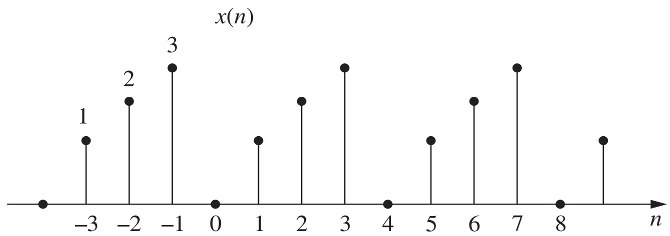


Figure 6.1 $x(n)$ for Example 6.1.

Solution: To determine the exponential form of the DFS, we have

$$W_N^k = e^{-j(2\pi/N)k}$$

Given $N = 4$

$\therefore$
$$W_4^0 = 1, W_4^1 = e^{-j(2\pi/4)1} = e^{-j(\pi/2)}$$

$$W_4^1 = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

$$W_4^2 = (W_4^1)(W_4^1) = (-j)(-j) = -1$$

$$W_4^3 = (W_4^2)(W_4^1) = (-1)(-j) = j$$

$$W_4^4 = (W_4^2)(W_4^2) = (-1)(-1) = 1$$

The exponential form of DFS is given by

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j(2\pi/N)nk} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-nk} \quad \text{for all } n$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j(\omega_0)nk}$$

where
$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk} = \sum_{n=0}^{N-1} x(n)e^{-j(2\pi/N)nk} \quad \text{for all } k$$

For $k = 0$,
$$X(0) = \sum_{n=0}^{3} x(n)W_4^{(0)n} = x(0) + x(1) + x(2) + x(3) = 0 + 1 + 2 + 3 = 6$$

For $k = 1$, $\quad X(1) = \sum_{n=0}^{3} x(n)W_4^{(1)n} = x(0)W_4^0 + x(1)W_4^1 + x(2)W_4^2 + x(3)W_4^3$

$$= 0(1) + (1)(-j) + (2)(-1) + (3)(j) = -2 + j2$$

For $k = 2$, $\quad X(2) = \sum_{n=0}^{3} x(n)W_4^{(2)n} = x(0)W_4^0 + x(1)W_4^2 + x(2)W_4^4 + x(3)W_4^6$

$$= 0 + 1(-1) + 2(1) + 3(-1) = -2$$

For $k = 3$, $\quad X(3) = \sum_{n=0}^{3} x(n)W_4^{(3)n} = x(0)W_4^0 + x(1)W_4^3 + x(2)W_4^6 + x(3)W_4^9$

$$= 0 + 1(j) + 2(-1) + 3(-j) = -2 - j2$$

The complex exponential form of the Fourier series is:

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_4^{-nk}$$

$$x(n) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, W_4^{-nk}$$

$$= \frac{1}{4}\left[X(0) + X(1)\, W_4^{-n} + X(2)W_4^{-2n} + X(3)\, W_4^{-3n}\right]$$

$$= \frac{1}{4}\left[6 + (-2 + j2)e^{j(2\pi/4)n} + (-2)e^{j(2\pi/4)2n} + (-2 - j2)e^{j(2\pi/4)3n}\right]$$

$$= \frac{1}{2}\left[3 + (-1 + j1)e^{(j\pi/2)n} - e^{j\pi n} - (1 + j)e^{j(3\pi/2)n}\right]$$

To determine the trigonometric form of DFS, the $A(0)$, $A(1)$, $B(1)$, and $A(2)$ are determined as:

$$A(0) = \frac{X(0)}{N} = \frac{6}{4} = \frac{3}{2}$$

$$A(1) = \frac{X(1) + X(4-1)}{4} = \frac{(-2 + j2) + (-2 - j2)}{4} = -1$$

$$B(1) = j\frac{X(1) - X(4-1)}{4} = j\frac{(-2 + j2) - (-2 - j2)}{4} = -1$$

$$A\left(\frac{4}{2}\right) = \frac{X(4/2)}{4} = -\frac{1}{2}$$

Therefore, the trigonometric form of the DFS is determined as:

$$x(n) = \frac{3}{2} - \cos\left(\frac{\pi}{2}n\right) - \sin\left(\frac{\pi}{2}n\right) - \frac{1}{2}\cos \pi n$$

6.3 PROPERTIES OF DFS

6.3.1 Linearity

Consider two periodic sequences $x_1(n)$, $x_2(n)$ both with period N, such that

$$\text{DFS}\,[x_1(n)] = X_1(k)$$

and

$$\text{DFS}\,[x_2(n)] = X_2(k)$$

Then,

$$\text{DFS}[ax_1(n) + bx_2(n)] = aX_1(k) + bX_2(k)$$

6.3.2 Time Shifting

If $x(n)$ is a periodic sequence with N samples and

$$\text{DFS}\,[x(n)] = X(k)$$

Then

$$\text{DFS}\,[x(n-m)] = e^{-j(2\pi/N)mk}\,X(k)$$

where $x(n - m)$ is a shifted version of $x(n)$.

6.3.3 Symmetry Property

We know that

$$\text{DFS}\,[x^*(n)] = X^*(-k) \quad \text{and} \quad \text{DFS}\,[x^*(-n)] = X^*(k)$$

Therefore,

$$\text{DFS}\,\{\text{Re}[x(n)]\} = \text{DFS}\left[\frac{x(n) + x^*(n)}{2}\right]$$

$$= \frac{1}{2}\left[X(k) + X^*(-k)\right] = X_e(k)$$

and

$$\text{DFS}\{j\,\text{Im}[x(n)]\} = \text{DFS}\left[\frac{x(n) - x^*(n)}{2}\right]$$

$$= \frac{1}{2}[X(k) - X^*(-k)] = X_o(k)$$

We know that $x(n) = x_e(n) + x_o(n)$

where

$$x_e(n) = \frac{1}{2}[x(n) + x^*(-n)]$$

and

$$x_o(n) = \frac{1}{2}[x(n) - x^*(-n)]$$

Then

$$\text{DFS}[x_e(n)] = \text{DFS}\left\{\frac{1}{2}[x(n) + x^*(-n)]\right\}$$

$$= \frac{1}{2}[X(k) + X^*(k)] = \text{Re}\,X(k)$$

and $$\text{DFS}\,[x_o(n)] = \text{DFS}\left\{\frac{1}{2}[x(n) - x^*(-n)]\right\}$$

$$= \frac{1}{2}[X(k) - X^*(k)] = j\,\text{Im}\,\{X(k)\}$$

6.3.4 Periodic Convolution

Let $x_1(n)$ and $x_2(n)$ be two periodic sequences with period N with

$$\text{DFS}\,[x_1(n)] = X_1(k)$$

and $$\text{DFS}\,[x_2(n)] = X_2(k)$$

If $X_3(k) = X_1(k)\,X_2(k)$, then the periodic sequence $x_3(n)$ with Fourier series coefficients $X_3(k)$ is:

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m)\,x_2(n-m)$$

Hence $$\text{DFS}\,[x_1(m)\,x_2(n-m)] = X_1(k)\,X_2(k)$$

The properties of discrete Fourier series are given in Table 6.1.

6.4 RELATION BETWEEN DFT AND Z-TRANSFORM

The Z-transform of N-point sequence $x(n)$ is given by

$$X(z) = \sum_{n=0}^{N-1} x(n)\,z^{-n}$$

Let us evaluate $X(z)$ at N equally spaced points on the unit circle, i.e., at $z = e^{j(2\pi/N)k}$.

$$\left|e^{j(2\pi/N)k}\right| = 1 \text{ and } \underline{\left|e^{j(2\pi/N)k}\right.} = \frac{2\pi}{N}k$$

Hence, when k is varied from 0 to $N - 1$, we get N equally spaced points on the unit circle in the z-plane.

$\therefore$ $$X(z)\Big|_{z=e^{j(2\pi/N)k}} = \sum_{n=0}^{N-1} x(n)\,z^{-n}\Big|_{z=e^{j(2\pi/N)k}} = \sum_{n=0}^{N-1} x(n)\,e^{-j(2\pi/N)kn}$$

By the definition of N-point DFT, we get

$$X(k) = \sum_{n=0}^{N-1} x(n)\,e^{-j(2\pi/N)kn}$$

From the above equations, we get

$$X(k) = X(z)\Big|_{z=e^{j(2\pi/N)k}}$$

Now, we can conclude that the N-point DFT of a finite duration sequence can be obtained from the Z-transform of the sequence at N equally spaced points around the unit circle.

TABLE 6.1 Properties of DFS

Property	*Periodic sequence (period N)*	*DFS coefficients*
Signal	$x(n)$	$X(k)$
Signal	$x_1(n),\ x_2(n)$	$X_1(k),\ X_2(k)$
Linearity	$ax_1(n) + bx_2(n)$	$aX_1(k) + bX_2(k)$
Time shifting	$x(n - m)$	$W_N^{km} X(k)$
	$x(n + m)$	$W_N^{-km} X(k)$
Frequency shifting	$W_N^{ln} x(n)$	$X(k + l)$
Periodic convolution	$\sum_{m=0}^{N-1} x_1(m)\, x_2(n-m)$	$X_1(k)\ X_2(k)$
Multiplication	$x_1(n)\ x_2(n)$	$\frac{1}{N}\sum_{l=0}^{N-1} X_1(l)\, X_2(k-l)$
Symmetry property	$x^*(n)$	$X^*(-k)$
	$x^*(-n)$	$X^*(k)$
	Re $[x(n)]$	$X_e(k) = \frac{1}{2}[X(k) + X^*(-k)]$
	j Im $[x(n)]$	$X_o(k) = \frac{1}{2}[X(k) - X^*(-k)]$
	$x_e(n) = \frac{1}{2}[x(n) + x^*(-n)]$	Re $[X(k)]$
	$x_o(n) = \frac{1}{2}[x(n) - x^*(-n)]$	j Im $[X(k)]$
	If $x(n)$ is real	$X(k) = X^*(-k)$
		Re $[X(k)]$ = Re $[X(-k)]$
		Im $[X(k)]$ = –Im $[X(-k)]$
		$\lvert X(k)\rvert = \lvert X^*(-k)\rvert$
		$\angle X(k) = \angle X(-k)$

EXAMPLE 6.2 Compute the DFT of each of the following finite length sequences considered to be of length N:

(a) $x(n) = \delta(n)$

(b) $x(n) = \delta(n - n_0)$, where $0 < n_0 < N$

(c) $x(n) = a^n$, $0 \le n \le N - 1$

(d) $x(n) = \begin{cases} 1 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$

Solution:

(a) Given $x(n) = \delta(n)$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)nk}$$

$$= \sum_{n=0}^{N-1} \delta(n) e^{-j(2\pi/N)nk} = 1$$

i.e. $X(k) = 1$ for $0 \le k \le N-1$.

(b) Given $x(n) = \delta(n - n_0)$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)nk} = \sum_{n=0}^{N-1} \delta(n-n_0) e^{-j(2\pi/N)nk}$$

$$= e^{-j(2\pi/N)n_0 k} \quad \text{for } 0 \le k \le N-1$$

(c) Given $x(n) = a^n$

$$X(k) = \sum_{n=0}^{N-1} a^n e^{-j(2\pi/N)nk} = \sum_{n=0}^{N-1} [ae^{-j(2\pi/N)k}]^n \quad \text{for } 0 \le k \le N-1$$

$$= \frac{1 - a^N e^{-j2\pi k}}{1 - ae^{-j(2\pi/N)k}}$$

(d) Given $x(n) = \begin{cases} 1 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)nk}$$

$$= \sum_{n=0}^{(N/2)-1} x(2n) e^{-j(2\pi/N)2nk} + \sum_{n=0}^{(N/2)-1} x(2n+1) e^{-j(2\pi/N)(2n+1)k}$$

$$= \sum_{n=0}^{(N/2)-1} x(2n) e^{-j(4\pi/N)nk} = \sum_{n=0}^{(N/2)-1} e^{-j4\pi kn/N}$$

$$= \frac{1 - e^{-j2\pi k}}{1 - e^{-j4\pi k/N}}$$

EXAMPLE 6.3 Find the Z-transform of the sequence $x(n) = u(n) - u(n-6)$ and sample it at 4 points on the unit circle using the relation:

$$X(k) = X(z)\Big|_{z=e^{j(2\pi/N)k}}, \quad k = 0, 1, 2, 3$$

Find the inverse DFT of $X(k)$ and compare it with $x(n)$ and comment.

Solution: Given $x(n) = u(n) - u(n-6) = \{1, 1, 1, 1, 1, 1\}$

$$= \delta(n) + \delta(n-1) + \delta(n-2) + \delta(n-3) + \delta(n-4) + \delta(n-5)$$

$$\therefore \quad X(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

$$X(k) = X(z)\Big|_{z=e^{j(2\pi/N)k}} = X(z)\Big|_{z=e^{j(2\pi/4)k}}, \quad k = 0, 1, 2, 3$$

$$= 1 + e^{-j(\pi/2)k} + e^{-j\pi k} + e^{-j(3\pi/2)k} + e^{-j2\pi k} + e^{-j(5\pi/2)k}$$

$$= 2 + 2e^{-j(\pi/2)k} + e^{-j\pi k} + e^{-j(3\pi/2)k}$$

The IDFT of $X(k)$ is:

$$x'(n) = \{2, 2, 1, 1\}$$

Comparing $x'(n)$ and $x(n)$, we can find that time domain aliasing occurs in the first two points because $X(z)$ is not sampled with sufficient number of points on the unit circle.

Note: The length of the sequence $N = 6$, and the number of samples taken on the unit circle to find $X(k)$ is 4. Since $N < L$, time domain aliasing occurs. That is last 2 samples $(L - N = 2)$ are added to first two samples due to under sampling.

EXAMPLE 6.4 (a) Find the 4-point DFT of $x(n) = \{1, -1, 2, -2\}$ directly.
(b) Find the IDFT of $X(k) = \{4, 2, 0, 4\}$ directly.

Solution:

(a) Given sequence is $x(n) = \{1, -1, 2, -2\}$. Here the DFT $X(k)$ to be found is $N =$ 4-point and length of the sequence $L = 4$. So no padding of zeros is required.

We know that the DFT $\{x(n)\}$ is given by

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} = \sum_{n=0}^{N-1} x(n)\, e^{-j(2\pi/N)nk} = \sum_{n=0}^{3} x(n) e^{-j(\pi/2)nk}, \quad k = 0, 1, 2, 3$$

$$\therefore \quad X(0) = \sum_{n=0}^{3} x(n)\, e^0 = x(0) + x(1) + x(2) + x(3) = 1 - 1 + 2 - 2 = 0$$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j(\pi/2)n} = x(0) + x(1) e^{-(j\pi/2)} + x(2) e^{-j\pi} + x(3) e^{-j(3\pi/2)}$$

$$= 1 + (-1)(0 - j) + 2(-1 - j0) - 2(0 + j)$$

$$= -1 - j$$

$$X(2)=\sum_{n=0}^{3}x(n)e^{-j\pi n}=x(0)+x(1)e^{-j\pi}+x(2)e^{-j2\pi}+x(3)e^{-j3\pi}$$

$$=1-1(-1-j0)+2(1-j0)-2(-1-j0)=6$$

$$X(3)=\sum_{n=0}^{3}x(n)e^{-j(3\pi/2)n}=x(0)+x(1)e^{-j(3\pi/2)}+x(2)e^{-j3\pi}+x(3)e^{-j(9\pi/2)}$$

$$=1-1(0+j)+2(-1-j0)-2(0-j)=-1+j$$

$\therefore\quad X(k) = \{0, -1-j, 6, -1+j\}$

(b) Given DFT is $X(k) = \{4, 2, 0, 4\}$. The IDFT of $X(k)$, i.e. $x(n)$ is given by

$$x(n)=\frac{1}{N}\sum_{k=0}^{N-1}X(k)W_N^{-nk} = \frac{1}{N}\sum_{k=0}^{N-1}X(k)e^{j(2\pi/N)nk}$$

i.e. $$x(n)=\frac{1}{4}\sum_{k=0}^{3}X(k)e^{j(\pi/2)nk}$$

$\therefore$ $$x(0)=\frac{1}{4}\sum_{k=0}^{3}X(k)e^{0}=\frac{1}{4}[X(0)+X(1)+X(2)+X(3)]$$

$$=\frac{1}{4}[4+2+0+4]=2.5$$

$$x(1)=\frac{1}{4}\sum_{k=0}^{3}X(k)e^{j(\pi/2)k}=\frac{1}{4}\left[X(0)+X(1)e^{j(\pi/2)}+X(2)e^{j\pi}+X(3)e^{j(3\pi/2)}\right]$$

$$=\frac{1}{4}[4+2(0+j)+0+4(0-j)]=1-j0.5$$

$$x(2)=\frac{1}{4}\sum_{k=0}^{3}X(k)e^{j\pi k}=\frac{1}{4}\left[X(0)+X(1)e^{j\pi}+X(2)e^{j2\pi}+X(3)e^{j3\pi}\right]$$

$$=\frac{1}{4}[4+2(-1+j0)+0+4(-1+j0)]=-0.5$$

$$x(3)=\frac{1}{4}\sum_{k=0}^{3}X(k)e^{j(3\pi/2)k}=\frac{1}{4}\left[X(0)+X(1)e^{j(3\pi/2)}+X(2)e^{j3\pi}+X(3)e^{j(9\pi/2)}\right]$$

$$=\frac{1}{4}[4+2(0-j)+0+4(0+j)]=1+j0.5$$

$\therefore\quad x_3(n) = \{2.5, 1-j0.5, -0.5, 1+j0.5\}$

EXAMPLE 6.5 (a) Find the 4-point DFT of $x(n) = \{1, -2, 3, 2\}$.
(b) Find the IDFT of $X(k) = \{1, 0, 1, 0\}$.

Solution:

(a) Given $x(n) = \{1, -2, 3, 2\}$.

Here $N = 4$, $L = 4$. The DFT of $x(n)$ is $X(k)$.

$$\therefore \quad X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} = \sum_{n=0}^{3} x(n) e^{-j(2\pi/4)nk} = \sum_{n=0}^{3} x(n) e^{-j(\pi/2)nk}, \quad k = 0, 1, 2, 3$$

$$X(0) = \sum_{n=0}^{3} x(n)e^0 = x(0) + x(1) + x(2) + x(3) = 1 - 2 + 3 + 2 = 4$$

$$X(1) = \sum_{n=0}^{3} x(n)e^{-j(\pi/2)n} = x(0) + x(1)e^{-j(\pi/2)} + x(2)e^{-j\pi} + x(3)e^{-j(3\pi/2)}$$

$$= 1 - 2(0 - j) + 3(-1 - j0) + 2(0 + j) = -2 + j4$$

$$X(2) = \sum_{n=0}^{3} x(n)e^{-j\pi n} = x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$$

$$= 1 - 2(-1 - j0) + 3(1 - j0) + 2(-1 - j0) = 4$$

$$X(3) = \sum_{n=0}^{3} x(n)e^{-j(3\pi/2)n} = x(0) + x(1)e^{-j(3\pi/2)} + x(2)e^{-j3\pi} + x(3)e^{-j(9\pi/2)}$$

$$= 1 - 2(0 + j) + 3(-1 - j0) + 2(0 - j) = -2 - j4$$

$$\therefore \quad X(k) = \{4, -2 + j4, 4, -2 - j4\}$$

(b) Given $X(k) = \{1, 0, 1, 0\}$

Let the IDFT of $X(k)$ be $x(n)$.

$$\therefore \quad x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-nk} = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j(2\pi/N)nk}$$

$$x(0) = \frac{1}{4} \sum_{k=0}^{3} X(k)e^0 = \frac{1}{4}\left[X(0) + X(1) + X(2) + X(3)\right] = \frac{1}{4}\left[1 + 0 + 1 + 0\right] = 0.5$$

$$x(1) = \frac{1}{4} \sum_{k=0}^{3} X(k)e^{j(\pi/2)k} = \frac{1}{4}\left[X(0) + X(1)e^{j(\pi/2)} + X(2)e^{j\pi} + X(3)e^{j(3\pi/2)}\right]$$

$$= \frac{1}{4}[1 + 0 + e^{j\pi} + 0] = \frac{1}{4}[1 + 0 - 1 + 0] = 0$$

$$x(2) = \frac{1}{4} \sum_{k=0}^{3} X(k)e^{j\pi k} = \frac{1}{4}\left[X(0) + X(1)e^{j\pi} + X(2)e^{j2\pi} + X(3)e^{j3\pi}\right]$$

$$= \frac{1}{4}[1 + 0 + e^{j2\pi} + 0] = \frac{1}{4}[1 + 0 + 1 + 0] = 0.5$$

$$x(3) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j(3\pi/2)k} = \frac{1}{4}\left[X(0) + X(1)e^{j(3\pi/2)} + X(2)e^{j3\pi} + X(3)e^{j(9\pi/2)}\right]$$

$$= \frac{1}{4}[1 + 0 + e^{j3\pi} + 0] = \frac{1}{4}[1 + 0 - 1 + 0] = 0$$

$\therefore$ The IDFT of $X(k) = \{1, 0, 1, 0\}$ is $x(n) = \{0.5, 0, 0.5, 0\}$.

EXAMPLE 6.6 Compute the DFT of the 3-point sequence $x(n) = \{2, 1, 2\}$. Using the same sequence, compute the 6-point DFT and compare the two DFTs.

Solution: The given 3-point sequence is $x(n) = \{2, 1, 2\}$, $N = 3$.

$$\text{DFT}\, x(n) = X(k) = \sum_{n=0}^{N-1} x(n)W_N^{nk} = \sum_{n=0}^{2} x(n)e^{-j(2\pi/3)nk}, \quad k = 0, 1, 2$$

$$= x(0) + x(1)e^{-j(2\pi/3)k} + x(2)e^{-j(4\pi/3)k}$$

$$= 2 + \left(\cos\frac{2\pi}{3}k - j\sin\frac{2\pi}{3}k\right) + 2\left(\cos\frac{4\pi}{3}k - j\sin\frac{4\pi}{3}k\right)$$

When $k = 0$, $\quad X(k) = X(0) = 2 + 1 + 2 = 5$

When $k = 1$, $\quad X(k) = X(1) = 2 + \left(\cos\frac{2\pi}{3} - j\sin\frac{2\pi}{3}\right) + 2\left(\cos\frac{4\pi}{3} - j\sin\frac{4\pi}{3}\right)$

$$= 2 + (-0.5 - j0.866) + 2(-0.5 + j0.866)$$

$$= 0.5 + j0.866$$

When $k = 2$, $\quad X(k) = X(2) = 2 + \left(\cos\frac{4\pi}{3} - j\sin\frac{4\pi}{3}\right) + 2\left(\cos\frac{8\pi}{3} - j\sin\frac{8\pi}{3}\right)$

$$= 2 + (-0.5 + j0.866) + 2(-0.5 - j0.866)$$

$$= 0.5 - j0.866$$

$\therefore$ 3-point DFT of $x(n) = X(k) = \{5, 0.5 + j0.866, 0.5 - j0.866\}$

To compute the 6-point DFT, convert the 3-point sequence $x(n)$ into 6-point sequence by padding with zeros.

$$x(n) = \{2, 1, 2, 0, 0, 0\}, \quad N = 6$$

$$\text{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)W_N^{nk} = \sum_{n=0}^{5} x(n)e^{-j(2\pi/N)nk}, \quad k = 0, 1, 2, 3, 4, 5$$

$$= x(0) + x(1)e^{-j(2\pi/6)k} + x(2)e^{-j(4\pi/6)k} + x(3)e^{-j(6\pi/6)k} + x(4)e^{-j(8\pi/6)k}$$
$$+ x(5)e^{-j(10\pi/6)k}$$

$$= 2 + e^{-j(\pi/3)k} + 2e^{-j(2\pi/3)k}$$

When $k = 0$, $\quad X(0) = 2 + 1 + 2 = 5$

When $k = 1$, $X(1) = 2 + e^{-j(\pi/3)} + 2e^{-j(2\pi/3)}$

$$= 2 + (0.5 - j0.866) + 2(-0.5 - j0.866) = 1.5 - j2.598$$

When $k = 2$, $X(2) = 2 + e^{-j(2\pi/3)} + 2e^{-j(4\pi/3)}$

$$= 2 + (-0.5 - j0.866) + 2(-0.5 + j0.866) = 0.5 + j0.866$$

When $k = 3$, $X(3) = x(0) + x(1)e^{-j(3\pi/3)} + x(2)e^{-j(6\pi/3)}$

$$= 2 + (\cos\pi - j\sin\pi) + 2(\cos 2\pi - j\sin 2\pi)$$
$$= 2 - 1 + 2 = 3$$

When $k = 4$, $X(4) = x(0) + x(1)e^{-j(4\pi/3)} + x(2)e^{-j(8\pi/3)}$

$$= 2 + \left(\cos\frac{4\pi}{3} - j\sin\frac{4\pi}{3}\right) + 2\left(\cos\frac{8\pi}{3} - j\sin\frac{8\pi}{3}\right)$$
$$= 2 + (-0.5 + j0.866) + 2(-0.5 - j0.866)$$
$$= 0.5 - j0.866$$

When $k = 5$, $X(5) = x(0) + x(1)e^{-j(5\pi/3)} + x(2)e^{-j(10\pi/3)}$

$$= 2 + \left(\cos\frac{5\pi}{3} - j\sin\frac{5\pi}{3}\right) + 2\left(\cos\frac{10\pi}{3} - j\sin\frac{10\pi}{3}\right)$$
$$= 2 + (0.5 - j0.866) + 2(-0.5 + j0.866) = 1.5 + j0.866$$

Tabulating the above 3-point and 6-point DFTs, we have

DFT	$X(0)$	$X(1)$	$X(2)$	$X(3)$	$X(4)$	$X(5)$
3-point	5	$0.5 + j0.866$	$0.5 - j0.866$	–	–	–
6-point	5	$1.5 - j2.598$	$0.5 + j0.866$	3	$0.5 - j0.866$	$1.5 + j0.866$

Since 6-point = 3 × 2-point
$X(k)$ of 3-point sequence = $X(2k)$ of 6-point sequence

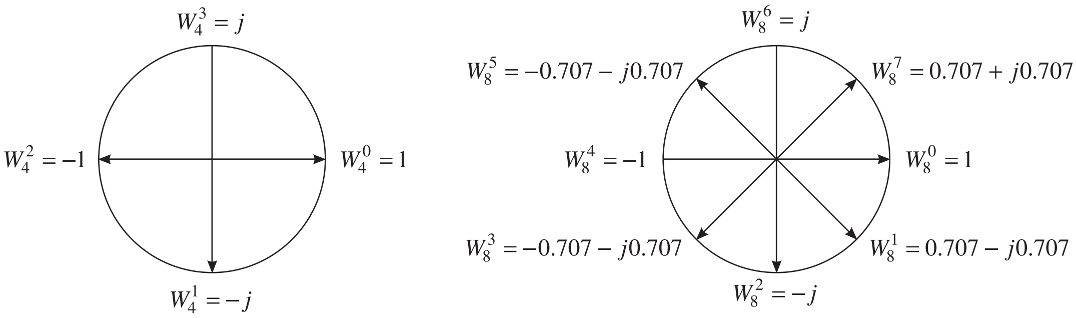
6.5 COMPARISON BETWEEN DTFT AND DFT

- DFT is a sampled version of DTFT, where the frequency term ω is sampled. But, we know that DTFT is obtained by using the sampled form of input signal $x(t)$. So, we find that DFT is obtained by the double sampling of $x(t)$.
- DFT gives only positive frequency values, whereas DTFT can give both positive and negative frequency values.
- DTFT and DFT coincide at intervals of $\omega = 2\pi k/N$, where $k = 0, 1, \ldots, N - 1$.
- To get more accurate values of DFT, number of samples N must be very high but when N is very high, the required computation time will also be very high.

6.6 A SLIGHTLY FASTER METHOD FOR COMPUTING DFT VALUES

The computation procedure given above is very lengthy and cumbersome. We can improve the situation somewhat by using the powers of twiddle factor (W_N) instead of the factors of $e^{-j(2\pi/N)}$. This procedure may be thought of as the forerunner of the method, using the Fast Fourier Transformation (FFT) algorithm (in Chapter 7).

As stated above, we use the powers of W_N in the multiplications instead of $e^{-j(2\pi/N)}$ and its factors. Figure 6.2 and Table 6.2 show the powers of W for various DFTs.



(a) 4-point DFT (b) 8-point DFT

Figure 6.2 Values of W_N^n for 4-point and 8-point DFTs

TABLE 6.2 Table showing powers of the factor W_N for 4-point and 8-point DFTs

Twiddle factor	*4-point DFT*	*Twiddle factor*	*8-point DFT*
W_4^0	1	W_8^0	1
W_4^1	$-j$	W_8^1	$0.707 - j0.707$
W_4^2	-1	W_8^2	$-j$
W_4^3	j	W_8^3	$-0.707 - j0.707$
$W_4^4 = W_4^0$	1	W_8^4	-1
$W_4^5 = W_4^1$	$-j$	W_8^5	$-0.707 - j0.707$
$W_4^6 = W_4^2$	-1	W_8^6	j
$W_4^7 = W_4^3$	j	W_8^7	$0.707 + j0.707$

EXAMPLE 6.7 Obtain DFT of the sequence $x(n) = \{x(0), x(1), x(2), x(3)\} = \{1, 0, -1, 2\}$.

Solution: Given $x(n) = \{x(0), x(1), x(2), x(3)\} = \{1, 0, -1, 2\}$

Here $N = 4$

$$\therefore \qquad X(k) = \sum_{n=0}^{N-1} x(n)W_N^{nk} = \sum_{n=0}^{3} x(n)W_4^{nk}$$

$$= x(0)W_4^0 + x(1)W_4^k + x(2)W_4^{2k} + x(3)W_4^{3k}, \qquad k = 0, 1, 2, 3$$

$$\therefore \quad X(0) = x(0)W_4^0 + x(1)W_4^0 + x(2)W_4^0 + x(3)W_4^0 = 1 + 0 - 1 + 2 = 2$$

$$X(1) = x(0)W_4^0 + x(1)W_4^1 + x(2)W_4^2 + x(3)W_4^3 = 1 + 0 + (-1)(-1) + 2j = 2 + j2$$

$$X(2) = x(0)W_4^0 + x(1)W_4^2 + x(2)W_4^4 + x(3)W_4^6 = 1 + 0 + (-1)(1) + 2(-1) = -2$$

$$X(3) = x(0)W_4^0 + x(1)W_4^3 + x(2)W_4^6 + x(3)W_4^9 = 1 + 0 + (-1)(-1) + 2(-j) = 2 - j2$$

We get the DFT sequence as $X(k) = \{2, 2 + j2, -2, 2 - j2\}$.

Thus, we find that our new method is faster than the direct method of evaluating the DFT.

6.7 MATRIX FORMULATION OF THE DFT AND IDFT

If we let $W_N = e^{-j(2\pi/N)}$, the defining relations for the DFT and IDFT may be written as:

$$X(k) = \sum_{n=0}^{N-1} x(n)W_N^{nk}, \qquad k = 0, 1, \ldots, N-1$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-nk}, \qquad n = 0, 1, 2, \ldots, N-1$$

The first set of N DFT equations in N unknowns may be expressed in matrix form as:

$$\mathbf{X} = \mathbf{W_N x}$$

Here $\mathbf{X}$ and $\mathbf{x}$ are $N \times 1$ matrices, and $\mathbf{W_N}$ is an $N \times N$ square matrix called the DFT matrix. The full matrix form is described by

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ W_N^0 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}$$

6.8 THE IDFT FROM THE MATRIX FORM

The matrix $\mathbf{x}$ may be expressed in terms of the inverse of $\mathbf{W_N}$ as:

$$\mathbf{x} = \mathbf{W_N^{-1}}\ \mathbf{x}$$

The matrix $\mathbf{W_N^{-1}}$ is called the IDFT matrix. We may also obtain $\mathbf{x}$ directly from the IDFT relation in matrix form, where the change of index from n to k and the change in the sign of the exponent in $e^{j(2\pi/N)nk}$ lead to the conjugate transpose of $\mathbf{W_N}$. We then have

$$\mathbf{x} = \frac{1}{\mathbf{N}}\left[\mathbf{W_N^*}\right]^T \mathbf{x}$$

Comparison of the two forms suggests that $\mathbf{W_N^{-1}} = \frac{1}{N}\left[\mathbf{W_N^*}\right]^T$.

This very important result shows that $\mathbf{W_N^{-1}}$ requires only conjugation and transposition of $\mathbf{W_N}$, an obvious computational advantage.

6.9 USING THE DFT TO FIND THE IDFT

Both the DFT and IDFT are matrix operations and there is an inherent symmetry in the DFT and IDFT relations. In fact, we can obtain the IDFT by finding the DFT of the conjugate sequence and then conjugating the results and dividing by N, Mathematically,

$$x(n) = \text{IDFT}[X(k)] = \frac{1}{N}[\text{DFT}\{X^*(k)\}]^*$$

This result involves the conjugate symmetry and duality of the DFT and IDFT, and suggests that the DFT algorithm itself can also be used to find the IDFT. In practice, this is indeed what is done.

EXAMPLE 6.8 Find the DFT of the sequence

$$x(n) = \{1, 2, 1, 0\}$$

Solution: The DFT $X(k)$ of the given sequence $x(n) = \{1, 2, 1, 0\}$ may be obtained by solving the matrix product as follows. Here $N = 4$.

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & W_N^3 \\ W_N^0 & W_N^2 & W_N^4 & W_N^6 \\ W_N^0 & W_N^3 & W_N^6 & W_N^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -j2 \\ 0 \\ j2 \end{bmatrix}$$

The result is DFT $\{x(n)\} = X(k) = \{4, -j2, 0, j2\}$.

EXAMPLE 6.9 Find the DFT of $x(n) = \{1, -1, 2, -2\}$.

Solution: The DFT, $X(k)$ of the given sequence $x(n) = \{1, -1, 2, -2\}$ can be determined using matrix as shown below.

$$X(k) = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1-j \\ 6 \\ -1+j \end{bmatrix}$$

$\therefore$ DFT $\{x(n)\} = X(k) = \{0, -1-j, 6, -1+j\}$

EXAMPLE 6.10 Find the 4-point DFT of $x(n)$ = {1, –2, 3, 2}.

Solution: Given $x(n)$ = {1, –2, 3, 2}, the 4-point DFT{$x(n)$} = $X(k)$ is determined using matrix as shown below.

$$\text{DFT}\{x(n)\} = X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2+j4 \\ 4 \\ -2-j4 \end{bmatrix}$$

$\therefore \quad \text{DFT}\{x(n)\} = X(k) = \{4, -2+j4, 4, -2-j4\}$

EXAMPLE 6.11 Find the 8-point DFT of $x(n)$ = {1, 1, 0, 0, 0, 0, 0, 0}. Use the property of conjugate symmetry.

Solution: For given $x(n)$ = {1, 1, 0, 0, 0, 0, 0, 0},

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j(2\pi/N)nk} = \sum_{n=0}^{7} x(n)\, e^{-j(2\pi/8)nk}$$

$$= 1 + e^{-j(\pi/4)k}, \qquad k = 0, 1, 2, \ldots, 7$$

Since N = 8, we need to compute $X(k)$ only for $k \le (8/2) = 4$.

$$X(0) = 1 + 1 = 2$$

$$X(1) = 1 + e^{-j(\pi/4)} = 1 + 0.707 - j0.707 = 1.707 - j0.707$$

$$X(2) = 1 + e^{-j(\pi/2)} = 1 + 0 - j = 1 - j$$

$$X(3) = 1 + e^{-j(3\pi/4)} = 1 - 0.707 - j0.707 = 0.293 - j0.707$$

$$X(4) = 1 + e^{-j\pi} = 1 - 1 = 0$$

With N = 8, conjugate symmetry says $X(k) = X^*(N - k) = X^*(8 - k)$ and we find

$$X(5) = X^*(8-5) = X^*(3) = 0.293 + j0.707$$

$$X(6) = X^*(8-6) = X^*(2) = 1 + j$$

$$X(7) = X^*(8-7) = X^*(1) = 1.707 + j0.707$$

Thus, $X(k) = \{2, 1.707 - j0.707, 1 - j, 0.293 - j0.707, 0, 0.293 + j0.707, 1 + j, 1.707 + j0.707\}$

EXAMPLE 6.12 Find the IDFT of $X(k)$={4, –j2, 0, j2} using DFT.

Solution: Given $X(k)$ = {4, –j2, 0, j2} $\therefore$ $X^*(k)$ = {4, j2, 0, –j2}

The IDFT of $X(k)$ is determined using matrix as shown below.

To find IDFT of $X(k)$ first find $X^*(k)$, then find DFT of $X^*(k)$, then take conjugate of DFT {$X^*(k)$} and divide by N.

$$\text{DFT}\,\{X^*(k)\} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 4 \\ j2 \\ 0 \\ -j2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 4 \\ 0 \end{bmatrix}$$

$$\therefore \quad \text{IDFT}\,[X(k)] = x(n) = \frac{1}{4}[4, 8, 4, 0]^* = \frac{1}{4}[4, 8, 4, 0] = [1, 2, 1, 0]$$

EXAMPLE 6.13 Find the IDFT of $X(k) = \{4, 2, 0, 4\}$ using DFT.

Solution: Given $X(k) = \{4, 2, 0, 4\}$

$$\therefore \qquad X^*(k) = \{4, 2, 0, 4\}$$

The IDFT of $X(k)$ is determined using matrix as shown below.

To find IDFT of $X(k)$, first find $X^*(k)$, then find DFT of $X^*(k)$, then take conjugate of DFT $\{X^*(k)\}$ and divide by N.

$$\text{DFT}\,[X^*(k)] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 4+j2 \\ -2 \\ 4-j2 \end{bmatrix}$$

$$\therefore \quad \text{IDFT}\,\{X(k)\} = x(n) = \frac{1}{4}[10, 4+j2, -2, 4-j2]^* = \{2.5, 1-j0.5, -0.5, 1+j0.5\}$$

EXAMPLE 6.14 Find the IDFT of $X(k) = \{1, 0, 1, 0\}$.

Solution: Given $X(k) = \{1, 0, 1, 0\}$, the IDFT of $X(k)$, i.e. $x(n)$ is determined using matrix as shown below.

$$X^*(k) = \{1, 0, 1, 0\}^* = \{1, 0, 1, 0\}$$

$$\text{DFT}\,\{X^*(k)\} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\therefore \quad \text{IDFT}\,\{X(k)\} = x(n) = \frac{1}{4}[\text{DFT}\,\{X^*(k)\}]^* = \frac{1}{4}\{2, 0, 2, 0\} = \{0.5, 0, 0.5, 0\}$$

6.10 PROPERTIES OF DFT

Like the Fourier and Z-transforms, the DFT has several important properties that are used to process the finite duration sequences. Some of those properties are discussed as follows:

6.10.1 Periodicity

If a sequence $x(n)$ is periodic with periodicity of N samples, then N-point DFT of the sequence, $X(k)$ is also periodic with periodicity of N samples.

Hence, if $x(n)$ and $X(k)$ are an N-point DFT pair, then

$$x(n+N) = x(n) \qquad \text{for all } n$$

$$X(k+N) = X(k) \qquad \text{for all } k$$

Proof: By definition of DFT, the $(k + N)$th coefficient of $X(k)$ is given by

$$X(k+N) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi n(k+N)/N} = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N} e^{-j2\pi nN/N}$$

But $e^{-j2\pi n} = 1$ for all n (Here n is an integer)

$$\therefore \qquad X(k+N) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N} = X(k)$$

6.10.2 Linearity

If $x_1(n)$ and $x_2(n)$ are two finite duration sequences and if

$$\text{DFT }\{x_1(n)\} = X_1(k)$$

and

$$\text{DFT }\{x_2(n)\} = X_2(k)$$

Then for any real valued or complex valued constants a and b,

$$\text{DFT}\{ax_1(n) + bx_2(n)\} = aX_1(k) + bX_2(k)$$

Proof:

$$\text{DFT}\{ax_1(n) + bx_2(n)\} = \sum_{n=0}^{N-1} [ax_1(n) + bx_2(n)]\, e^{-j2\pi nk/N}$$

$$= a\sum_{n=0}^{N-1} x_1(n)e^{-j2\pi nk/N} + b\sum_{n=0}^{N-1} x_2(n)e^{-j2\pi nk/N}$$

$$= aX_1(k) + bX_2(k)$$

6.10.3 DFT of Even and Odd Sequences

The DFT of an even sequence is purely real, and the DFT of an odd sequence is purely imaginary. Therefore, DFT can be evaluated using cosine and sine transforms for even and odd sequences respectively.

For even sequence, $X(k) = \sum_{n=0}^{N-1} x(n) \cos\left(\dfrac{2\pi nk}{N}\right)$

For odd sequence, $X(k) = \sum_{n=0}^{N-1} x(n) \sin\left(\dfrac{2\pi nk}{N}\right)$

6.10.4 Time Reversal of the Sequence

The time reversal of an N-point sequence $x(n)$ is obtained by wrapping the sequence $x(n)$ around the circle in the clockwise direction. It is denoted as $x[(-n), \text{ mod } N]$ and

$$x[(-n), \text{ mod } N] = x(N-n), \qquad 0 \le n \le N-1$$

If DFT $\{x(n)\} = X(k)$, then

$$\begin{aligned}\text{DFT }\{x(-n), \text{mod } N\} &= \text{DFT }\{x(N-n)\} \\ &= X[(-k), \text{mod } N] = X(N-k)\end{aligned}$$

Proof: $\text{DFT }\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n)e^{-j2\pi nk/N}$

Changing index from n to m, where $m = N - n$, we have $n = N - m$.

Now,
$$\begin{aligned}\text{DFT }\left\{x(N-n)\right\} &= \sum_{m=0}^{N-1} x(m)e^{-j2\pi k(N-m)/N} \\ &= \sum_{m=0}^{N-1} x(m)e^{-j(2\pi/N)kN}e^{j(2\pi/N)km} \\ &= \sum_{m=0}^{N-1} x(m)e^{j(2\pi/N)km} \\ &= \sum_{m=0}^{N-1} x(m)e^{j(2\pi/N)km}e^{-j2\pi m} \\ &= \sum_{m=0}^{N-1} x(m)e^{-j2\pi m[(N-k)/N]} = X(N-k)\end{aligned}$$

6.10.5 Circular Frequency Shift

If
$$\text{DFT }\{x(n)\} = X(k)$$

Then,
$$\text{DFT }\{x(n)\, e^{j2\pi ln/N}\} = X\{(k-l), (\text{mod } N)]$$

Proof:
$$\begin{aligned}\text{DFT }\{x(n)e^{j2\pi l\, n/N}\} &= \sum_{n=0}^{N-1} x(n)e^{j2\pi l\, n/N}e^{-j2\pi kn/N} \\ &= \sum_{n=0}^{N-1} x(n)e^{-j2\pi n(k-l)/N} \\ &= \sum_{n=0}^{N-1} x(n)e^{-j2\pi n(N+k-l)/N} \\ &= X(N+k-l) = X[(k-l), (\text{mod } N)]\end{aligned}$$

6.10.6 Complex Conjugate Property

If $$\text{DFT }\{x(n)\} = X(k)$$

Then $$\text{DFT }\{x^*(n)\} = X^*(N-k) = X^*[(-k), \text{mod } N]$$

Proof:
$$\text{DFT}\{x^*(n)\} = \sum_{n=0}^{N-1} x^*(n)\, e^{-j2\pi kn/N}$$
$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{j2\pi nk/N}\right]^* = \left[\sum_{n=0}^{N-1} x(n) e^{-j2\pi n(N-k)/N}\right]^* = X^*(N-k)$$

$$\text{DFT}\{x^*(N-n)\} = X^*(k)$$

Proof:
$$\text{IDFT}\{X^*(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k) e^{j2\pi kn/N}$$
$$= \frac{1}{N}\left[\sum_{k=0}^{N-1} X(k) e^{-j2\pi kn/N}\right]^* = \frac{1}{N}\left[\sum_{k=0}^{N-1} X(k) e^{j2\pi k(N-n)/N}\right]^* = x^*(N-n)$$

6.10.7 DFT of Delayed Sequence (Circular time shift of a sequence)

Let $x(n)$ be a discrete sequence, and $x'(n)$ be a delayed or shifted sequence of $x(n)$ by n_0 units of time.

If $$\text{DFT }\{x(n)\} = X(k)$$

Then, $$\text{DFT }\{x'(n)\} = \text{DFT}\{x[(n-n_0), \text{mod } N]\} = X(k)\, e^{-j2\pi n_0 k/N}$$

Proof: By the definition of IDFT,

$$\text{IDFT }\{X(k)\} = x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}nk}$$

Replacing n by $n - n_0$, we have

$$x(n-n_0) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}(n-n_0)k}$$
$$= \frac{1}{N}\sum_{k=0}^{N-1}\left[X(k)\, e^{-j\frac{2\pi}{N}n_0 k}\right] e^{j\frac{2\pi}{N}nk}$$
$$= \text{IDFT}\left[X(k)\, e^{-j\frac{2\pi}{N}n_0 k}\right]$$

On taking DFT on both sides, we get

$$\text{DFT }[x(n-n_0)] = X(k)e^{-j\frac{2\pi}{N}kn_0}$$

6.10.8 DFT of Real Valued Sequence

Let $x(n)$ be a real sequence. By definition of DFT,

$$\text{DFT }\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} x(n)\left(\cos\frac{2\pi}{N}nk - j\sin\frac{2\pi}{N}nk\right)$$

$$= \sum_{n=0}^{N-1} x(n)\cos\frac{2\pi}{N}nk - j\sum_{n=0}^{N-1} x(n)\sin\frac{2\pi}{N}nk$$

Also $X(k) = X_R(k) + jX_I(k)$

Therefore, we can say

Real part $\quad X_R(k) = \sum_{n=0}^{N-1} x(n)\cos\left(\frac{2\pi}{N}nk\right), \quad \text{for } 0 \le k \le N-1$

Imaginary part $\quad X_I(k) = -\sum_{n=0}^{N-1} x(n)\sin\left(\frac{2\pi}{N}nk\right), \quad \text{for } 0 \le k \le N-1$

When $x(n)$ is real, then $X(k)$ will have the following features:

(a) $X(k)$ has complex conjugate symmetry, i.e. $X(k) = X^*(N-k)$
(b) Real component is even function, i.e. $X_R(k) = X_R(N-k)$
(c) Imaginary component is odd function, i.e. $X_I(k) = -X_I(N-k)$
(d) Magnitude function is even function, i.e. $|X(k)| = |X(N-k)|$
(e) Phase function is odd function, i.e. $\underline{|X}(k) = -\underline{|X}(N-k)$
(f) If $x(n) = x(-n)$ (even sequence), then $X(k)$ is purely real.
(g) If $x(n) = -x(-n)$ (odd sequence), then $X(k)$ is purely imaginary.

6.10.9 Multiplication of Two Sequences

If $\quad \text{DFT }[x_1(n)] = X_1(k)$

and $\quad \text{DFT }[x_2(n)] = X_2(k)$

Then $\quad \text{DFT}[x_1(n)x_2(n)] = \frac{1}{N}[X_1(k) \oplus X_2(k)]$

6.10.10 Circular Convolution of Two Sequences

The convolution property of DFT says that, the multiplication of DFTs of two sequences is equivalent to the DFT of the circular convolution of the two sequences.

Let DFT $[x_1(n)] = X_1(k)$ and DFT $[x_2(n)] = X_2(k)$, then by the convolution property $X_1(k)X_2(k) = \text{DFT}\{x_1(n) \oplus x_2(n)\}$.

Proof: Let $x_1(n)$ and $x_2(n)$ be two finite duration sequences of length N. The N-point DFTs of the two sequences are:

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi}{N}nk}, \quad k = 0, 1, \ldots, N-1$$

$$X_2(k) = \sum_{l=0}^{N-1} x_2(l) e^{-j\frac{2\pi}{N}lk}, \quad k = 0, 1, \ldots, N-1$$

On multiplying the above two DFTs, we obtain the result as another DFT, say, $X_3(k)$. Now, $X_3(k)$ will be N-point DFT of a sequence $x_3(m)$.

$$\therefore \quad X_3(k) = X_1(k)X_2(k) \quad \text{and} \quad \text{IDFT}\{X_3(k)\} = x_3(m)$$

By the definition of IDFT,

$$x_3(m) = \frac{1}{N}\sum_{k=0}^{N-1} X_3(k)\, e^{j\frac{2\pi}{N}mk}, \quad m = 0, 1, 2, \ldots, N-1$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)X_2(k)\, e^{j\frac{2\pi}{N}mk}$$

Using the above equations for $X_1(k)$ and $X_2(k)$, the equation for $x_3(m)$ is:

$$x_3(m) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi}{N}nk} \sum_{l=0}^{N-1} x_2(l) e^{-j\frac{2\pi}{N}lk}\, e^{j\frac{2\pi}{N}mk}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(m-n-l)}$$

Let $m - n - l = PN$ where P is an integer.

$$\therefore \quad e^{j\frac{2\pi}{N}k(m-n-l)} = e^{j\frac{2\pi}{N}kPN} = e^{j2\pi kP} = (e^{j2\pi P})^k$$

We know that

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi k}{N}(m-n-l)} = \sum_{k=0}^{N-1} (e^{j2\pi P})^k = \sum_{k=0}^{N-1} 1^k = \sum_{k=0}^{N-1} 1 = N$$

Therefore, the above equation for $x_3(m)$ can be written as:

$$x_3(m) = \frac{1}{N}\sum_{n=0}^{N-1} x_1(n)\sum_{l=0}^{N-1} x_2(l)N = \sum_{n=0}^{N-1} x_1(n)\sum_{l=0}^{N-1} x_2(l)$$

If $x_2(l)$ is a periodic sequence with periodicity of N samples, then $x_2(l \pm PN) = x_2(l)$

Here $\quad m - n - l = PN$

$\therefore \quad l = m - n - PN$

$\therefore \quad x_2(l) = x_2(m - n - PN) = x_2(m - n) = x_2[(m - n), \bmod N]$

Therefore, $x_3(m)$ can be

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)\sum_{n=0}^{N-1} x_2(m-n) = \sum_{n=0}^{N-1} x_1(n)x_2(m-n)$$

Replacing m by n and n by k, we have $x_3(n) = \sum_{k=0}^{N-1} x_1(k)x_2(n-k)$

Note: For simplicity, $x_2[(m - n), \bmod N]$ is represented as $x_2(m - n)$.

The equation for $x_2(l)$ is in the form of convolution sum. Since the equation for $x_2(l)$ involves the index $[(m - n), \bmod N]$, it is called circular convolution.

Hence, we conclude that multiplication of the DFTs of two sequences is equivalent to the DFT of the circular convolution of the two sequences.

$$X_1(k)X_2(k) = \text{DFT}\,\{x_1(n) \oplus x_2(n)\}$$

6.10.11 Parseval's Theorem

Parseval's theorem says that the DFT is an energy-conserving transformation and allows us to find the signal energy either from the signal or its spectrum. This implies that the sum of squares of the signal samples is related to the sum of squares of the magnitude of the DFT samples.

If $\quad \text{DFT}\,\{x_1(n)\} = X_1(k)$

and $\quad \text{DFT}\,\{x_2(n)\} = X_2(k)$

Then $\quad \sum_{n=0}^{N-1} x_1(n)x_2^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)X_2^*(k)$

6.10.12 Circular Correlation

For complex valued sequences $x(n)$ and $y(n)$,

If $\quad \text{DFT}\,\{x(n)\} = X(k)$

and $\quad \text{DFT}\,\{y(n)\} = Y(k)$

Then $\quad \text{DFT }\{r_{xy}(l)\} = \text{DFT}\left[\sum_{n=0}^{N-1} x(n)\, y^*((n-l), \text{mod } N)\right] = X(k)Y^*(k)$

where, $r_{xy}(l)$ is the circular cross correlation sequence. The properties of DFT are summarized in Table 6.3.

TABLE 6.3 Properties of the DFT

Property	*Time domain*	*Frequency domain*
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$ax_1(n) + bx_2(n)$	$aX_1(k) + bX_2(k)$
Time reversal	$x((-n), \text{mod } N) = x(N-n)$	$X(N-k)$
Circular time shift (delayed sequence)	$x((n-l), \text{mod } N)$	$X(k)e^{-j2\pi kl/N}$
Circular frequency shift	$x(n)e^{j2\pi ln/N}$	$X((k-l), \text{mod } N)$
Circular convolution	$x_1(n) \oplus x_2(n)$	$X_1(k)X_2(k)$
Multiplication of two sequences	$x_1(n)x_2(n)$	$\frac{1}{N}(X_1(k) \oplus X_2(k))$
Complex conjugate	$x^*(n)$	$X^*(N-k)$
Circular correlation	$x_1(n) \oplus y^*(-n)$	$X(k)Y^*(k)$
Parseval's theorem	$\sum_{n=0}^{N-1} x(n)\,y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

Central ordinates

The computation of the DFT at the indices $k = 0$ and (for even N) at $k = N/2$ can be simplified using the central ordinate theorems that arise as a direct consequence of the defining relations. In particular, we find that $X(0)$ equals the sum of the N signal samples $x(n)$ and $X(N/2)$ equals the sum of $(-1)^n x(n)$ (with alternating sign changes). This also implies that, if $x(n)$ is real valued, so are $X(0)$ and $X(N/2)$. Similar results hold for the IDFT.

The DFT of a real sequence possesses conjugate symmetry about the origin with $X(-k) = X^*(k)$. Since DFT is periodic, $X(-k) = X(N-k)$. This also implies conjugate symmetry about the index $k = 0.5N$, and thus $X(k) = X^*(N-k)$.

If N is odd, the conjugate symmetry is about the half integer value $0.5N$. The index $k = 0.5N$ is called the folding index. This is illustrated in Figure 6.3.

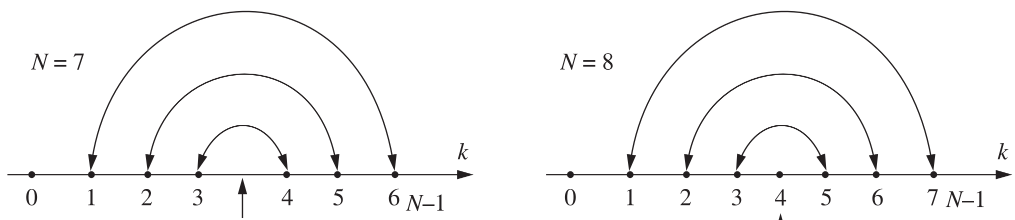


Figure 6.3 Conjugate symmetry for (a) N odd and (b) N even.

The conjugate symmetry suggests that we need compute only half the DFT values to find the entire DFT.

EXAMPLE 6.15 Let $X(k)$ be a 12-point DFT of a length 12 real sequence $x(n)$. The first 7 samples of $X(k)$ are given by $X(0) = 8$, $X(1) = -1 + j2$, $X(2) = 2 + j3$, $X(3) = 1 - j4$, $X(4) = -2 + j2$, $X(5) = 3 + j1$, $X(6) = -1 - j3$. Determine the remaining samples of $X(k)$.

Solution: Given $N = 12$, we have $X(k) = X^*(N - k) = X^*(12 - k)$. The first 7 samples of $X(k)$ are given, the remaining samples are:

$$X(7) = X^*(12 - 7) = X^*(5) = 3 - j1$$
$$X(8) = X^*(12 - 8) = X^*(4) = -2 - j2$$
$$X(9) = X^*(12 - 9) = X^*(3) = 1 + j4$$
$$X(10) = X^*(12 - 10) = X^*(2) = 2 - j3$$
$$X(11) = X^*(12 - 11) = X^*(1) = -1 - j2$$

EXAMPLE 6.16 The DFT of a real signal is $\{1, A, -1, B, 0, -j2, C, -1 + j\}$. Find A, B and C.

Solution: Given $X(k) = \{1, A, -1, B, 0, -j2, C, -1 + j\}$, using conjugate symmetry, we have

$$\begin{aligned} X(k) &= \{X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)\} \\ &= \{X(0), X^*(8-1), X(2), X^*(8-3), X(4), X(5), X^*(8-6), X(7)\} \\ &= \{1, -1-j,\ -1, j2, 0,\ -j2,\ -1,\ -1+j\} \end{aligned}$$

$\therefore$ $$A = -1 - j, \quad B = j2, \quad \text{and} \quad C = -1$$

EXAMPLE 6.17 Let $x(n) = \{A, 2, 3, 4, 5, 6, 7, B\}$. If $X(0) = 20$ and $X(4) = 0$, find A and B.

Solution: Using central ordinates, we have

$$X(0) = \sum_{n=0}^{7} x(n) = A + 2 + 3 + 4 + 5 + 6 + 7 + B = A + B + 27 = 20$$

and $$X(4) = X\left(\frac{N}{2}\right) = \sum_{n=0}^{7} (-1)^n x(n) = A - 2 + 3 - 4 + 5 - 6 + 7 - B = 0$$

i.e. $$A - B + 3 = 0$$

Therefore, $A + B + 27 = 20$ and $A - B + 3 = 0$

Solving the above two equations, we have $A = -5$ and $B = -2$.

EXAMPLE 6.18 The DFT of a real signal is $X(k) = \{1, A, -1, B, -7, -j2, C, -1 + j\}$. What is its signal energy?

Solution: Using the conjugate symmetry property, we have

$$X(k) = \{1, A, -1, B, -7, -j2, C, -1 + j\}$$

$$= \{1, X^*(8-1), -1, X^*(8-3), -7, -j2, X^*(8-6), -1+j\}$$

$$= \{1, -1-j, -1, j2, -7, -j2, -1, -1+j\}$$

Using Parseval's theorem, we have signal energy

$$= \sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |X(k)|^2$$

$$= \frac{1}{8}\left[|1|^2 + |(-1-j)|^2 + |-1|^2 + |j2|^2 + |-7|^2 + |-j2|^2 + |-1|^2 + |-1+j|^2\right]$$

$$= \frac{1}{8}[1 + 2 + 1 + 4 + 49 + 4 + 1 + 4] = 8.25$$

EXAMPLE 6.19 Let $x(n)$ be a real valued sequence of length N, and let $X(k)$ be its DFT with real and imaginary parts $X_R(k)$ and $X_I(k)$ respectively. Show that if $x(n)$ is real, $X_R(k) = X_R(N-k)$ and $X_I(k) = -X_I(N-k)$ for $k = 1, 2, ..., N-1$.

Solution: Since $x(n)$ is real, $x^*(n) = x(n)$. We have

$$\text{DFT}\left[x^*(n)\right] = X(k) = \sum_{n=0}^{N-1} x^*(n)\, e^{-j\frac{2\pi}{N}nk}$$

$$= \left(\sum_{n=0}^{N-1} x(n)\, e^{j\frac{2\pi}{N}nk}\right)^*$$

$$= \left(\sum_{n=0}^{N-1} x(n)\, e^{j\frac{2\pi}{N}nk}\; e^{-j\frac{2\pi}{N}Nn}\right)^*$$

$$= \left(\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}(N-k)}\right)^*$$

$$= X^*(N-k)$$

$$\therefore \qquad X(k) = X^*(N-k)$$

$$X(k) = X_R(k) + jX_I(k) = [X_R(N-k) + jX_I(N-k)]^*$$

$$= X_R^*(N-k) - jX_I^*(N-k)$$

$\therefore \qquad X_R(k) = X_R^*(N-k)$

and $\qquad X_I(k) = X_I^*(N-k)$

EXAMPLE 6.20 Show that with $x(n)$ as an N-point sequence and $X(k)$ as its N-point DFT,

$$\sum_{n=0}^{N-1} \left|x(n)\right|^2 = \frac{1}{N}\sum_{k=0}^{N-1} \left|X(k)\right|^2$$

Solution:

$$\sum_{n=0}^{N-1} \left|x(n)\right|^2 = \sum_{n=0}^{N-1} x(n)x^*(n)$$

$$= \sum_{n=0}^{N-1} x(n)\left[\frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}nk}\right]^*$$

$$= \sum_{n=0}^{N-1} x(n)\left[\frac{1}{N}\sum_{k=0}^{N-1} X^*(k)\, e^{-j\frac{2\pi}{N}nk}\right]$$

$$= \sum_{k=0}^{N-1} X^*(k)\left[\frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}nk}\right]$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)\, X(k)$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} \left|X(k)\right|^2$$

EXAMPLE 6.21 Consider the length-6 sequence defined for $0 \le n < 6$.

$$x(n) = \{1, -2, 3, 0, -1, 1\}$$

with a 8-point DFT $X(k)$. Evaluate the following functions of $X(k)$ without computing DFT:

(a) $X(0)$, (b) $X(3)$, (c) $\sum_{k=0}^{5} X(k)$, (d) $\sum_{k=0}^{5} \left|X(k)\right|^2$

Solution:

(a) We know that $X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}nk}, \quad k = 0, 1, \ldots, N-1$

$$\therefore \qquad X(0) = \sum_{n=0}^{5} x(n) e^{0} = \sum_{n=0}^{5} x(n) = 1 - 2 + 3 + 0 - 1 + 1 = 2$$

(b) $$X(3) = \sum_{n=0}^{5} x(n)\, e^{-j\frac{2\pi}{6}3n} = \sum_{n=0}^{5} x(n)\,(-1)^n$$

$$= x(0)\,(1) + x(1)(-1) + x(2)(1) + x(3)(-1) + x(4)(1) + x(5)(-1)$$
$$= (1)(1) - 2(-1) + 3(1) + (0)(-1) - 1(1) + 1(-1)$$
$$= 4$$

(c) We have $x(n) = \dfrac{1}{N} \displaystyle\sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}nk}, \quad n = 0, 1, \ldots, N-1$

$$\therefore \qquad \sum_{k=0}^{N-1} X(k) = Nx(0) = 6(1) = 6$$

(d) From Parseval's theorem

$$\sum_{k=0}^{5} |X(k)|^2 = N \sum_{n=0}^{5} |x(n)|^2 = 6\left[(1)^2 + (-2)^2 + (3)^2 + (0)^2 + (-1)^2 + (1)^2\right]$$
$$= 6(1 + 4 + 9 + 0 + 1 + 1) = 96$$

EXAMPLE 6.22 If the DFT $\{x(n)\} = X(k) = \{4, -j2, 0, j2\}$, using properties of DFT, find

(a) DFT of $x(n-2)$
(b) DFT of $x(-n)$
(c) DFT of $x^*(n)$
(d) DFT of $x^2(n)$
(e) DFT of $x(n) \oplus x(n)$
(f) Signal energy

Solution:

(a) Using the time shift property of DFT, we have for $N = 4$.

$$\text{DFT}\,\{x(n-2)\} = e^{-j\frac{2\pi}{4}2k}\, X(k) = e^{-j\pi k}\, X(k)$$
$$= \{X(0)e^{0}, X(1)\, e^{-j\pi}, X(2)\, e^{-j2\pi}, X(3)\, e^{-j3\pi}\}$$
$$= \{4(1), -j2(-1), 0(1), j2(-1)\} = \{4, j2, 0, -j2\}$$

(b) Using the flipping (time reversal) property of DFT, we have

$$\text{DFT}\,\{x(-n)\} = X(-k) = X^*(k) = \{4, -j2, 0, j2\}^* = \{4, j2, 0, -j2\}$$

(c) Using the conjugation property of DFT, we have

$$\text{DFT } \{x^*(n)\} = X^*(-k) = \{4, j2, 0, -j2\}^* = \{4, -j2, 0, j2\}$$

Since DFT $\{x^*(n)\} = \text{DFT } \{x(n)\}$, we can say that $x(n)$ is real valued.

(d) Using the property of convolution of product of two signals, we have

$$\text{DFT } \{x(n)\, x(n)\} = \frac{1}{N} [X(k) \oplus X(k)] = \frac{1}{4} [(4, -j2, 0, j2) \oplus (4, -j2, 0, j2)]$$

$$= \{6, -j4, 0, j4\}$$

(e) Using the circular convolution property of DFT, we have

$$\text{DFT } \{x(n) \oplus x(n)\} = [X(k)\, X(k)] = \{4, -j2, 0, j2\}\, \{4, -j2, 0, j2\}$$

$$= \{16, -4, 0, -4\}$$

(f) Using Parseval's theorem, we have

$$\text{Signal energy } = \frac{1}{4} \sum |X(k)|^2 = \frac{1}{4} \sum |\{4, -j2, 0, j2\}|^2$$

$$= \frac{1}{4} [16 + 4 + 0 + 4] = 6$$

EXAMPLE 6.23 If IDFT $\{X(k)\} = x(n) = \{1, 2, 1, 0\}$, using properties of DFT, find

(a) IDFT $\{X(k-1)\}$
(b) IDFT $\{X(k) \oplus X(k)\}$
(c) IDFT $\{X(k)X(k)\}$
(d) Signal energy

Solution: Given IDFT $\{X(k)\} = x(n) = \{1, 2, 1, 0\}$

(a) Using modulation property, we have

$$\text{IDFT } \{X(k-1)\} = x(n)\, e^{j2\pi n/4} = x(n) e^{j\pi \frac{n}{2}}$$

$$= \left\{ x(0) e^0, x(1)\, e^{\frac{j\pi}{2}}, x(2) e^{j\pi}, x(3) e^{j\frac{3\pi}{2}} \right\}$$

$$= 1(1), 2(j), 1(-1), 0(-j) = \{1, j2, -1, 0\}$$

(b) Using periodic convolution property, we have

$$\text{IDFT } \{X(k) \oplus X(k)\} = Nx^2(n) = 4\{1, 2, 1, 0\}^2 = \{4, 16, 4, 0\}$$

(c) Using the convolution in time domain property, we have

$$\text{IDFT } \{X(k)\, X(k)\} = \{1, 2, 1, 0\} \oplus \{1, 2, 1, 0\} = \{2, 4, 6, 4\}$$

(d) Signal energy $= \sum_{n=0}^{N-1} |x(n)|^2 = |x(0)|^2 + |x(1)|^2 + |x(2)|^2 + |x(3)|^2$

$$= (1)^2 + (2)^2 + (1)^2 + (0)^2 = 6$$

6.11 METHODS OF PERFORMING LINEAR CONVOLUTION

In chapter 2, we have discussed four methods of finding the linear convolution of two sequences, viz. (i) Graphical method, (ii) Tabular array method, (iii) Tabular method, and (iv) Matrices method. Now, we discuss one more method, i.e. linear convolution using DFT.

6.11.1 Linear Convolution Using DFT

The DFT supports only circular convolution. When two numbers of N-point sequence are circularly convolved, it produces another N-point sequence. For circular convolution, one of the sequence should be periodically extended. Also the resultant sequence is periodic with period N.

The linear convolution of two sequences of length N_1 and N_2 produces an output sequence of length $N_1 + N_2 - 1$. To perform linear convolution using DFT, both the sequences should be converted to $N_1 + N_2 - 1$ sequences by padding with zeros. Then take $N_1 + N_2 - 1$-point DFT of both the sequences and determine the product of their DFTs. The resultant sequence is given by the IDFT of the product of DFTs. [Actually the response is given by the circular convolution of the $N_1 + N_2 - 1$ sequences].

Let $x(n)$ be an N_1-point sequence and $h(n)$ be an N_2-point sequence. The linear convolution of $x(n)$ and $h(n)$ produces a sequence $y(n)$ of length $N_1 + N_2 - 1$. So pad $x(n)$ with $N_2 - 1$ zeros and $h(n)$ with $N_1 - 1$ zeros and make both of them of length $N_1 + N_2 - 1$.

Let $X(k)$ be an $N_1 + N_2 - 1$-point DFT of $x(n)$, and $H(k)$ be an $N_1 + N_2 - 1$-point DFT of $h(n)$. Now, the sequence $y(n)$ is given by the inverse DFT of the product $X(k)\ H(k)$.

$$\therefore \qquad y(n) = \text{IDFT}\ \{X(k)H(k)\}$$

This technique of convolving two finite duration sequences using DFT techniques is called fast convolution. The convolution of two sequences by convolution sum formula $y(n) = \sum_{k=-\infty}^{\infty} x(k)\ h(n-k)$ is called direct convolution or slow convolution. The term fast is used because the DFT can be evaluated rapidly and efficiently using any of a large class of algorithms called Fast Fourier Transform (FFT).

In a practical sense, the size of DFTs need not be restricted to $N_1 + N_2 - 1$-point transforms. Any number L can be used for the transform size subject to the restriction $L \geq (N_1 + N_2 - 1)$. If $L > (N_1 + N_2 - 1)$, then $y(n)$ will have zero valued samples at the end of the period.

EXAMPLE 6.24 Find the linear convolution of the sequences $x(n)$ and $h(n)$ using DFT.

$$x(n) = \{1, 2\},\ h(n) = \{2, 1\}$$

Solution: Let $y(n)$ be the linear convolution of $x(n)$ and $h(n)$. $x(n)$ and $h(n)$ are of length 2 each. So the linear convolution of $x(n)$ and $h(n)$ will produce a 3 sample sequence (2 + 2 – 1 = 3). To avoid time aliasing, we convert the 2 sample input sequences into 3 sample sequences by padding with zeros.

$$\therefore \qquad x(n) = \{1, 2, 0\} \text{ and } h(n) = \{2, 1, 0\}$$

By the definition of N-point DFT, the 3-point DFT of $x(n)$ is:

$$X(k) = \sum_{n=0}^{2} x(n)\, e^{-j\frac{2\pi}{3}kn} = x(0)e^0 + x(1)e^{-j\frac{2\pi}{3}k} + x(2)e^{-j\frac{4\pi}{3}k} = 1 + 2\, e^{-j\frac{2\pi}{3}k}$$

When $k = 0$, $X(0) = 1 + 2e^0 = 3$

When $k = 1$, $X(1) = 1 + 2e^{-j\frac{2\pi}{3}} = 1 + 2(-0.5 - j0.866) = -j1.732$

When $k = 2$, $X(2) = 1 + 2e^{-j\frac{4\pi}{3}} = 1 + 2(-0.5 + j0.866) = j1.732$

By the definition of N-point DFT, the 3-point DFT of $h(n)$ is:

$$H(k) = \sum_{n=0}^{2} h(n)e^{-j\frac{2\pi}{3}nk} = h(0)e^0 + h(1)e^{-j\frac{2\pi}{3}k} + h(2)e^{-j\frac{4\pi}{3}k} = 2 + e^{-j\frac{2\pi}{3}k}$$

When $k = 0$, $H(0) = 2 + 1 = 3$

When $k = 1$, $H(1) = 2 + e^{-j\frac{2\pi}{3}} = 2 + (-0.5 - j0.866) = 1.5 - j0.866$

When $k = 2$, $H(2) = 2 + e^{-j\frac{4\pi}{3}} = 2 + (-0.5 + j0.866) = 1.5 + j0.866$

Let $\qquad Y(k) = X(k)H(k) \quad \text{for} \quad k = 0, 1, 2$

When $k = 0$, $Y(0) = X(0)H(0) = (3)(3) = 9$

When $k = 1$, $Y(1) = X(1)H(1) = (-j1.732)(1.5 - j0.866) = -1.5 - j2.598$

When $k = 2$, $Y(2) = X(2)H(2) = (j1.732)(1.5 + j0.866) = -1.5 + j2.598$

$$\therefore \qquad Y(k) = \{9,\ -1.5 - j2.598, -1.5 + j2.598\}$$

The sequence $y(n)$ is obtained from IDFT of $Y(k)$.

By definition of IDFT,

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\, e^{j\frac{2\pi}{N}nk}; \quad \text{for} \quad n = 0, 1, 2, \ldots, N-1$$

$$\therefore \qquad y(n) = \frac{1}{3}\sum_{k=0}^{2} Y(k)e^{j\frac{2\pi}{3}nk} = \frac{1}{3}\left[Y(0)e^0 + Y(1)e^{j\frac{2\pi}{3}n} + Y(2)e^{j\frac{4\pi}{3}n}\right] \quad \text{for} \quad n = 0, 1, 2$$

When $n = 0,\ y(0) = \dfrac{1}{3}[Y(0) + Y(1) + Y(2)]$

$$= \frac{1}{3}[9 + (-1.5 - j2.598) + (-1.5 + j2.598)]$$

$$= \frac{1}{3}[6] = 2$$

When $n = 1,\ y(1) = \dfrac{1}{3}\left[Y(0) + Y(1)\,e^{j\frac{2\pi}{3}} + Y(2)\,e^{j\frac{4\pi}{3}}\right]$

$$= \frac{1}{3}[9 + (-1.5 - j2.598)(-0.5 + j0.866) + (-1.5 + j2.598)(-0.5 - j0.866)]$$

$$= \frac{1}{3}[9 + 0.75 + 2.25 + 0.75 + 2.25] = 5$$

When $n = 2,\ y(2) = \dfrac{1}{3}\left[Y(0) + Y(1)\,e^{j\frac{4\pi}{3}} + Y(2)\,e^{j\frac{8\pi}{3}}\right]$

$$= \frac{1}{3}[9 + (-1.5 - j2.598)(-0.5 + j0.866) + (-1.5 + j2.598)(-0.5 - j0.866)]$$

$$= \frac{1}{3}[9 + 0.75 - 2.25 + 0.75 - 2.25] = 2$$

$\therefore \qquad y(n) = \{2, 5, 2\}$

Verification by tabular method

The linear convolution of $x(n) = \{1, 2\}$ and $h(n) = \{2, 1\}$ is obtained using the tabular method as shown below.

$h(n)$ \ $x(n)$	1	2
2	2	4
1	1	2

From the above table, $y(n) = \{2, 1 + 4, 2\} = \{2, 5, 2\}$.

EXAMPLE 6.25 Find the linear convolution of the sequences $x(n)$ and $h(n)$ using DFT.

$$x(n) = \{1, 0, 2\},\ h(n) = \{1, 1\}$$

Solution: Let $y(n)$ be the linear convolution of $x(n)$ and $h(n)$. $x(n)$ is of length 3 and $h(n)$ is of length 2. So the linear convolution of $x(n)$ and $h(n)$ will produce a 4-sample sequence $(3 + 2 - 1 = 4)$. To avoid time aliasing, we convert the 2-sample and 3-sample sequences into 4-sample sequences by padding with zeros.

$$\therefore \qquad x(n) = \{1, 0, 2, 0\} \quad \text{and} \quad h(n) = \{1, 1, 0, 0\}$$

By the definition of N-point DFT, the 4-point DFT of $x(n)$ is:

$$X(k) = \sum_{n=0}^{3} x(n)\, e^{-j\frac{2\pi}{4}kn} = x(0)e^0 + x(1)e^{-j\frac{\pi}{2}k} + x(2)e^{-j\pi k} + x(3)\, e^{-j\frac{3\pi}{2}k}$$

$$= 1 + 2e^{-j\pi k} \qquad k = 0, 1, 2, 3$$

When $k = 0$, $X(0) = 1 + 2e^0 = 1 + 2 = 3$

When $k = 1$, $X(1) = 1 + 2e^{-j\pi} = 1 + 2(-1) = -1$

When $k = 2$, $X(2) = 1 + 2e^{-j2\pi} = 1 + 2(1) = 3$

When $k = 3$, $X(3) = 1 + 2e^{-j3\pi} = 1 + 2(-1) = -1$

$$\therefore \qquad X(k) = \{3, -1, 3, -1\}$$

By the definition of N-point DFT, the 4-point DFT of $h(n)$ is:

$$H(k) = \sum_{n=0}^{3} h(n)\, e^{-j\frac{2\pi}{4}nk} = h(0)e^0 + h(1)e^{-j\frac{\pi}{2}k} + h(2)e^{-j\pi k} + h(3)\, e^{-j\frac{3\pi}{2}k}$$

$$= 1 + e^{-j\frac{\pi}{2}k} \qquad k = 0, 1, 2, 3$$

When $k = 0$, $H(0) = 1 + 1 = 2$

When $k = 1$, $H(1) = 1 + e^{-j\frac{\pi}{2}} = 1 - j$

When $k = 2$, $H(2) = 1 + e^{-j\pi} = 1 - 1 = 0$

When $k = 3$, $H(3) = 1 + e^{-j\frac{3\pi}{2}} = 1 + j$

$$\therefore \qquad H(k) = \{2, 1 - j, 0, 1 + j\}$$

Let $\qquad Y(k) = X(k)H(k) \quad \text{for } k = 0, 1, 2$

$$\therefore \qquad Y(k) = X(k)\, H(k) = \{3, -1, 3, -1\}\, \{2, 1 - j, 0, 1 + j\} = \{6, -1 + j, 0, -1 - j\}$$

The sequence $y(n)$ is obtained from IDFT of $Y(k)$.

By definition of IDFT,

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\, e^{j\frac{2\pi}{N}nk}, \quad \text{for} \quad n = 0, 1, 2, 3$$

$$\therefore \quad y(n) = \frac{1}{4}\sum_{k=0}^{3} Y(k)\, e^{j\frac{\pi}{2}nk}$$

$$= \frac{1}{4}\left[Y(0)e^0 + Y(1)e^{j\frac{\pi}{2}n} + Y(2)e^{j\pi n} + Y(3)e^{j\frac{3\pi}{2}n}\right], \quad \text{for} \quad n = 0, 1, 2, 3$$

i.e.
$$y(n) = \frac{1}{4}[6 + (-1 + j)\, e^{j\frac{\pi}{2}n} + (-1 - j)\, e^{j\frac{3\pi}{2}n}$$

When $n = 0$, $y(0) = \frac{1}{4}[6 + (-1 + j) + (-1 - j)] = 1$

When $n = 1$, $y(1) = \frac{1}{4}\left[6 + (-1 + j)e^{j\frac{\pi}{2}} + (-1 - j)e^{j\frac{3\pi}{2}}\right]$

$$= \frac{1}{4}[6 + (-1 + j)(j) + (-1 - j)(-j)]$$

$$= \frac{1}{4}[6 - j - 1 + j - 1] = 1$$

When $n = 2$, $y(2) = \frac{1}{4}[6 + (-1 + j)e^{j\pi} + (-1 - j)e^{j3\pi}]$

$$= \frac{1}{4}[6 + (-1 + j)(-1) + (-1 - j)(-1)]$$

$$= \frac{1}{4}[6 + 1 - j + 1 - j] = 2$$

When $n = 3$, $y(3) = \frac{1}{4}\left[6 + (-1 + j)e^{j\frac{3\pi}{2}} + (-1 - j)e^{j\frac{9\pi}{2}}\right]$

$$= \frac{1}{4}[6 + (-1 + j)(-j) + (-1 - j)(j)]$$

$$= \frac{1}{4}[6 + j + 1 - j + 1] = 2$$

Therefore, the linear convolution of $x(n)$ and $h(n)$ is:

$$y(n) = x(n) * h(n) = \{1, 1, 2, 2\}$$

Verification by tabular method

The linear convolution of $x(n) = \{1, 0, 2\}$ and $h(n) = \{1, 1\}$ is obtained using the tabular method as shown below.

		$x(n)$	
	1	0	2
$h(n)$ 1	1	0	2
1	1	0	2

From the above table, $y(n) = \{1, 1, 2, 2\}$.

6.12 METHODS OF PERFORMING CIRCULAR CONVOLUTION

In chapter 2, we have discussed three methods of finding circular convolution of discrete-time sequences: (i) Graphical method, (ii) Tabular array method, (iii) Tabular method. Now, we discuss a method of finding circular convolution using DFT and IDFT.

6.12.1 Circular Convolution Using DFT and IDFT

Let $x_1(n)$ and $x_2(n)$ be the given sequences. Let $x_3(n)$ be the sequence obtained by circular convolution of $x_1(n)$ and $x_2(n)$. The following procedure can be used to get the sequence $x_3(n)$:

Step 1: Take the N-point DFT of $x_1(n)$ and $x_2(n)$.

Let $X_1(k) = \text{DFT}\,[x_1(n)]$ and $X_2(k) = \text{DFT}\,[x_2(n)]$

Step 2: Determine the product of $X_1(k)$ and $X_2(k)$.

Let this product be $X_3(k)$, i.e. $X_3(k) = X_1(k)X_2(k)$

Step 3: By convolution theorem of DFT, we get

$$\text{DFT}\,\{x_1(n) \oplus x_2(n)\} = X_1(k)X_2(k)$$

Here $x_1(n) \oplus x_2(n) = x_3(n)$ and $X_1(k)X_2(k) = X_3(k)$

$\therefore$ $$\text{DFT}\,\{x_3(n)\} = X_3(k)$$

On taking IDFT, we get

$$x_3(n) = \text{IDFT}\,\{X_3(k)\} = \text{IDFT}\,\{X_1(k)\,X_2(k)\}$$

i.e., the sequence $x_3(n)$ is obtained by taking IDFT of the product sequence $X_1(k)X_2(k)$.

EXAMPLE 6.26 Perform circular convolution of the following sequences using DFT and IDFT:

$$x_1(n) = \{1, 2, 1, 2\} \quad \text{and} \quad x_2(n) = \{4, 3, 2, 1\}$$

Solution: To perform circular convolution of $x_1(n)$ and $x_2(n)$ using DFT and IDFT, find $X_1(k)$, the DFT of $x_1(n)$, $X_2(k)$, the DFT of $x_2(n)$, then find $X_3(k) = X_1(k)X_2(k)$ and take IDFT of $X_3(k)$ to get $x_3(n)$, the circular convolution of $x_1(n)$ and $x_2(n)$.

The 4-point DFT of $x_1(n)$ is:

$$\text{DFT}\{x_1(n)\} = X_1(k) = \sum_{n=0}^{3} x_1(n)\, e^{-j\frac{2\pi}{N}nk}, \quad k = 0,1,2,3$$

$$= x_1(0)\,e^0 + x_1(1)\,e^{-j\frac{2\pi}{4}k} + x_1(2)\,e^{-j\frac{4\pi}{4}k} + x_1(3)\,e^{-j\frac{6\pi}{4}k}$$

$$= 1 + 2e^{-j\frac{\pi}{2}k} + e^{-j\pi k} + 2e^{-j\frac{3\pi}{2}k}$$

When $k = 0$, $X_1(0) = 1 + 2 + 1 + 2 = 6$

When $k = 1$, $X_1(1) = 1 + 2e^{-j\frac{\pi}{2}} + e^{-j\pi} + 2e^{-j\frac{3\pi}{2}} = 1 - j2 - 1 + j2 = 0$

When $k = 2$, $X_1(2) = 1 + 2e^{-j\pi} + e^{-j2\pi} + 2e^{-j3\pi} = 1 - 2 + 1 - 2 = -2$

When $k = 3$, $X_1(3) = 1 + 2e^{-j\frac{3\pi}{2}} + e^{-j3\pi} + 2e^{-j\frac{9\pi}{2}} = 1 + j2 - 1 - j2 = 0$

$\therefore$ $$X_1(k) = \{6, 0, -2, 0\}$$

The 4-point DFT of $x_2(n)$ is:

$$\text{DFT}\,\{x_2(n)\} = X_2(k) = \sum_{n=0}^{3} x_2(n)\,e^{-j\frac{2\pi}{4}nk}, \quad k = 0, 1, 2, 3$$

$$= x_2(0)e^0 + x_2(1)e^{-j\frac{\pi}{2}k} + x_2(2)\,e^{-j\pi k} + x_2(3)\,e^{-j\frac{3\pi}{2}k}$$

$$= 4 + 3e^{-j\frac{\pi k}{2}} + 2e^{-j\pi k} + e^{-j\frac{3\pi}{2}k}$$

When $k = 0$, $X_2(0) = 4 + 3 + 2 + 1 = 10$

When $k = 1$, $X_2(1) = 4 + 3e^{-j\frac{\pi}{2}} + 2e^{-j\pi} + e^{-j\frac{3\pi}{2}} = 4 - j3 - 2 + j = 2 - j2$

When $k = 2$, $X_2(2) = 4 + 3e^{-j\pi} + 2e^{-j2\pi} + e^{-j3\pi} = 4 - 3 + 2 - 1 = 2$

When $k = 3$, $X_2(3) = 4 + 3e^{-j\frac{3\pi}{2}} + 2e^{-j3\pi} + e^{-j\frac{9\pi}{2}} = 4 + j3 - 2 - j = 2 + j2$

$$\therefore \qquad X_2(k) = \{10,\ 2 - j2,\ 2,\ 2 + j2\}$$

Let $X_3(k)$ be the product of $X_1(k)$ and $X_2(k)$.

$$\therefore \qquad X_3(k) = X_1(k)X_2(k) = \{6, 0, -2, 0\}\{10,\ 2 - j2,\ 2,\ 2 + j2\}$$

$$= \{60, 0, -4, 0\}$$

$$\therefore \qquad x_3(n) = \text{IDFT}\{X_3(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X_3(k)\, e^{j\frac{2\pi}{N}nk}; \quad n = 0, 1, 2, 3$$

$$= \frac{1}{4}\left\{X_3(0)\, e^0 + X_3(1)\, e^{j\frac{2\pi}{4}n} + X_3(2)\, e^{j\frac{4\pi}{4}n} + X_3(3)\, e^{j\frac{6\pi}{4}n}\right\}$$

$$= \frac{1}{4}\{60 - 4e^{j\pi n}\}$$

When $n = 0$, $x_3(0) = \frac{1}{4}(60 - 4) = 14$

When $n = 1$, $x_3(1) = \frac{1}{4}(60 - 4e^{j\pi}) = \frac{1}{4}(60 + 4) = 16$

When $n = 2$, $x_3(2) = \frac{1}{4}(60 - 4e^{j2\pi}) = \frac{1}{4}(60 - 4) = 14$

When $n = 3$, $x_3(3) = \frac{1}{4}(60 - 4e^{j3\pi}) = \frac{1}{4}(60 + 4) = 16$

$$\therefore \qquad x_3(n) = \{14, 16, 14, 16\}$$

EXAMPLE 6.27 Find the circular convolution of $x(n) = \{1, 0.5\}$; $h(n) = \{0.5, 1\}$ by DFT and IDFT method.

Solution:

The circular convolution $y(n)$ of $x(n)$ and $h(n)$ is computed by DFT method as shown below.

Given $x(n) = \{1, 0.5\}$

$$X(k) = \sum_{n=0}^{1} x(n)\, e^{-j\frac{2\pi}{2}nk}, \quad k = 0, 1$$

i.e. $X(k) = x(0) + x(1)\, e^{-j\pi k}$

$\therefore$ $X(0) = 1 + 0.5e^0 = 1.5$

and $X(1) = x(0) + x(1)e^{-j\pi} = 1 + 0.5(-1) = 0.5$

$\therefore$ $X(k) = \{1.5,\ 0.5\}$

Given $h(n) = \{0.5,\ 1\}$

$$H(k) = \sum_{n=0}^{1} h(n) e^{-j\frac{2\pi}{2}nk}, \quad k = 0, 1$$

$\therefore$ $H(k) = h(0) + h(1)\, e^{-j\pi k}$

$H(0) = 0.5 + 1e^0 = 0.5 + 1 = 1.5$

$H(1) = 0.5 + 1(e^{-j\pi}) = 0.5 - 1 = -0.5$

$\therefore$ $H(k) = \{1.5,\ -0.5\}$

$$\therefore \quad Y(k) = X(k)H(k) = \{1.5, 0.5\}\{1.5,\ -0.5\} = \{2.25,\ -0.25\}$$

$$\therefore \quad y(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}nk} = \frac{1}{2}\sum_{k=0}^{1} X(k) e^{j\frac{2\pi}{2}nk}, \quad n = 0, 1$$

$$= \frac{1}{2}[X(0) + X(1)\, e^{j\pi n}]$$

$$y(0) = \frac{1}{2}[2.25 - 0.25e^0] = 1$$

$$y(1) = \frac{1}{2}[2.25 - 0.25e^{j\pi}] = \frac{1}{2}[2.25 + 0.25] = 1.25$$

$$\therefore \quad y(n) = x(n) \oplus h(n) = \{y(0), y(1)\} = \{1, 1.25\}$$

6.13 CONVOLUTION OF LONG SEQUENCES (SECTIONED CONVOLUTIONS)

The response of an LTI system for any arbitrary input is given by linear convolution of the input and the impulse response of the system. While implementing linear convolution in FIR filters, the input signal sequence $x(n)$ is much longer than the impulse response $h(n)$ of a DSP system. If one of the sequence is very much longer than the other, then it is very difficult to compute the linear convolution using DFT for the following reasons:

1. The entire sequence must be available before convolution can be carried out. This makes long delay in getting the output.
2. Large amounts of memory is required and computation of DFT becomes cumbersome.

To overcome these problems, we go to sectioned convolutions. In this technique, the longer sequence is sectioned (or splitted) into the size of smaller sequence. If required, the

longer sequence may be padded with zeros. Then the linear convolution of each section (block) of longer sequence and the smaller sequence is performed. The output sequences obtained from the convolutions of all the sections are combined to get the overall output sequence. There are two methods of sectioned convolutions. They are overlap-add method and overlap-save method.

6.13.1 Overlap-add Method

In overlap-add method, the longer sequence $x(n)$ of length L is split into m number of smaller sequences of length N equal to the size of the smaller sequence $h(n)$. (If required zero padding may be done to L so that $L = mN$).

The linear convolution of each section (of length N) of longer sequence with the smaller sequence of length N is performed. This gives an output sequence of length $2N - 1$. In this method, the last $N - 1$ samples of each output sequence overlaps with the first $N - 1$ samples of next section. While combining the output sequences of the various sectioned convolutions, the corresponding samples of overlapped regions are added and the samples of non-overlapped regions are retained as such.

If the linear convolution is to be performed by DFT (or FFT), since DFT supports only circular convolution and not linear convolution directly, we have to pad each section of the longer sequence (of length N) and also the smaller sequence (of length N) with $N - 1$ zeros before computing the circular convolution of each section with the smaller sequence. The steps for this fast convolution by overlap-add method are as follows:

Step 1: $N - 1$ zeros are padded at the end of the impulse response sequence $h(n)$ which is of length N and a sequence of length $2N - 1$ is obtained. Then the $2N - 1$ point FFT is performed and the output values are stored.

Step 2: Split the data, i.e. $x(n)$ into m blocks each of length N and pad $N - 1$ zeros to each block to make them $2N - 1$ sequence blocks and find the FFT of each block.

Step 3: The stored frequency response of the filter, i.e. the FFT output sequence obtained in Step 1 is multiplied by the FFT output sequence of each of the selected block in Step 2.

Step 4: A $2N - 1$ point inverse FFT is performed on each product sequence obtained in Step 3.

Step 5: The first $(N - 1)$ IFFT values obtained in Step 4 for each block, overlapped with the last $N - 1$ values of the previous block. Therefore, add the overlapping values and keep the non-overlapping values as they are. The result is the linear convolution of $x(n)$ and $h(n)$.

6.13.2 Overlap-save Method

In overlap-save method, the results of linear convolution of the various sections are obtained using circular convolution. Let $x(n)$ be a longer sequence of length L and $h(n)$ be a smaller sequence of length N. The regular convolution of sequences of length L and N has $L + N - 1$ samples. If $L > N$, we have to zero pad the second sequence $h(n)$ to length L. So their linear convolution will have $2L - 1$ samples. Its first $N - 1$ samples are contaminated by

wraparound and the rest corresponds to the regular convolution. To understand this let L = 12 and N = 5. If we pad N by 7 zeros, their regular convolution has 23 (or $2L - 1$) samples with 7 trailing zeros ($L - N = 7$). For periodic convolution, 11 samples ($L - 1 = 11$) are wrapped around. Since the last 7 (or $L - N$) are zeros only, first four samples $(2L - 1) - (L) - (L - N) = N - 1 = 5 - 1 = 4$ of the periodic convolution are contaminated by wraparound. This idea is the basis of overlap-save method. First, we add $N - 1$ leading zeros to the longer sequence $x(n)$ and section it into k overlapping (by $N - 1$) segments of length M. Typically we choose $M = 2N$. Next, we zero pad $h(n)$ (with trailing zeros) to length M, and find the periodic convolution of $h(n)$ with each section of $x(n)$. Finally, we discard the first $N - 1$ (contaminated) samples from each convolution and glue (concatenate) the results to give the required convolution.

Step 1: N zeros are padded at the end of the impulse response $h(n)$ which is of length N and a sequence of length $M = 2N$ is obtained. Then the $2N$ point FFT is performed and the output values are stored.

Step 2: A $2N$ point FFT on each selected data block is performed. Here each data block begins with the last $N - 1$ values in the previous data block, except the first data block which begins with $N - 1$ zeros.

Step 3: The stored frequency response of the filter, i.e. the FFT output sequence obtained in Step 1 is multiplied by the FFT output sequence of each of the selected blocks obtained in Step 2.

Step 4: A $2N$ point inverse FFT is performed on each of the product sequences obtained in Step 3.

Step 5: The first $N - 1$ values from the output of each block are discarded and the remaining values are stored. That gives the response $y(n)$.

In either of the above two methods, the FFT of the shorter sequence need be found only once, stored, and reused for all subsequent partial convolutions. Both methods allow on-line implementation if we can tolerate a small processing delay that equals the time required for each section of the long sequence to arrive at the processor.

EXAMPLE 6.28 Perform the linear convolution of the following sequences using (a) overlap-add method, (b) overlap-save method.

$$x(n) = \{1, -2, 2, -1, 3, -4, 4, -3\} \text{ and } h(n) = \{1, -1\}$$

Solution: (a) *Overlap-add method*

Here the longer sequence is $x(n) = \{1, -2, 2, -1, 3, -4, 4, -3\}$ of length L = 8 and the smaller sequence is $h(n) = \{1, -1\}$ of length N = 2. So $x(n)$ is sectioned into 4 blocks $x_1(n), x_2(n), x_3(n)$ and $x_4(n)$ each of length 2 samples as shown below.

$$\begin{array}{r|r|r|r} x_1(n) = 1;\ n = 0 & x_2(n) = 2;\ n = 2 & x_3(n) = 3;\ n = 4 & x_4(n) = 4;\ n = 6 \\ = -2;\ n = 1 & = -1;\ n = 3 & = -4;\ n = 5 & = -3;\ n = 7 \end{array}$$

Let $y_1(n)$, $y_2(n)$, $y_3(n)$ and $y_4(n)$ be the output of linear convolution of $x_1(n)$, $x_2(n)$, $x_3(n)$ and $x_4(n)$ respectively with $h(n)$.

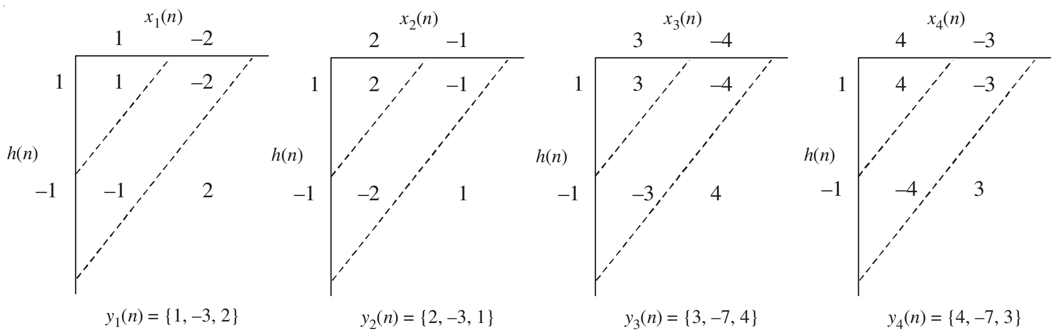
Here $h(n)$ starts at $n = n_h = 0$.

$x_1(n)$ starts at $n = n_1 = 0$, $\therefore\ y_1(n) = x_1(n) * h(n)$ starts at $n = n_1 + n_h = 0 + 0 = 0$
$x_2(n)$ starts at $n = n_2 = 2$, $\therefore\ y_2(n) = x_2(n) * h(n)$ starts at $n = n_2 + n_h = 2 + 0 = 2$
$x_3(n)$ starts at $n = n_3 = 4$, $\therefore\ y_3(n) = x_3(n) * h(n)$ starts at $n = n_3 + n_h = 4 + 0 = 4$
$x_4(n)$ starts at $n = n_4 = 6$, $\therefore\ y_4(n) = x_4(n) * h(n)$ starts at $n = n_4 + n_h = 6 + 0 = 6$

The convolution of each section (of 2 samples) with $h(n)$ (of 2 samples) yields a 3-sample output. This can be obtained by any of the methods discussed in Chapter 2.

$$y_1(n) = x_1(n) * h(n) = \{1, -2\} * \{1, -1\} = \{1, -3, 2\}$$
$$y_2(n) = x_2(n) * h(n) = \{2, -1\} * \{1, -1\} = \{2, -3, 1\}$$
$$y_3(n) = x_3(n) * h(n) = \{3, -4\} * \{1, -1\} = \{3, -7, 4\}$$
$$y_4(n) = x_4(n) * h(n) = \{4, -3\} * \{1, -1\} = \{4, -7, 3\}$$

The computation by the tabular method is shown below.



In overlap-add method, the last $N - 1 = 2 - 1 = 1$ sample in an output sequence overlaps with the first $N - 1 = 2 - 1 = 1$ sample of the next output sequence. The overall output $y(n)$ is obtained by combining the outputs $y_1(n)$, $y_2(n)$, $y_3(n)$, and $y_4(n)$ as shown in Table 6.4. Here the overlapping samples are added.

TABLE 6.4 Combining the output of the convolution of each section

n	0	1	2	3	4	5	6	7	8
$y_1(n)$	1	–3	2						
$y_2(n)$			2	–3	1				
$y_3(n)$					3	–7	4		
$y_4(n)$							4	–7	3
$y(n)$	1	–3	4	–3	4	–7	8	–7	3

$\therefore$ $$y(n) = \{1, -3, 4, -3, 4, -7, 8, -7, 3\}$$

Verification: The direct convolution of $y(n) = x(n) * h(n)$, as shown below, gives

$$y(n) = \{1, -1-2, 2+2, -2-1, 1+3, -3-4, 4+4, -4-3, 3\}$$
$$= \{1, -3, 4, -3, 4, -7, 8, -7, 3\}$$

	$x(n)$							
	1	−2	2	−1	3	−4	4	−3
$h(n)$ 1	1	−2	2	−1	3	−4	4	−3
−1	−1	2	−2	1	−3	4	−4	3

which is same as obtained by the overlap-add method.

(b) *Overlap-save method*

Given $x(n) = \{1, -2, 2, -1, 3, -4, 4, -3\}$ $\therefore L = 8$

and $$h(n) = \{1, -1\}; \quad N = 2$$

$$M = 2N = 4$$

Add $N - 1 = 2 - 1 = 1$ leading zero to the longer sequence $x(n)$

$\therefore$ $$x(n) = \{0, 1, -2, 2, -1, 3, -4, 4, -3\}$$

If we choose $M = 4$, we get three overlapping sections of $x(n)$ (we need to zero pad the last one) described by $x_1(n) = \{0, 1, -2, 2\}$, $x_2(n) = \{2, -1, 3, -4\}$, $x_3(n) = \{-4, 4, -3, 0\}$.

$h(n)$ starts at $n = n_h = 0$

$x_1(n)$ starts at $n = n_1 = -1$, $\therefore$ $y_1(n) = x_1(n) \oplus h(n)$ starts at $n = n_1 + n_h = -1 + 0 = -1$

$x_2(n)$ starts at $n = n_2 = 2$, $\therefore$ $y_2(n) = x_2(n) \oplus h(n)$ starts at $n = n_2 + n_h = 2 + 0 = 2$

$x_3(n)$ starts at $n = n_3 = 5$, $\therefore$ $y_3(n) = x_3(n) \oplus h(n)$ starts at $n = n_3 + n_h = 5 + 0 = 5$

The zero padded $h(n)$ becomes $h(n) = \{1, -1, 0, 0\}$. Periodic convolutions of $x_1(n)$, $x_2(n)$ and $x_3(n)$ with $h(n)$, i.e. $y_1(n)$, $y_2(n)$ and $y_3(n)$ respectively can be computed by any of the methods discussed in Chapter 2. The computation by matrices method is shown below.

$$x_1(n) \oplus h(n) = \{0, 1, -2, 2\} \oplus \{1, -1, 0, 0\} = \{-2, 1, -3, 4\}$$

$$x_2(n) \oplus h(n) = \{2, -1, 3, -4\} \oplus \{1, -1, 0, 0\} = \{6, -3, 4, -7\}$$

$$x_3(n) \oplus h(n) = \{-4, 4, -3, 0\} \oplus \{1, -1, 0, 0\} = \{-4, 8, -7, 3\}$$

$$\begin{bmatrix} 0 & 2 & -2 & 1 \\ 1 & 0 & 2 & -2 \\ -2 & 1 & 0 & 2 \\ 2 & -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & -4 & 3 & -1 \\ -1 & 2 & -4 & 3 \\ 3 & -1 & 2 & -4 \\ -4 & 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 4 \\ -7 \end{bmatrix}, \quad \begin{bmatrix} -4 & 0 & -3 & 4 \\ 4 & -4 & 0 & -3 \\ -3 & 4 & -4 & 0 \\ 0 & -3 & 4 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \\ -7 \\ 3 \end{bmatrix}$$

In overlap-save method, the first $N - 1 = 2 - 1 = 1$ sample of each output overlaps with the last $N - 1 = 2 - 1 = 1$ sample of previous output. So discard the first sample in each section and save the remaining samples. The overall output is obtained by combining $y_1(n)$, $y_2(n)$ and $y_3(n)$ as given in Table 6.5.

TABLE 6.5 Combining the output of the convolution of each section

n	–1	0	1	2	3	4	5	6	7	8
$y_1(n)$	–2	1	–3	4						
$y_2(n)$				6	–3	4	–7			
$y_3(n)$							–4	8	–7	3
$y(n)$		1	–3	4	–3	4	–7	8	–7	3

$$\therefore \qquad y(n) = \{1, -3, 4, -3, 4, \ -7, 8, -7, 3\}$$

The result is same as that obtained by the overlap-add method.

EXAMPLE 6.29 Perform the linear convolution of the following sequences by (a) overlap-add method, (b) overlap-save method

$$x(n) = \{1, -2, 3, 2, -3, 4, 3, -4\} \text{ and } h(n) = \{1, 2, \ -1\}$$

Solution: (a) *Overlap-add method*

Here the longer sequence is $x(n) = \{1, -2, 3, 2, -3, 4, 3, -4\}$ of length 8 and the smaller sequence is $h(n) = \{1, 2, -1\}$ of length 3. So pad a zero to $x(n)$ and make $x(n)$ of length 9. So $x(n) = \{1, -2, 3, 2, -3, 4, 3, -4, 0\}$.

Section $x(n)$ into 3 blocks $x_1(n)$, $x_2(n)$ and $x_3(n)$, each of length 3 samples as shown below.

$$\begin{array}{rr|rr|rr}
x_1(n) = 1; & n = 0 & x_2(n) = 2; & n = 3 & x_3(n) = 3; & n = 6 \\
= -2; & n = 1 & = -3; & n = 4 & = -4; & n = 7 \\
= 3; & n = 2 & = 4; & n = 5 & = 0; & n = 8
\end{array}$$

Let $y_1(n)$, $y_2(n)$ and $y_3(n)$ be the output of linear convolution of $x_1(n)$, $x_2(n)$ and $x_3(n)$ respectively with $h(n)$.

Here, $h(n)$ starts at $n = n_h = 0$

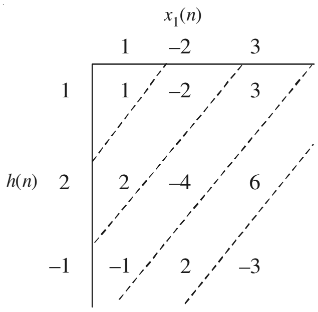
$x_1(n)$ starts at $n = n_1 = 0$, $\therefore$ $y_1(n)$ starts at $n = n_1 + n_h = 0 + 0 = 0$

$x_2(n)$ starts at $n = n_2 = 3$, $\therefore$ $y_2(n)$ starts at $n = n_2 + n_h = 3 + 0 = 3$

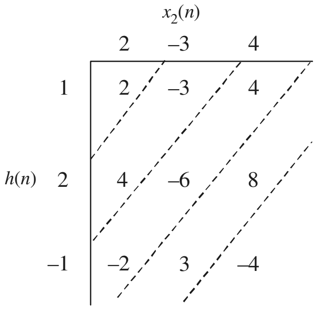
$x_3(n)$ starts at $n = n_3 = 6$, $\therefore$ $y_3(n)$ starts at $n = n_3 + n_h = 6 + 0 = 6$

The convolution of each section (of 3 samples) with $h(n)$ (of 3 samples) yields a 5-sample output. This can be obtained by any of the methods discussed in Chapter 2. The computation of $y_1(n)$, $y_2(n)$ and $y_3(n)$ by the tabular method is shown below.

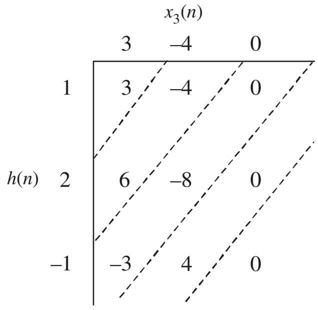
$$y_1(n) = x_1(n) * h(n) = \{1, -2, 3\} * \{1, 2, -1\} = \{1, 0, -2, 8, -3\}$$
$$y_2(n) = x_2(n) * h(n) = \{2, -3, 4\} * \{1, 2, -1\} = \{2, 1, -4, 11, -4\}$$
$$y_3(n) = x_3(n) * h(n) = \{3, -4, 0\} * \{1, 2, -1\} = \{3, 2, -11, 4, 0\}$$



$y_1(n) = \{1, 0, -2, 8, -3\}$



$y_2(n) = \{2, 1, -4, 11, -4\}$



$y_3(n) = \{3, 2, -11, 4, 0\}$

Since, in overlap-add method, the last $N-1 = 3 - 1 = 2$ samples in an output sequence overlaps with the first $N-1 = 3-1 = 2$ samples of next output sequence, the overall output $y(n)$ is obtained by combining the outputs $y_1(n)$, $y_2(n)$ and $y_3(n)$ as shown in Table 6.6. Here the overlap samples are added.

TABLE 6.6 Combining the output of the convolution of each section

n	0	1	2	3	4	5	6	7	8	9	10
$y_1(n)$	1	0	–2	8	–3						
$y_2(n)$				2	1	–4	11	–4			
$y_3(n)$							3	2	–11	4	0
$y(n)$	1	0	–2	10	–2	–4	14	–2	–11	4	0

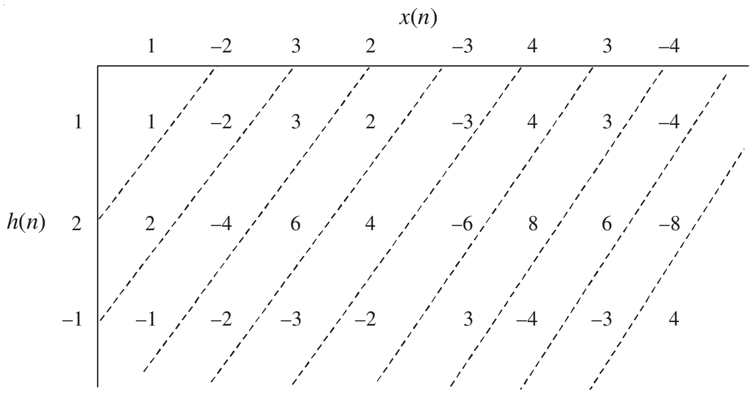
The last zero is discarded because $x(n)$ was padded with one zero.

$$\therefore \qquad y(n) = \{1, 0, -2, 10, -2, -4, 14, -2, -11, 4\}$$

Verification: The direct convolution of $y(n) = x(n) * h(n)$, as shown below, gives

$$y(n) = \{1, 2-2, -1-4+3, 2+6+2, -3+4-3, -2-6+4, 3+8+3, -4+6-4, -3-8, 4\}$$

$$= \{1, 0, -2, 10, -2, -4, 14, -2, -11, 4\}$$



which is the same as obtained by the overlap-add method.

(b) *Overlap-save method*

Given $x(n) = \{1, -2, 3, 2, -3, 4, 3, -4\}$ $\therefore L = 8$

and $h(n) = \{1, 2, -1\}$ $\therefore N = 3$

$$M = 2N = 6$$

Add $N - 1 = 3 - 1 = 2$ leading zeros to the longer sequence $x(n)$

$\therefore$ $x(n) = \{0, 0, 1, -2, 3, 2, -3, 4, 3, -4\}$

If we choose $M = 6$, we get three overlapping sections of $x(n)$ (we need to zero pad the last one) described by

$$x_1(n) = \{0, 0, 1, -2, 3, 2\},\ x_2(n) = \{3, 2, -3, 4, 3, -4\},\ x_3(n) = \{3, -4, 0, 0, 0, 0\}$$

$h(n)$ starts at $n = n_h = 0$

$x_1(n)$ starts at $n = n_1 = -2$, $\therefore y_1(n)$ starts at $n = n_1 + n_h = -2 + 0 = -2$

$x_2(n)$ starts at $n = n_2 = 2$, $\therefore y_2(n)$ starts at $n = n_2 + n_h = 2 + 0 = 2$

$x_3(n)$ starts at $n = n_3 = 6$, $\therefore y_3(n)$ starts at $n = n_3 + n_h = 6 + 0 = 6$

The zero padded $h(n)$ becomes $h(n) = \{1, 2, -1, 0, 0, 0\}$. Periodic convolutions of $x_1(n)$, $x_2(n)$ and $x_3(n)$ with $h(n)$, i.e. $y_1(n)$, $y_2(n)$ and $y_3(n)$ can be computed by any of the methods discussed in Chapter 2. The computation by the matrix method is shown below.

$$\begin{bmatrix} 0 & 2 & 3 & -2 & 1 & 0 \\ 0 & 0 & 2 & 3 & -2 & 1 \\ 1 & 0 & 0 & 2 & 3 & -2 \\ -2 & 1 & 0 & 0 & 2 & 3 \\ 3 & -2 & 1 & 0 & 0 & 2 \\ 2 & 3 & -2 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ -2 \\ 10 \end{bmatrix}, \begin{bmatrix} 3 & -4 & 3 & 4 & -3 & 2 \\ 2 & 3 & -4 & 3 & 4 & -3 \\ -3 & 2 & 3 & -4 & 3 & 4 \\ 4 & -3 & 2 & 3 & -4 & 3 \\ 3 & 4 & -3 & 2 & 3 & -4 \\ -4 & 3 & 4 & -3 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -8 \\ 12 \\ -2 \\ -4 \\ 14 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 0 & 0 & 0 & -4 \\ -4 & 3 & 0 & 0 & 0 & 0 \\ 0 & -4 & 3 & 0 & 0 & 0 \\ 0 & 0 & -4 & 3 & 0 & 0 \\ 0 & 0 & 0 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -11 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

The overall output is obtained by combining $y_1(n)$, $y_2(n)$ and $y_3(n)$ as shown in Table 6.7.

TABLE 6.7 Combining the output of the convolution of each section.

n	–2	–1	0	1	2	3	4	5	6	7	8	9	10	11
$y_1(n)$	1	–2	1	0	–2	10								
$y_2(n)$					–8	12	–2	–4	14	–2				
$y_3(n)$									3	2	–11	4	0	0
$y(n)$			1	0	–2	10	–2	–4	14	–2	–11	4		

$$\therefore \qquad y(n) = \{1, 0, -2, 10, -2, -4, 14, -2, -11, 4\}$$

The result is same as that obtained earlier by the overlap-add method.

SHORT QUESTIONS WITH ANSWERS

1. Define DFT of a discrete-time sequence.
Ans. The DFT of a discrete-time sequence designated by $X(k)$ is defined as:

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi nk}{N}}, \qquad k = 0, 1, 2, \ldots, N-1$$

It is used to convert a finite discrete-time sequence $x(n)$ to an N-point frequency domain sequence $X(k)$.

2. Define IDFT.
Ans. The IDFT of an N-point frequency domain sequence $X(k)$ is defined as:

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi nk}{N}}, \quad n = 0, 1, 2, \ldots, N-1$$

It is used to convert an N-point frequency domain sequence $X(k)$ to a finite time domain sequence $x(n)$.

3. What is the relation between DTFT and DFT?
Ans. The DTFT is a continuous periodic function of ω. The DFT is obtained by sampling DTFT at a finite number of equally spaced points over one period.

4. What is the drawback of discrete-time Fourier transform and how is it overcome?
Ans. The drawback of DTFT is, it is a continuous function of ω and so it cannot be processed by a digital system. DFT is sampled version of DTFT. Since it is in discrete form, it can be processed by a digital system.

5. Give any two applications of DFT.
Ans. Two important applications of DFT are:
1. It allows us to determine the frequency content of a signal, that is, to perform spectral analysis.
2. It is used to perform filtering operation in the frequency domain.

6. When is an N-point periodic sequence said to be even or odd sequence?
Ans. An N-point periodic sequence is called even, if it satisfies the condition,

$$x(n - N) = x(n) \qquad \text{for } 0 \le n \le (N-1)$$

An N-point periodic sequence is called odd, if it satisfies the condition,

$$x(n - N) = -x(n) \qquad \text{for } 0 \le n \le N-1$$

7. What is the relation between Z-transform and DFT?
Ans. The DFT is the Z-transform evaluated at a finite number of equally spaced points along the unit circle centred at the origin of the z-plane.

i.e. $$X(k) = X(z)\Big|_{z=e^{j\frac{2\pi k}{N}}}; \quad k = 0, 1, 2, \ldots, N-1$$

8. What is zero padding? Why it is needed?
Ans. Appending zeros to a sequence in order to increase the size or length of the sequence is called zero padding. For circular convolution, the length of the sequences must be same. If the length of the sequences are different, they can be made equal by zero padding.

9. List the differences between linear convolution and circular convolution.
Ans. The following are the differences between linear convolution and circular convolution:

Linear convolution	*Circular convolution*
1. The length of the input sequences can be different.	1. The length of the input sequences should be same.
2. Zero padding is not required.	2. If the length of input sequences are different, then zero padding is required.
3. The input sequences need not be periodic.	3. At least one of the input sequences should be periodic or extended.
4. The output sequence is non-periodic.	4. The output sequence is periodic. The periodicity is same as that of the input sequence.
5. The length of output sequence will be greater than the length of input sequence.	5. The length of the output sequence is same as that of the input sequence.

10. Why linear convolution is important in digital signal processing (DSP)?
Ans. Linear convolution is important in DSP because the response or output of LTI discrete system for any input $x(n)$ is given by linear convolution of $x(n)$ and the impulse response $h(n)$ of the system. That means, if the impulse response of a system is known, then the response of the system for any input can be determined by convolution operation.

11. Why circular convolution is important in DSP?
Ans. The response of a discrete system is given by linear convolution of input and impulse response. For fast convolution, we use DFT. DFT and FFT play a vital

role in DSP. The DFT supports only circular convolution. So circular convolution is important in DSP.

12. What is sectioned convolution?

Ans. In linear convolution of two sequences, if one of the sequences is very much longer than the other, then it is very difficult to compute the linear convolution using DFT. In such cases, the longer sequence is sectioned into smaller sequences and the linear convolution of each section of the longer sequence with the smaller sequence is computed. The output sequence is obtained by combining all these convolutions.

13. Why sectioned convolution is performed?

Ans. In linear convolution of two sequences, if one of the sequences is very much longer than the other, then it is very difficult to compute the linear convolution using DFT for the following reasons:

1. The entire sequence should be available before convolution can be carried out. This makes long delay in getting the output.
2. Large amount of memory is required to store the sequences.

14. What are the two methods of sectioned convolution?

Ans. The two methods of sectioned convolution are overlap-add method and overlap-save method.

15. Give any two applications of DFT?

Ans. The applications of DFT are as follows:

1. The DFT is used for spectral analysis of signals using a digital computer.
2. The DFT is used to perform filtering operations on signals using digital computer.

16. Why DFT is preferred over DTFT?

Ans. DFT is preferred over DTFT because DFT allows us to perform frequency analysis on a digital computer, whereas DTFT does not.

17. How IDFT can be obtained from DFT?

Ans. To obtain IDFT of the discrete frequencies sequence $X(k)$, find the DFT of the conjugate of $X(k)$, and then conjugate the results and divide by N. Mathematically,

$$x(n) = \text{IDFT}\,[X(k)] = \frac{1}{N}[\text{DFT}\,\{X^*(k)\}]^*.$$

REVIEW QUESTIONS

1. Write the properties of DFT.

2. How do you find IDFT using DFT?

3. State and prove the time reversal property of DFT.

4. State and prove the circular time shift property of the DFT.

5. Prove that DFT $\{x_1(n) \oplus x_2(n)\} = X_1(k)\,X_2(k)$.

6. Discuss the overlap-add method of sectioned convolution.
7. Discuss the overlap-save method of sectioned convolution.

FILL IN THE BLANKS

1. The DTFT is a periodic _______ function of ω with a period of _______.
2. _______ allows us to perform frequency analysis on a digital computer.
3. The _______ is obtained by sampling one period of the Fourier transform $X(\omega)$.
4. The relation between DTFT $X(\omega)$ and DFT $X(k)$ is _______.
5. The DTFT is nothing but the Z-transform evaluated along the _______ centred at the origin of z-plane.
6. The DTFT is nothing but the Z-transform evaluated at a _______ of equally spaced points on the _______ centred at the origin of z-plane.
7. The DFT $X(k)$ of a discrete-time sequence $x(n)$ is defined as _______.
8. The IDFT $x(n)$ of the sequence $X(k)$ is defined as _______.
9. The relation between Z-transform $X(z)$ and DFT $X(k)$ is _______.
10. _______ is known as the twiddle factor.
11. _______ is called the DFT matrix and _______ is called the IDFT matrix.
12. IDFT can be found using DFT by the formula $x(n)$ = _______.
13. The periodicity property of DFT says that _______.
14. The linearity property of DFT says that _______.
15. The DFT of an even sequence $x(n)$ is _______.
16. The DFT of an odd sequence $x(n)$ is _______.
17. The DFT of a time reversed sequence $\{x(-n), \text{mod } N\}$ is _______.
18. The DFT of $\left\{x(n)\, e^{j\frac{2\pi nl}{N}}\right\}$ is _______.
19. The DFT of delayed sequence $x(n - n_0)$ is _______.
20. The DFT of $x_1(n)\, x_2(n)$ is _______.
21. The DFT of $x_1(n) \oplus x_2(n)$ is _______.
22. Parseval's theorem states that $\sum_{n=0}^{N-1} x_1(n)\, x_2^*(n) =$ _______.
23. The DFT of circular correlation of sequences $x(n)$ and $y(n)$ is _______.
24. The central ordinate theorem says that $X(0)$ = _______.

25. The central ordinate theorem says that $X(N/2)$ = ________.
26. The DFT supports only ________ convolution.
27. The technique of convolving two finite duration sequences using DFT techniques is called ________.
28. The convolution of two sequences by convolution sum formula $y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$ is called direct convolution or ________.
29. Convolution of long sequences can be done using ________ convolutions.
30. The two methods of sectioned convolution are ________ method and ________ method.
31. In overlap-add method, the last ________ samples of each output sequence overlaps with the first ________ samples of next section.
32. In overlap-add method, the ________ regions are added and the ________ regions are retained.
33. In overlap-save method, we use ________ convolution.
34. In overlap-save method, we discard the first ________ samples in each convolution and glue the results to give the required convolution.

OBJECTIVE TYPE QUESTIONS

1. DTFT is a periodic function with a period of
   (a) π (b) 0 (c) 2π (d) infinity
2. DFT performs filtering operation in
   (a) time domain (b) frequency domain
   (c) both time and frequency domains (d) none of these
3. The DFT of $x(n)$, i.e. $X(k)$ is defined as $X(k)$ =
   (a) $X(\omega)\big|_{\omega=\frac{2\pi k}{N}}$ (b) $X(\omega)\big|_{\omega=2\pi k}$ (c) $X(\omega)\big|_{\omega=\frac{2\pi n}{N}}$ (d) $X(\omega)\big|_{\omega=2\pi n}$
4. The DTFT is the Z-transform evaluated along the
   (a) imaginary axis of z-plane (b) real axis of z-plane
   (c) unit circle in z-plane (d) entire z-plane
5. DFT $\{x(n)\}$ is given by $X(k)$ =
   (a) $\sum_{n=0}^{N-1} x(n)\,W_N^{nk}$, $k = 0, 1, 2, \ldots, N-1$ (b) $\sum_{n=0}^{N-1} x(n)\,W_N^{-nk}$, $k = 0, 1, 2, \ldots, N-1$
   (c) $\sum_{n=0}^{N-1} x(n)\,W_N^{nk}$, $n = 0, 1, 2, \ldots, N-1$ (d) $\sum_{n=0}^{N-1} x(n)\,W_N^{-nk}$, $n = 0, 1, 2, \ldots, N-1$

6. The IDFT of $X(k)$ is given by $x(n) =$

(a) $\frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{nk}, \quad k = 0, 1, 2, \ldots, N-1$

(b) $\frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-nk}, \quad k = 0, 1, 2, \ldots, N-1$

(c) $\frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{nk}, \quad n = 0, 1, 2, \ldots, N-1$

(d) $\frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-nk}, \quad n = 0, 1, 2, \ldots, N-1$

7. The twiddle factor is $W_N =$

(a) $e^{j\frac{2\pi}{N}}$ (b) $e^{\frac{j\pi}{N}}$ (c) $e^{-j\frac{2\pi}{N}}$ (d) $e^{\frac{-j\pi}{N}}$

8. The DFT of $x(n)$, i.e. $X(k)$ is given by $X(k) =$

(a) $X(z)\big|_{z=e^{-j\frac{2\pi}{N}nk}}$ (b) $X(z)\big|_{z=e^{j\frac{2\pi}{N}nk}}$ (c) $X(z)\big|_{z=0}$ (d) $X(z)\big|_{z=e^{\frac{2\pi nk}{N}}}$

9. DFT $\{\delta(n)\}$ =
(a) 2π (b) π (c) 1 (d) 0

10. The IDFT of $X(k)$ is given by $x(n) =$

(a) $\frac{1}{N}[\text{DFT}\{X^*(k)\}]^*$ (b) $\frac{1}{N}[\text{IDFT}\{X^*(k)\}]^*$

(c) $\frac{1}{N}[\text{DFT}\{X(k)\}]^*$ (d) $\frac{1}{N}[\text{IDFT}\{X(k)\}]^*$

11. DFT $[x_1(n) \oplus x_2(n)]$ =

(a) $\frac{1}{N}[X_1(k)X_2(k)]$ (b) $N[X_1(k)X_2(k)]$

(c) $X_1(k)X_2(k)$ (d) $X_1(k) \oplus X_2(k)$

12. DFT $[x_1(n)\ x_2(n)]$ =

(a) $\frac{1}{N}[X_1(k)X_2(k)]$ (b) $\frac{1}{N}[X_1(k) \oplus X_2(k)]$

(c) $N[X_1(k)X_2(k)]$ (d) $N[X_1(k) \oplus X_2(k)]$

13. Parseval's theorem states that $\sum_{n=0}^{N-1} x_1(n) x_2^*(n) =$

(a) $\frac{1}{N}\sum_{k=0}^{N-1} X_1(k) X_2(k)$ (b) $\frac{1}{N}\sum_{k=0}^{N-1} X_1^*(k) X_2^*(k)$

(c) $\frac{1}{N}\sum_{k=0}^{N-1} X_1(k) X_2^*(k)$ (d) $\frac{1}{N}\sum_{k=0}^{N-1} X_1^*(k) X_2(k)$

14. The circular correlation of $x(n)$ and $y(n)$ is $r_{xy}(l)$ =

(a) $X(k)^* Y(k)$ (b) $X^*(k)Y(k)$ (c) $X^*(k)Y^*(k)$ (d) $X(k)Y(k)$

15. As per central ordinates theorem, if DFT $[x(n)] = X(k)$, then $X(0)$ =

(a) $\sum_{n=0}^{N-1} x(n)$ (b) $\sum_{n=0}^{N-1} x(0)$ (c) $\sum_{k=0}^{N-1} X(k)$ (d) $\sum_{n=0}^{N-1} x(-n)$

16. As per central ordinates theorem, if DFT $[x(n)] = X(k)$, then $X(N/2)$ =

(a) $\sum_{n=0}^{N-1} (-1)^n x(n)$ (b) $\sum_{n=0}^{N-1} x(n/2)$ (c) $\sum_{n=0}^{N/2} x(n)$ (d) $\sum_{n=0}^{(N/2)-1} (-1)^n x(n)$

17. If the DFT of a real signal $x(n)$ is $X(k) = \{1, 2 - j, A, 5, 3 + j2, 2 + j\}$, the value of A is

(a) $3 + j2$ (b) $3 - j2$ (c) $-3 - j2$ (d) $-3 + j2$

18. Let $x(n) = \{0, 1, A, 2, 3, 4\}$. If $X(0) = 10$, A = ?

(a) $A = 0$ (b) $A = 1$ (c) $A = 2$ (d) $A = 3$

19. The DFT of a real signal is $X(k) = \{1, 2 - j, 2, 2 + j\}$. What is its signal energy?

(a) 15 (b) 7 (c) 12 (d) not defined

20. The DFT of a real signal is $X(k) = \{1, A, -2, B, -3, -j2, c, 1 + j\}$. What is its signal energy?

(a) 30 (b) 3.75 (c) 10 (d) 8

PROBLEMS

1. Find the DFT of the sequences (a) $x(n) = \{1, 1, 0, 0\}$, (b) $x(n) = 1/5, \quad \text{for } -1 \le n \le 1.$
$\qquad = 0, \quad \text{otherwise}$

2. Find the DFT of the discrete-time sequence $x(n) = \{1, 1, 2, 2, 3, 3\}$ and determine the corresponding amplitude and phase spectrum.

3. Find the 4-point DFT of the sequence $x(n) = \cos n\pi/4$.

4. Find the IDFT of $X(k) = \{1, 2, 3, 4\}$.

5. Find the DFT of $x(n) = \{0.5, 0, 0.5, 0\}$.

6. Find the IDFT of $X(k) = \{3, 2 + j, 1, 2 - j\}$.

7. Find the DFT of $x(n) = \{2, 0, 0, 1\}$.

8. Find the IDFT of $X(k) = \{2, 1 - j, 0, 1 + j\}$.

9. Find the DFT of $x(n) = \{2.5, 1 - j2, -0.5, 1 + j0.5\}$.

10. Find the IDFT of $X(k) = \{0, -1 - j, 6, -1 + j\}$

11. Find the IDFT of $X(k) = \{4, -2 + j4, 4, -2 - j4\}$.

12. Find the IDFT of $X(k) = \{5, 0.5 + j0.866, 0.5 - j0.866\}$.

13. Let $X(k)$ be a 14-point DFT of a length 14 real sequence $x(n)$. The first 8 samples of $X(k)$ are given by $X(0) = 8$, $X(1) = -2 + j2$, $X(2) = 1 + j3$, $X(3) = 3 - j4$, $X(4) = -1 - j5$, $X(5) = 3 + j6$, $X(6) = -4 - j3$, $X(7) = 5$. Determine the remaining samples of $X(k)$.

14. The DFT of a real signal is $\{2, 1 - j3, A, 2 + j1, 0, B, 3 - j5, C\}$. Find A, B and C.

15. Let $x(n) = \{2, A, 3, 0, 4, 0, B, 5\}$. If $X(0) = 18$ and $X(4) = 0$, find A and B.

16. The DFT of a real signal is $X(k) = \{1, A, -2, B, -5, j3, C, 2 - j\}$. What is its signal energy?

17. Consider the length-4 sequence $x(n) = \{2, -4, 6, 1\}$ defined for $0 \le n < 4$ with a 4-point DFT $X(k)$. Evaluate the following functions of $X(k)$ without computing DFT:

(a) $X(0)$ (b) $X(2)$

(c) $\sum_{k=0}^{3} X(k)$ (d) $\sum_{k=0}^{3} |X(k)|^2$

18. If the DFT $\{x(n)\} = X(k) = \{2, -j3, 0, j3\}$, using properties of DFT, find

(a) DFT of $x(n-2)$ (b) DFT of $x(-n)$

(c) DFT of $x^*(n)$ (d) DFT of $x^2(n)$

(e) DFT of $x(n) \oplus x(n)$ (f) Signal energy

19. If IDFT $\{X(k)\} = x(n) = \{2, 1, 2, 0\}$, using properties of DFT, find

(a) IDFT $\{X(k-1)\}$ (b) IDFT $\{X(k) \oplus X(k)\}$

(c) IDFT $\{X(k)\, X(k)\}$ (d) Signal energy

20. Find the linear convolution of the following sequences using DFT:

(a) $x(n) = \{1, -2, 4\}$, $h(n) = \{2, 1, 2, 1\}$

(b) $x(n) = \{2, 3, 4\}$, $h(n) = \{3, 7, 0, 5\}$

(c) $x(n) = \{1, 2, 1\}$, $h(n) = \{2, 0, 1\}$

21. Find the circular convolution of the following sequences using DFT:

(a) $x(n) = \{1, -1, 1, -1\}$, $h(n) = \{1, 2, 3, 4\}$

(b) $x(n) = \{1, 2, 0, 1\}$, $h(n) = \{2, 2, 1, 1\}$

(c) $x(n) = \{1, 2, 1, 2\}$, $h(n) = \{4, 3, 2, 1\}$

22. Find the linear convolution of the following sequences using (a) overlap-add method, and (b) overlap-save method:

(a) $x(n) = \{4, 4, 3, 3, 2, 2, 1, 1\}$, $h(n) = \{-1, 1\}$

(b) $x(n) = \{1, 3, 2, 4, 4, 2, 3, 1\}$, $h(n) = \{1, -1, 1\}$

(c) $x(n) = \{2, -3, 1, -4, 3, -2, 4, -1\}$, $h(n) = \{2, -1\}$

(d) $x(n) = \{5, 0, 4, 0, 3, 0, 2, 0\}$, $h(n) = \{1, 0, 1\}$

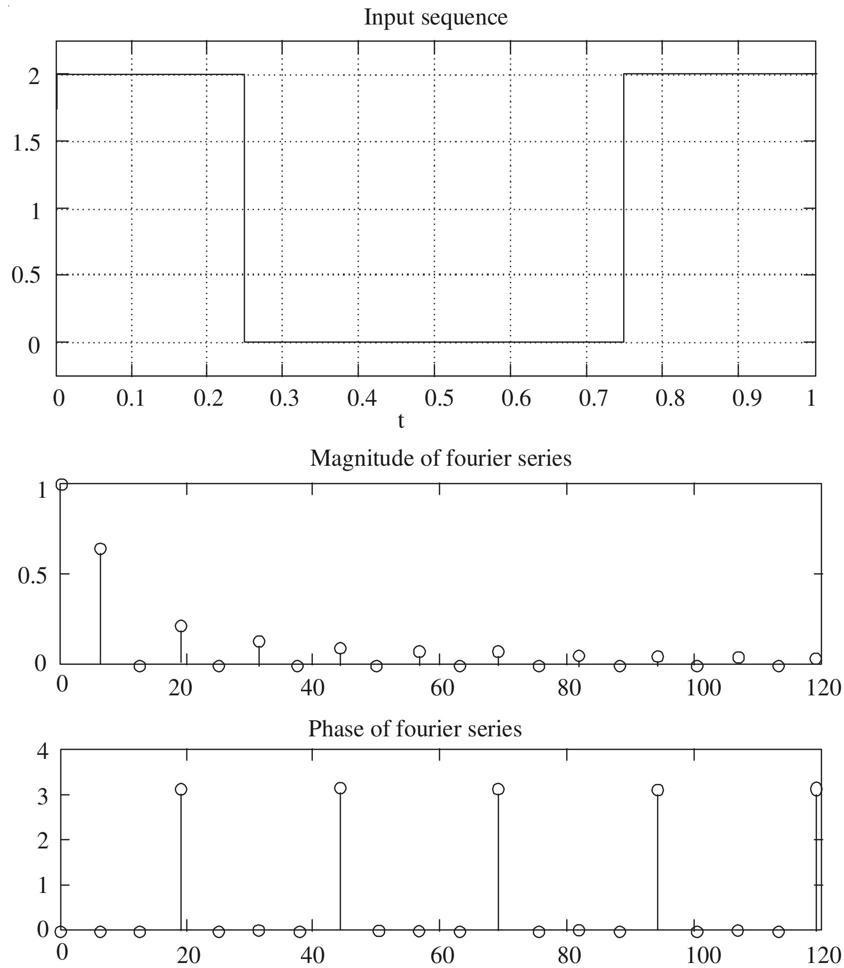
MATLAB PROGRAMS

Program 6.1

% Fourier series representation of a train of pulses

```
clc; clear all; close all;
syms t % symbolic Fourier Series computation
T0 = 1; % T0: period
m = heaviside(t)–heaviside(t–T0/4)+ heaviside(t–3*T0/4);
x = 2 * m;  %  periodic signal
% [X,w] = fourierseries(x,T0,20);  % X,w: Fourier series coefficients at harmonic
frequencies
N=20;  % N: number of harmonics
% computation of N Fourier series coefficients
for k = 1:N,
X1(k) = int(x*exp(–j* 2 *pi*(k–1)*t/T0), t, 0, T0)/T0;
X(k) = subs(X1(k));
w(k) = (k–1)*2*pi/T0; % harmonic frequencies
end
ezplot(x,[0 T0]),grid;
title('input sequence')
figure
subplot(2,1,1), stem(w,abs(X));
title('magnitude of fourier series')
subplot(2,1,2), stem(w,angle(X));
title('phase of fourier series')
```

Output:



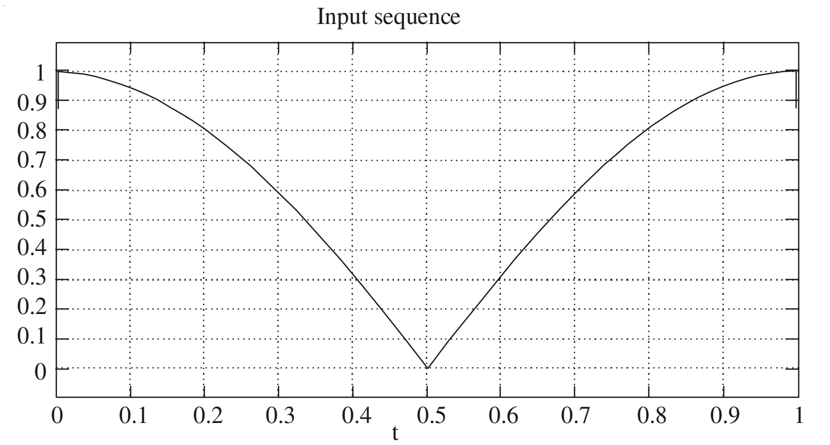
Program 6.2

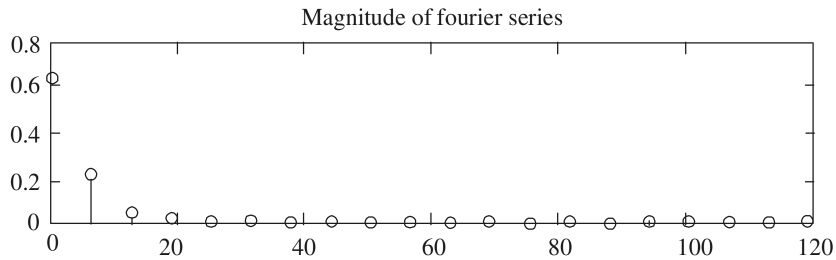
% Fourier series representation of a full wave rectified wave

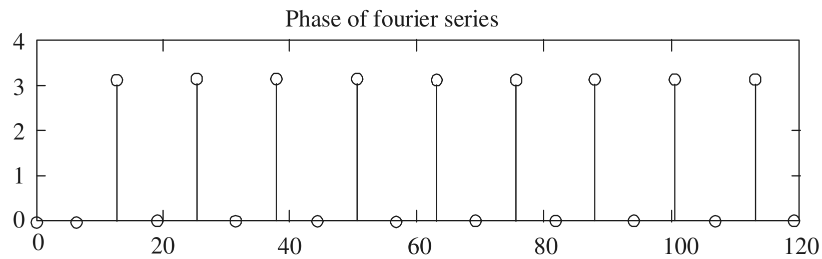
```
clc; clear all; close all;
syms t
T0 = 1;
m = heaviside(t)–heaviside(t– T0);
x = abs(cos(pi*t))* m;
N=20;
% computation of N Fourier series coefficients
for k = 1:N,
X1(k) = int(x*exp(–j* 2 *pi*(k–1)*t/T0), t, 0, T0)/T0;
```

```
X(k) = subs(X1(k));
w(k) = (k–1)*2*pi/T0;   % harmonic frequencies
end
ezplot(x,[0 T0]);grid;
title('input sequence')
figure
subplot(2,1,1);stem(w,abs(X));
title('magnitude of fourier series')
subplot(2,1,2); stem(w,angle(X));
title('phase of fourier series')
```

Output:







Program 6.3

```
% Direct computation of Discrete Fourier transform (matrix formulation)
clc; clear all; close all;
x=[1 –1 2 –2];
l=length(x);
y=x*dftmtx(l);
disp('the discrete fourier transform of the input sequence is')
disp(y)
l1=length(y);
y1=y*conj(dftmtx(l1))/l1;
disp('the Inverse discrete fourier transform of the input sequence is')
disp(y1)
```

Output:

```
the discrete fourier transform the input sequence is
  0              –1.0000 – 1.0000i    6.0000              –1.0000 + 1.0000i

the Inverse discrete fourier transform the input sequence is
     1    –1     2    –2
```

Program 6.4

```
% Linear convolution using DFT

clc; clear all; close all;
x=[1 2];
h=[2 1];
x1=[x zeros(1,length(h)–1)];
h1=[h zeros(1,length(x)–1)];
X=fft(x1);
H=fft(h1);
y=X.*H;
y1=ifft(y);
disp('the linear convolution of the given sequence')
disp(y1)
```

Output:

```
the linear convolution of the given sequence
     2     5     2
```

Program 6.5

% Circular convolution using DFT

```
clc; clear all; close all;
x=[1 2 1 2];
h=[4 3 2 1];
X=fft(x);
H=fft(h);
y=X.*H;
y1=real(ifft(y));
disp('the circular convolution of the given sequence')
disp(y1)
```

Output:

```
the circular convolution of the given sequence
    14    16    14    16
```

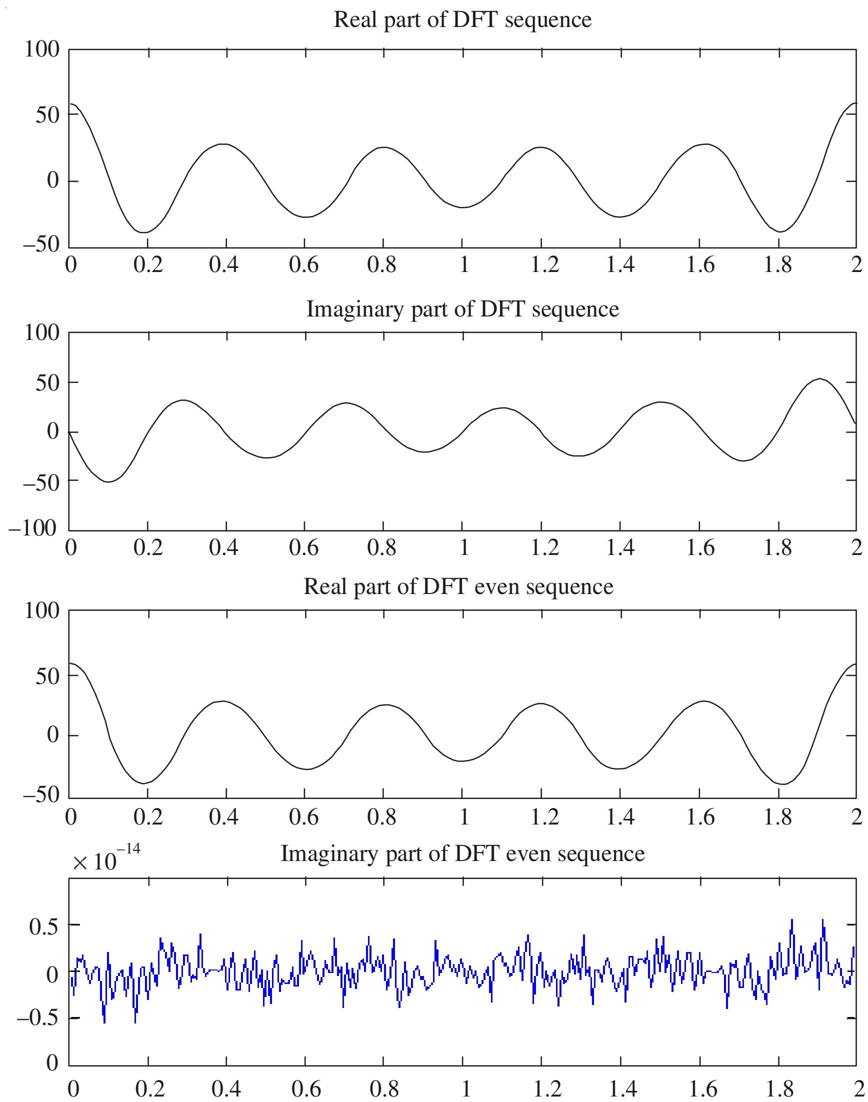
Program 6.6

% Relation between DFTs of the periodic even and odd parts of a real sequence

```
clc; clear all; close all;
x=[1 2 4 2 6 32 6 4 2 zeros(1,247)];
x1=[x(1) x(256:–1:2)];
xe=0.5*(x+x1);
xf=fft(x);
xef=fft(xe);
k=0:255;
subplot(2,1,1),plot(k/128,real(xf));
title('real part of DFT sequence')
```

```
subplot(2,1,2),plot(k/128,imag(xf));
title('imaginary part of DFT sequence')
figure
subplot(2,1,1), plot(k/128,real(xef));
title('real part of DFT even sequence')
subplot(2,1,2), plot(k/128,imag(xef);
title('imaginary part of DFT evensequence')
```

Output:



Program 6.7

```
% Parseval's relation of DFT
% sum(x(t)^2)=(1/N)*(sum(x(w)).^2)

clc; clear all; close all;
x=[(1:128) (128:–1:1)];
y=fft(x);
y1=sum(x.*x);
y2=length(x);
y3=abs(y);
y4=[sum(y3.*y3)]/y2;
y5=y1–y4;
disp('energy in time domain')
disp(y1)
disp('energy in Frequency domain')
disp(y4)
disp('error')
disp(y5)
```

Output:

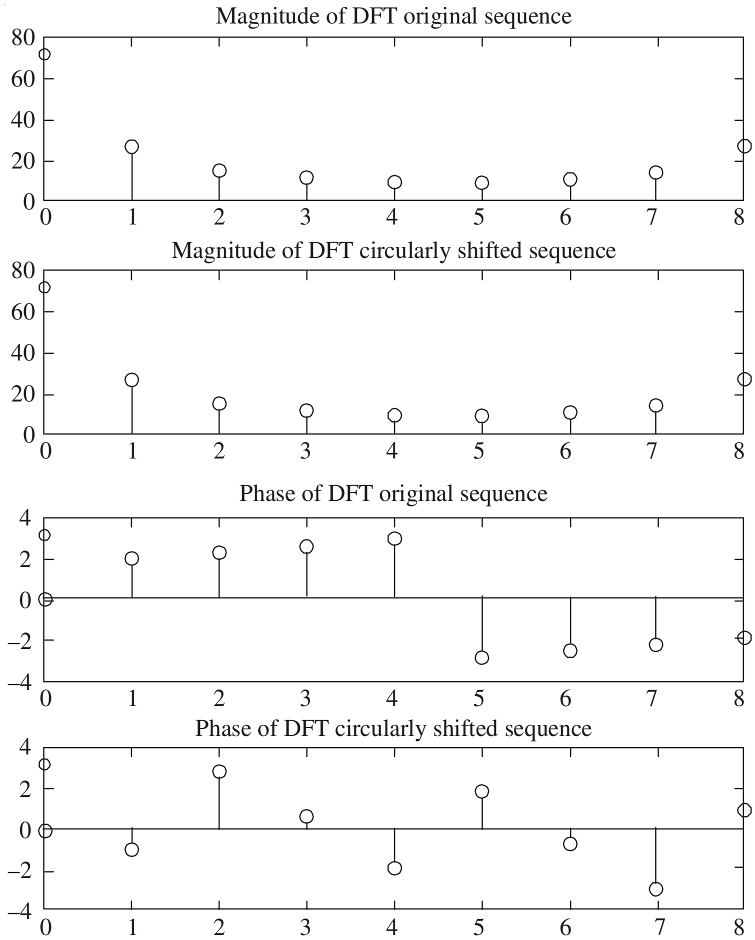
```
energy in time domain
    1414528

energy in Frequency domain
     1.4145e+06

error
      4.6566e–10
```

Program 6.8

% Circular time shifting property of DFT

```
clc; clear all; close all;
x=[0 2 4 6 8 10 12 14 16];
N=length(x)–1;
n=0:N;
M=5 ;%Samples
if abs(M)> length(x)
    M=rem(M,length(x));
end
if M<0
    M=M+length(x);
end
y=[x(M+1:length(x)) x(1:M)];
xf=fft(x);
yf=fft(y);
subplot(2,1,1),stem(n,abs(xf));
title('Magnitude of DFT original sequence')
subplot(2,1,2),stem(n,abs(yf));
title('Magnitude of DFT Circularly shifted sequence')
figure
subplot(2,1,1),stem(n,angle(xf));
title('Phase of DFT original sequence')
subplot(2,1,2),stem(n,angle(yf));
title('phase of DFT Circularly shifted sequence')
```

Output:



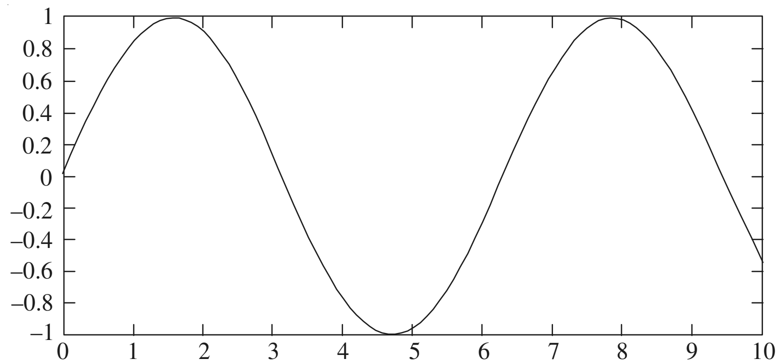
Program 6.9

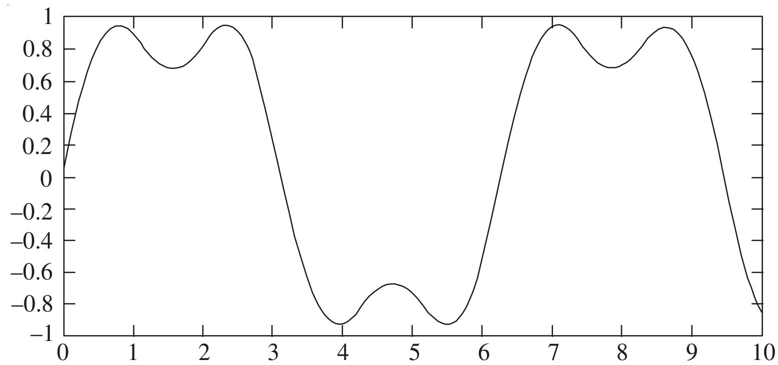
% Gibbs phenomenon

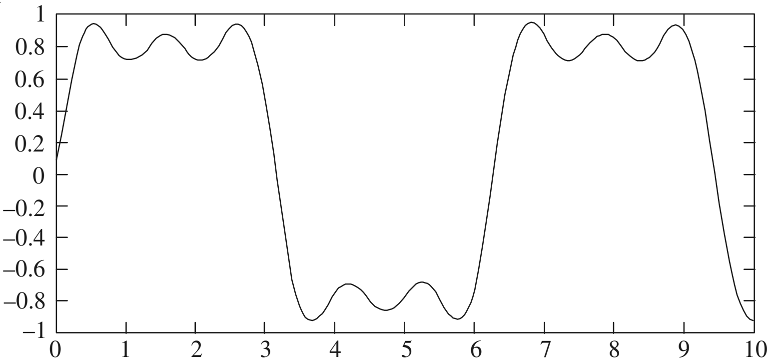
```
clc; clear all; close all;
t = 0:.1:10;
y1= sin(t);
plot(t,y1)
y2 = sin(t) + sin(3*t)/3;
figure;plot(t,y2);
y3 = sin(t) + sin(3*t)/3 + sin(5*t)/5 ;
```

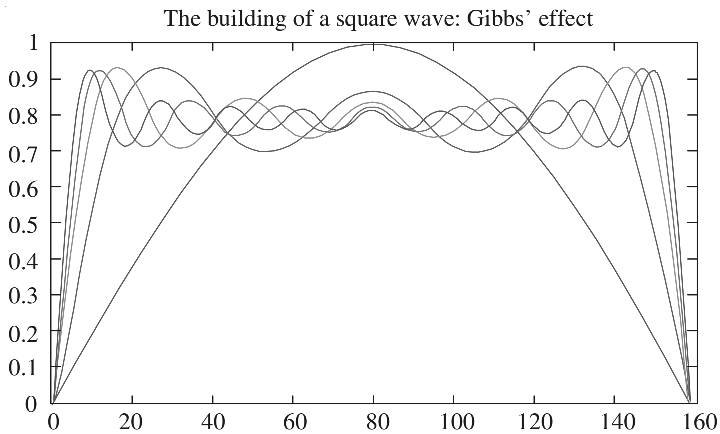
```
figure;plot(t,y3);
y4= sin(t) + sin(3*t)/3 + sin(5*t)/5 + sin(7*t)/7;
figure;plot(t,y4);
t = 0:.02:3.14;
y = zeros(10,length(t));
x = zeros(size(t));
for k=1:2:19
    x = x + sin(k*t)/k;
    y5((k+1)/2,:) = x;
end
figure;plot(y5(1:2:9,:)');
title('The building of a square wave: Gibbs'' effect')
```

Output:






1
0.8
0.6
0.4
0.2
0
–0.2
–0.4
–0.6
–0.8
–1
0 1 2 3 4 5 6 7 8 9 10


The building of a square wave: Gibbs' effect
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0 20 40 60 80 100 120 140 160

7

Fast Fourier Transform

7.1 INTRODUCTION

The N-point DFT of a sequence $x(n)$ converts the time domain N-point sequence $x(n)$ to a frequency domain N-point sequence $X(k)$. The direct computation of an N-point DFT requires $N \times N$ complex multiplications and $N(N-1)$ complex additions. Many methods were developed for reducing the number of calculations involved. The most popular of these is the Fast Fourier Transform (FFT), a method developed by Cooley and Turkey. The FFT may be defined as an algorithm (or a method) for computing the DFT efficiently (with reduced number of calculations). The computational efficiency is achieved by adopting a divide and conquer approach. This approach is based on the decomposition of an N-point DFT into successively smaller DFTs and then combining them to give the total transform. Based on this basic approach, a family of computational algorithms were developed and they are collectively known as FFT algorithms. Basically there are two FFT algorithms; Decimation-in-time (DIT) FFT algorithm and Decimation-in-frequency (DIF) FFT algorithm. In this chapter, we discuss DIT FFT and DIF FFT algorithms and the computation of DFT by these methods.

7.2 FAST FOURIER TRANSFORM

The DFT of a sequence $x(n)$ of length N is expressed by a complex-valued sequence $X(k)$ as

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N}, \qquad k = 0, 1, 2, \ldots, N-1$$

Let W_N be the complex valued phase factor, which is an Nth root of unity given by

$$W_N = e^{-j2\pi/N}$$

Thus, $X(k)$ becomes

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}, \quad k = 0, 1, 2, \ldots, N-1$$

Similarly, IDFT is written as

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-nk}, \quad n = 0, 1, 2, \ldots, N-1$$

From the above equations for $X(k)$ and $x(n)$, it is clear that for each value of k, the direct computation of $X(k)$ involves N complex multiplications ($4N$ real multiplications) and $N - 1$ complex additions ($4N - 2$ real additions). Therefore, to compute all N values of DFT, N^2 complex multiplications and $N(N - 1)$ complex additions are required. In fact the DFT and IDFT involve the same type of computations.

If $x(n)$ is a complex-valued sequence, then the N-point DFT given in equation for $X(k)$ can be expressed as

$$X(k) = X_R(k) + jX_I(k)$$

or
$$X(k) = \sum_{n=0}^{N-1} \left[x_R(n) + jx_I(n)\right]\left[\cos\frac{2\pi nk}{N} - j\sin\frac{2\pi nk}{N}\right]$$

Equating the real and imaginary parts of the above two equations for $X(k)$ we have

$$X_R(k) = \sum_{n=0}^{N-1} \left[x_R(n)\cos\frac{2\pi nk}{N} + x_I(n)\sin\frac{2\pi nk}{N}\right]$$

$$X_I(k) = -\sum_{n=0}^{N-1} \left[x_R(n)\sin\frac{2\pi nk}{N} - x_I(n)\cos\frac{2\pi nk}{N}\right]$$

The direct computation of the DFT needs $2N^2$ evaluations of trigonometric functions, $4N^2$ real multiplications and $4N(N-1)$ real additions. Also this is primarily inefficient as it cannot exploit the symmetry and periodicity properties of the phase factor W_N, which are

Symmetry property $W_N^{k+\frac{N}{2}} = -W_N^k$

Periodicity property $W_N^{k+N} = W_N^k$

FFT algorithm exploits the above two symmetry properties and so is an efficient algorithm for DFT computation.

By adopting a divide and conquer approach, a computationally efficient algorithm can be developed. This approach depends on the decomposition of an N-point DFT into successively smaller size DFTs. An N-point sequence, if N can be expressed as $N = r_1 r_2 r_3, \ldots, r_m$

where $r_1 = r_2 = r_3 = ... = r_m$, then $N = r^m$, can be decimated into r-point sequences. For each r-point sequence, r-point DFT can be computed. Hence the DFT is of size r. The number r is called the radix of the FFT algorithm and the number m indicates the number of stages in computation. From the results of r-point DFT, the r^2-point DFTs are computed. From the results of r^2-point DFTs, the r^3-point DFTs are computed and so on, until we get r^m-point DFT. If $r = 2$, it is called radix-2 FFT.

7.3 DECIMATION IN TIME (DIT) RADIX-2 FFT

In Decimation in time (DIT) algorithm, the time domain sequence $x(n)$ is decimated and smaller point DFTs are computed and they are combined to get the result of N-point DFT.

In general, we can say that, in DIT algorithm the N-point DFT can be realized from two numbers of $N/2$-point DFTs, the $N/2$-point DFT can be realized from two numbers of $N/4$-point DFTs, and so on.

In DIT radix-2 FFT, the N-point time domain sequence is decimated into 2-point sequences and the 2-point DFT for each decimated sequence is computed. From the results of 2-point DFTs, the 4-point DFTs, from the results of 4-point DFTs, the 8-point DFTs and so on are computed until we get N-point DFT.

For performing radix-2 FFT, the value of r should be such that, $N = 2^m$. Here, the decimation can be performed m times, where $m = \log_2 N$. In direct computation of N-point DFT, the total number of complex additions are $N(N-1)$ and the total number of complex multiplications are N^2. In radix-2 FFT, the total number of complex additions are reduced to $N \log_2 N$ and the total number of complex multiplications are reduced to $(N/2) \log_2 N$.

Let $x(n)$ be an N-sample sequence, where N is a power of 2. Decimate or break this sequence into two sequences $f_1(n)$ and $f_2(n)$ of length $N/2$, one composed of the even indexed values of $x(n)$ and the other of odd indexed values of $x(n)$.

Given sequence $x(n)$: $x(0), x(1), x(2), \ldots, x\left(\frac{N}{2}-1\right), \ldots, x(N-1)$

Even indexed sequence $f_1(n) = x(2n)$: $x(0), x(2), x(4), \ldots, x(N-2)$

Odd indexed sequence $f_2(n) = x(2n+1)$: $x(1), x(3), x(5), \ldots, x(N-1)$

We know that the transform $X(k)$ of the N-point sequence $x(n)$ is given by

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}, \qquad k = 0, 1, 2, \ldots, N-1$$

Breaking the sum into two parts, one for the even and one for the odd indexed values, gives

$$X(k) = \sum_{n \text{ even}}^{N-2} x(n)\, W_N^{nk} + \sum_{n \text{ odd}}^{N-1} x(n)\, W_N^{nk}, \qquad k = 0, 1, 2, \ldots, N-1$$

When n is replaced by $2n$, the even numbered samples are selected and when n is replaced by $2n + 1$, the odd numbered samples are selected. Hence,

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(2n)W_N^{k(2n)} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1)\,W_N^{k(2n+1)}$$

Rearranging each part of $X(k)$ into $(N/2)$-point transforms using

$$W_N^{2nk} = (W_N^2)^{nk} = \left(e^{-j\frac{2\pi}{N}}\right)^{2nk} = \left(e^{-j\frac{2\pi}{N/2}}\right)^{nk} = W_{N/2}^{nk}$$

and $W_N^{(2n+1)k} = W_N^k\,W_{N/2}^{nk}$, we can write

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} f_1(n)\,W_{N/2}^{nk} + W_N^k \sum_{n=0}^{\frac{N}{2}-1} f_2(n)\,W_{N/2}^{nk}$$

By definition of DFT, the $N/2$-point DFT of $f_1(n)$ and $f_2(n)$ is given by

$$F_1(k) = \sum_{n=0}^{\frac{N}{2}-1} f_1(n)W_{N/2}^{kn} \quad \text{and} \quad F_2(k) = \sum_{n=0}^{\frac{N}{2}-1} f_2(n)W_{N/2}^{kn}$$

$$\therefore \qquad X(k) = F_1(k) + W_N^k F_2(k), \qquad k = 0, 1, 2, \ldots, N-1$$

The implementation of this equation for $X(k)$ is shown in Figure 7.1.

This first step in the decomposition breaks the N-point transform into two $(N/2)$-point transforms and the W_N^k provides the N-point combining algebra.

The DFT of a sequence is periodic with period given by the number of points of DFT. Hence, $F_1(k)$ and $F_2(k)$ will be periodic with period $N/2$.

$$\therefore \qquad F_1\left(k + \frac{N}{2}\right) = F_1(k) \quad \text{and} \quad F_2\left(k + \frac{N}{2}\right) = F_2(k)$$

In addition, the phase factor $W_N^{\left(k + \frac{N}{2}\right)} = -W_N^k$.

Therefore, for $k \geq N/2$, $X(k)$ is given by

$$X(k) = F_1\left(k - \frac{N}{2}\right) - W_N^k F_2\left(k - \frac{N}{2}\right)$$

The implementation using the periodicity property is also shown in Figure 7.1.

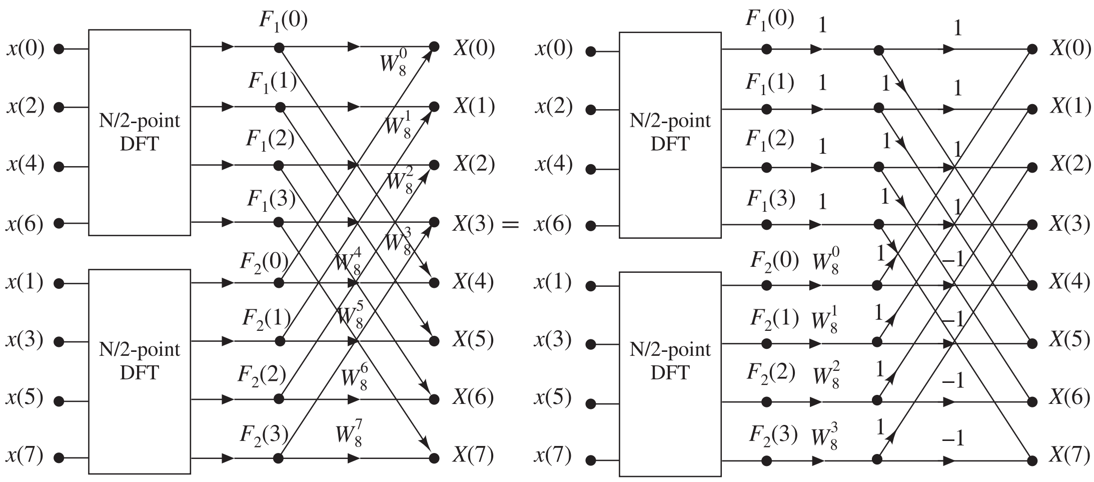


Figure 7.1 Illustration of flow graph of the first stage DIT FFT algorithm for $N = 8$.

The total number of complex multiplications, η_1, required to evaluate the N-point transform with this first decimation becomes $N + N^2/2$ determined as follows

$$\eta_1 = \left(\frac{N}{2}\right)^2 + \left(\frac{N}{2}\right)^2 + N = N + \frac{N^2}{2}$$

The first term in the sum is the number of complex multiplications for the direct calculation of the $(N/2)$-point DFT of the even indexed sequence; the second term is the number of complex multiplications for the direct calculation of the $(N/2)$-point DFT of the odd indexed sequence; and the third term is the number of complex multiplications required for the combining algebra.

Having performed the decimation in time once, we can repeat the process for each of the sequences $f_1(n)$ and $f_2(n)$. Thus $f_1(n)$ would result in two $(N/4)$-point sequences and $f_2(n)$ would result in another two $(N/4)$-point sequences.

Let the decimated $(N/4)$-point sequences of $f_1(n)$ be $g_{11}(n)$ and $g_{12}(n)$.

$\therefore$ $$g_{11}(n) = f_1(2n); \quad \text{for } n = 0, 1, 2, \ldots, \left(\frac{N}{4} - 1\right)$$

and $$g_{12}(n) = f_1(2n+1); \quad \text{for } n = 0, 1, 2, \ldots, \left(\frac{N}{4} - 1\right)$$

Let the decimated $(N/4)$-point sequences of $f_2(n)$ be $g_{21}(n)$ and $g_{22}(n)$.

$\therefore$ $$g_{21}(n) = f_2(2n); \quad \text{for } n = 0, 1, 2, \ldots, \left(\frac{N}{4} - 1\right)$$

and $$g_{22}(n) = f_2(2n+1); \quad \text{for } n = 0, 1, 2, \ldots, \left(\frac{N}{4} - 1\right)$$

Let $G_{11}(k) = N/4$-point DFT of $g_{11}(n)$ | $G_{21}(k) = N/4$-point DFT of $g_{21}(n)$

$G_{12}(k) = N/4$-point DFT of $g_{12}(n)$ | $G_{22}(k) = N/4$-point DFT of $g_{22}(n)$

Then like earlier analysis, we can show that

$$F_1(k) = G_{11}(k) + W_{N/2}^k\, G_{12}(k);\quad \text{for } k = 0, 1, 2, ..., \left(\frac{N}{2} - 1\right)$$

$$F_2(k) = G_{21}(k) + W_{N/2}^k\, G_{22}(k);\quad \text{for } k = 0, 1, 2, ..., \left(\frac{N}{2} - 1\right)$$

Hence the $(N/2)$-point DFTs are obtained from the $(N/4)$-point DFTs.

The implementation of these equations for $F_1(k)$ and $F_2(k)$ is shown in Figure 7.2.

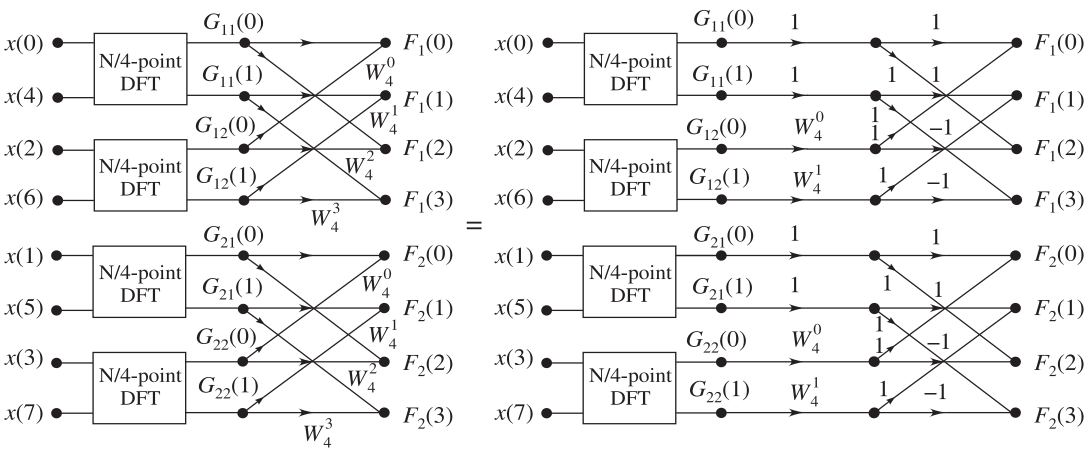


Figure 7.2 Illustration of flow graph of the second stage DIT FFT algorithm for $N = 8$.

This second step in the decomposition breaks the $N/2$-point transforms into $N/4$-point transforms and $W_{N/2}^k$ provides the $N/2$-point combining algebra. Since the DFT of a sequence is periodic with period given by the number of points of DFT, $G_{11}(k)$, $G_{12}(k)$, $G_{21}(k)$ and $G_{22}(k)$ will be periodic with period $N/4$.

$$\therefore \qquad G_{11}\left(k + \frac{N}{4}\right) = G_{11}(k) \quad \text{and} \quad G_{12}\left(k + \frac{N}{4}\right) = G_{12}(k)$$

$$G_{21}\left(k + \frac{N}{4}\right) = G_{21}(k) \quad \text{and} \quad G_{22}\left(k + \frac{N}{4}\right) = G_{22}(k)$$

In addition, the phase factor $W_N^{\left(k+\frac{N}{4}\right)} = -W_{N/2}^k$

Therefore, for $k \geq N/4$, $F_1(k)$ and $F_2(k)$ are given by

$$F_1(k) = G_{11}\left(k - \frac{N}{4}\right) - W_{N/2}^{k}\, G_{12}\left(k - \frac{N}{4}\right)$$

$$F_2(k) = G_{21}\left(k - \frac{N}{4}\right) - W_{N/2}^{k}\, G_{21}\left(k - \frac{N}{4}\right)$$

The implementation using periodicity property is also shown in Figure 7.2.

This decimation of the data sequences is repeated again and again until the resulting sequences are reduced to two point sequences.

Figure 7.3 shows the last stage without and with the consideration of periodicity.

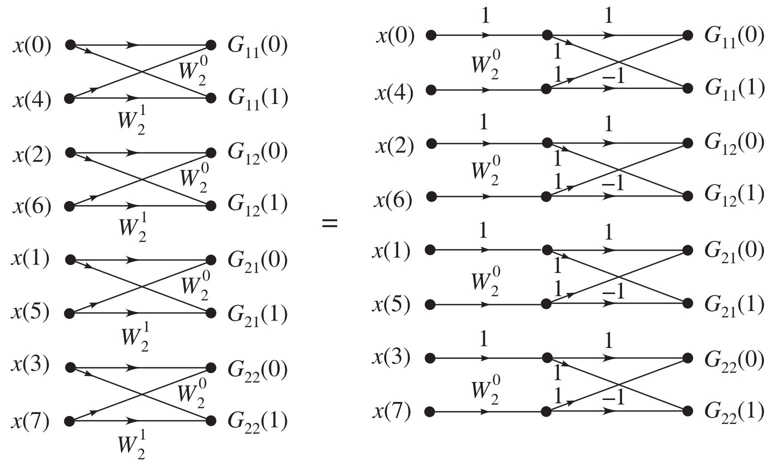


Figure 7.3 Illustration of flow graph of the third stage DIT FFT algorithm for $N = 8$.

Figure 7.4 illustrates the complete flow graph for 3 stages for $N = 8$. The same graph drawn in a way to remember easily considering the symmetry is shown in Figure 7.8.

7.4 THE 8-POINT DFT USING RADIX-2 DIT FFT

The computation of 8-point DFT using radix-2 FFT involves three stages of computation. Here $N = 8 = 2^3$, therefore, $r = 2$ and $m = 3$. The given 8-point sequence is decimated into four 2-point sequences. For each 2-point sequence, the two point DFT is computed. From the results of four 2-point DFTs, two 4-point DFTs are obtained and from the results of two 4-point DFTs, the 8-point DFT is obtained.

Let the given 8-sample sequence $x(n)$ be $\{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$. The 8-samples should be decimated into sequences of two samples. Before decimation they are arranged in bit reversed order as shown in Table 7.1.

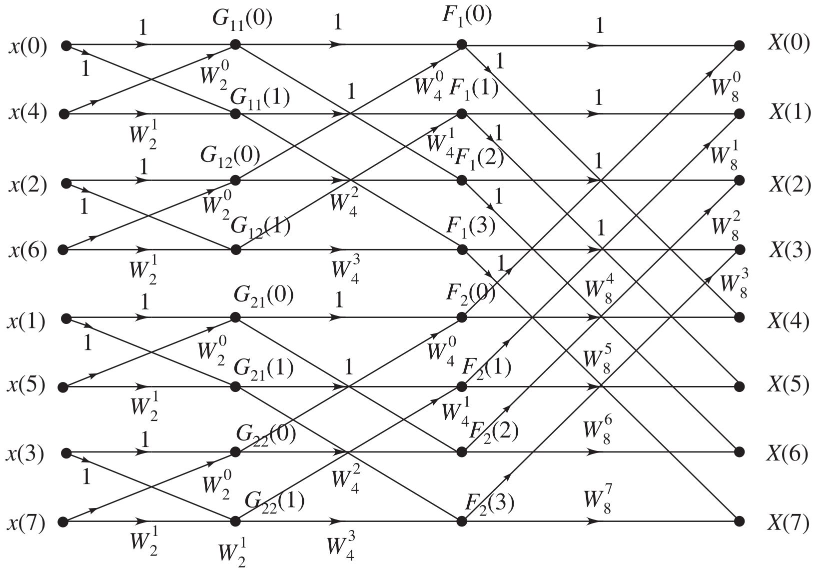


Figure 7.4 Illustration of complete flow graph obtained by combining all the three stages for $N = 8$.

TABLE 7.1 Normal and bit reversed order for $N = 8$.

Normal order		*Bit reversed order*	
$x(0)$	$x(000)$	$x(0)$	$x(000)$
$x(1)$	$x(001)$	$x(4)$	$x(100)$
$x(2)$	$x(010)$	$x(2)$	$x(010)$
$x(3)$	$x(011)$	$x(6)$	$x(110)$
$x(4)$	$x(100)$	$x(1)$	$x(001)$
$x(5)$	$x(101)$	$x(5)$	$x(101)$
$x(6)$	$x(110)$	$x(3)$	$x(011)$
$x(7)$	$x(111)$	$x(7)$	$x(111)$

The $x(n)$ in bit reversed order is decimated into 4 numbers of 2-point sequences as shown below.

(i) $x(0)$ and $x(4)$
(ii) $x(2)$ and $x(6)$
(iii) $x(1)$ and $x(5)$
(iv) $x(3)$ and $x(7)$

Using the decimated sequences as input, the 8-point DFT is computed. Figure 7.5 shows the three stages of computation of an 8-point DFT.

The computation of 8-point DFT of an 8-point sequence in detail is given below. The 8-point sequence is decimated into 4-point sequences and 2-point sequences as shown below.

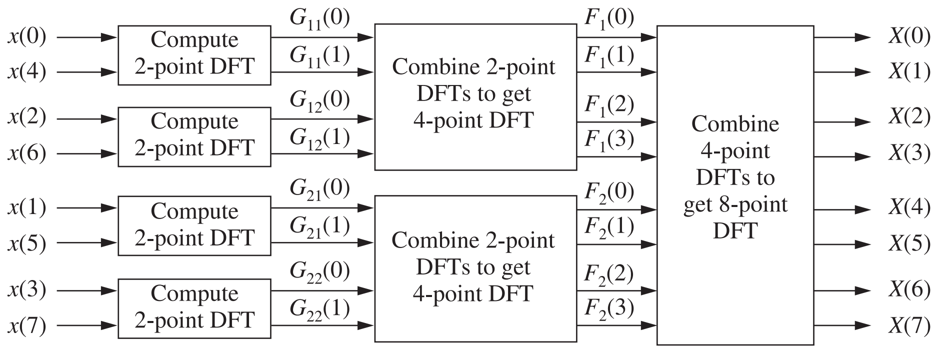


Figure 7.5 Three stages of computation in 8-point DFT.

Let $x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$ be the 8-point sequence.

$f_1(n) = \{x(0), x(2), x(4), x(6)\}$, $f_2(n) = \{x(1), x(3), x(5), x(7)\}$ 4-point sequences obtained from $x(n)$.

$g_{11}(n) = \{x(0), x(4)\}$, $g_{12}(n) = \{x(2), x(6)\}$, 2-point sequences obtained from $f_1(n)$

$g_{21}(n) = \{x(1), x(5)\}$, $g_{22}(n) = \{x(3), x(7)\}$, 2-point sequences obtained from $f_2(n)$

The relations between the samples of various sequences are given below.

$$g_{11}(0) = f_1(0) = x(0) \qquad g_{21}(0) = f_2(0) = x(1)$$
$$g_{11}(1) = f_1(2) = x(4) \qquad g_{21}(1) = f_2(2) = x(5)$$
$$g_{12}(0) = f_1(1) = x(2) \qquad g_{22}(0) = f_2(1) = x(3)$$
$$g_{12}(1) = f_1(3) = x(6) \qquad g_{22}(1) = f_2(3) = x(7)$$

The first stage of computation

In the first stage of computation, 2-point DFTs of the 2-sample sequences are computed.

Let $G_{11}(k) = \text{DFT}\{g_{11}(n)\}$. The 2-point DFT of $g_{11}(n)$ is given by

$$G_{11}(k) = \sum_{n=0}^{1} g_{11}(n) W_{N/4}^{nk}; \quad \text{for } k = 0, 1$$

$$\therefore \quad G_{11}(0) = g_{11}(0) W_2^0 + g_{11}(1) W_2^0 = g_{11}(0) + g_{11}(1) = x(0) + x(4) = x(0) + x(4) W_2^0$$

$$G_{11}(1) = g_{11}(0) W_2^0 + g_{11}(1) W_2^1 = g_{11}(0) - g_{11}(1) = x(0) - x(4) = x(0) - x(4) W_2^0$$

Let $G_{12}(k) = \text{DFT}\{g_{12}(n)\}$. The 2-point DFT of $g_{12}(n)$ is given by

$$G_{12}(k) = \sum_{n=0}^{1} g_{12}(n) W_{N/4}^{nk}; \quad \text{for } k = 0, 1$$

$$\therefore \quad G_{12}(0) = g_{12}(0) W_2^0 + g_{12}(1) W_2^0 = g_{12}(0) + g_{12}(1) = x(2) + x(6) = x(2) + x(6) W_2^0$$

$$G_{12}(1) = g_{12}(0) W_2^0 + g_{12}(1) W_2^1 = g_{12}(0) - g_{12}(1) = x(2) - x(6) = x(2) - x(6) W_2^0$$

Let $G_{21}(k)$ = DFT$\{g_{21}(n)\}$. The 2-point DFT of $g_{21}(n)$ is given by

$$G_{21}(k) = \sum_{n=0}^{1} g_{21}(n) W_{N/4}^{nk}; \quad \text{for } k = 0, 1$$

$$\therefore \quad G_{21}(0) = g_{21}(0) W_2^0 + g_{21}(1) W_2^0 = g_{21}(0) + g_{21}(1) = x(1) + x(5) = x(1) + x(5) W_2^0$$

$$G_{21}(1) = g_{21}(0) W_2^0 + g_{21}(1) W_2^1 = g_{21}(0) - g_{21}(1) = x(1) - x(5) = x(1) - x(5) W_2^0$$

Let $G_{22}(k)$ = DFT$\{g_{22}(n)\}$. The 2-point DFT of $g_{22}(n)$ is given by

$$G_{22}(k) = \sum_{n=0}^{1} g_{22}(n) W_{N/4}^{nk}; \quad \text{for } k = 0, 1$$

$$\therefore \quad G_{22}(0) = g_{22}(0) W_2^0 + g_{22}(1) W_2^0 = g_{22}(0) + g_{22}(1) = x(3) + x(7) = x(3) + x(7) W_2^0$$

$$G_{22}(1) = g_{22}(0) W_2^0 + g_{22}(1) W_2^1 = g_{22}(0) - g_{22}(1) = x(3) - x(7) = x(3) - x(7) W_2^0$$

The computation of first stage is shown in Figure 7.6(a).

The second stage of computation

In the second stage of computations, 4-point DFTs are computed using the 2-point DFTs obtained in stage one as inputs.

Let $F_1(k)$ = DFT$\{f_1(n)\}$. The 4-point DFT of $f_1(n)$ can be computed using the equation

$$F_1(k) = G_{11}(k) + W_{N/2}^k G_{12}(k); \quad \text{for } k = 0, 1, 2, 3$$

$$\therefore \quad F_1(0) = G_{11}(0) + W_4^0 G_{12}(0) = G_{11}(0) + G_{12}(0) W_4^0$$

$$F_1(1) = G_{11}(1) + W_4^1 G_{12}(1) = G_{11}(1) + G_{12}(1) W_4^1$$

$$F_1(2) = G_{11}(2) + W_4^2 G_{12}(2) = G_{11}(0) - G_{12}(0) W_4^0$$

$$F_1(3) = G_{11}(3) + W_4^3 G_{12}(3) = G_{11}(1) - G_{12}(1) W_4^1$$

Let $F_2(k)$ = DFT$\{f_2(n)\}$. The 4-point DFT of $f_2(n)$ can be computed using the equation

$$F_2(k) = G_{21}(k) + W_4^k G_{22}(k); \quad \text{for } k = 0, 1, 2, 3$$

$$\therefore \quad F_2(0) = G_{21}(0) + W_4^0 G_{22}(0) = G_{21}(0) + G_{22}(0) W_4^0$$

$$F_2(1) = G_{21}(1) + W_4^1 G_{22}(1) = G_{21}(1) + G_{22}(1) W_4^1$$

$$F_2(2) = G_{21}(2) + W_4^2 G_{22}(2) = G_{21}(0) - G_{22}(0) W_4^0$$

$$F_2(3) = G_{21}(3) + W_4^3 G_{22}(3) = G_{21}(1) - G_{22}(1) W_4^1$$

Here $G_{11}(k)$ and $G_{12}(k)$ are periodic with periodicity of 2.

i.e. $$G_{11}(k + 2) = G_{11}(k) \quad \text{and} \quad G_{12}(k + 2) = G_{12}(k)$$

Here $G_{21}(k)$ and $G_{22}(k)$ are periodic with periodicity of 2.

i.e. $$G_{21}(k+2) = G_{21}(k) \quad \text{and} \quad G_{22}(k+2) = G_{22}(k)$$

The computation of second stage is shown in Figure 7.6(b).

The third stage of computation

In the third stage of computations, the 8-point DFT is computed using the 4-point DFTs obtained in second stage as inputs.

Let $X(k)$ = DFT$\{x(n)\}$. The 8-point DFT of $x(n)$ can be computed using the equation.

$$X(k) = F_1(k) + W_8^k F_2(k); \quad \text{for } k = 0, 1, 2, 3, 4, 5, 6, 7$$

$$\therefore \quad X(0) = F_1(0) + W_8^0 F_2(0)$$

$$X(1) = F_1(1) + W_8^1 F_2(1)$$

$$X(2) = F_1(2) + W_8^2 F_2(2)$$

$$X(3) = F_1(3) + W_8^3 F_2(3)$$

$$X(4) = F_1(4) + W_8^4 F_2(4) = F_1(0) - F_2(0) W_8^0$$

$$X(5) = F_1(5) + W_8^5 F_2(5) = F_1(1) - F_2(1) W_8^1$$

$$X(6) = F_1(6) + W_8^6 F_2(6) = F_1(2) - F_2(2) W_8^2$$

$$X(7) = F_1(7) + W_8^7 F_2(7) = F_1(3) - F_2(3) W_8^3$$

Here $F_1(k)$ and $F_2(k)$ are periodic with periodicity of 4.

i.e. $$F_1(k+4) = F_1(k) \quad \text{and} \quad F_2(k+4) = F_2(k)$$

The computation of third stage is shown in Figure 7.6(c).

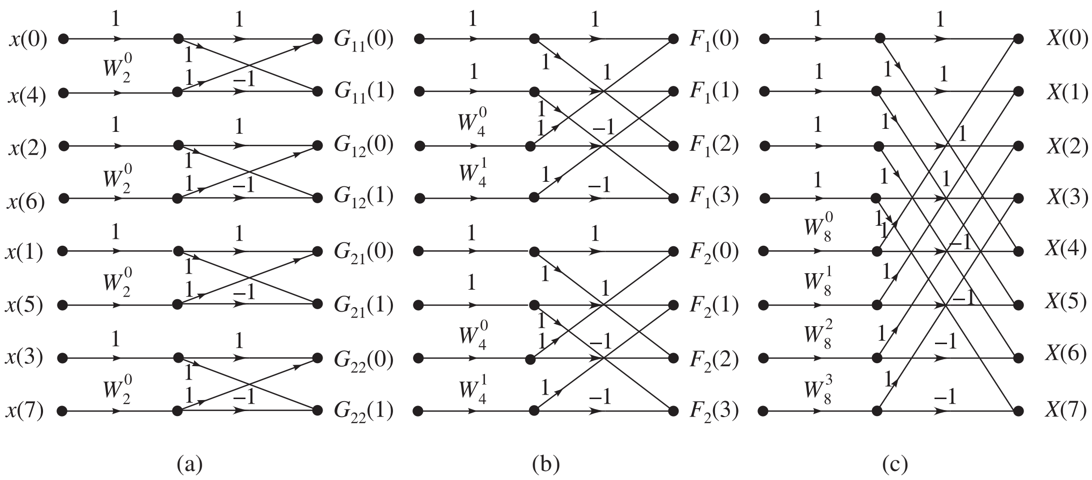


Figure 7.6 (a)–(c) Flow graphs for implementation of 1st, 2nd and 3rd stages of computation.

7.4.1 Butterfly Diagram

Observing the basic computations performed at each stage, we can arrive at the following conclusions:

(i) In each computation, two complex numbers a and b are considered.
(ii) The complex number b is multiplied by a phase factor W_N^k.
(iii) The product bW_N^k is added to the complex number a to form a new complex number A.
(iv) The product bW_N^k is subtracted from complex number a to form new complex number B.

The above basic computation can be expressed by a signal flow graph shown in Figure 7.7. The signal flow graph is also called butterfly diagram since it resembles a butterfly.

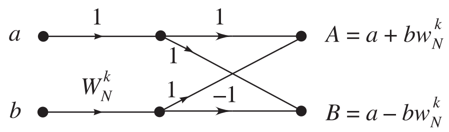


Figure 7.7 Basic butterfly diagram or flow graph of radix-2 DIT FFT.

The complete flow graph for 8-point DIT FFT considering periodicity drawn in a way to remember easily is shown in Figure 7.8. In radix-2 FFT, $N/2$ butterflies per stage are required to represent the computational process. In the butterfly diagram for 8-point DFT shown in Figure 7.8, for symmetry, W_2^0, W_4^0 and W_8^0 are shown on the graph eventhough they are unity. The subscript 2 indicates that it is the first stage of computation. Similarly, subscripts 4 and 8 indicate the second and third stages of computation.

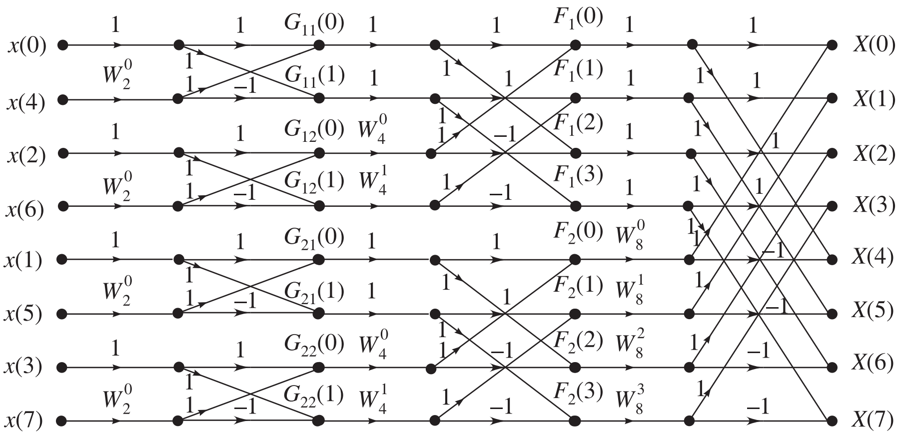


Figure 7.8 The signal flow graph or butterfly diagram for 8-point radix-2 DIT FFT.

7.5 DECIMATION IN FREQUENCY (DIF) RADIX-2 FFT

In decimation in frequency algorithm, the frequency domain sequence $X(k)$ is decimated. In this algorithm, the N-point time domain sequence is converted to two numbers of $N/2$-point

sequences. Then each $N/2$-point sequence is converted to two numbers of $N/4$-point sequences. This process is continued until we get $N/2$ numbers of 2-point sequences. Finally, the 2-point DFT of each 2-point sequence is computed. The 2-point DFTs of $N/2$ numbers of 2-point sequences will give N-samples, which is the N-point DFT of the time domain sequence. Here the equations for $N/2$-point sequences, $N/4$-point sequences, etc., are obtained by decimation of frequency domain sequences. Hence this method is called DIF.

To derive the decimation-in-frequency form of the FFT algorithm for N, a power of 2, we can first divide the given input sequence $x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)$ into the first half and last half of the points so that its DFT $X(k)$ is

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} = \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{kn} + \sum_{n=N/2}^{N-1} x(n) W_N^{kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n + \frac{N}{2}\right) W_N^{k\left(n+\frac{N}{2}\right)}$$

It is important to observe that while the above equation for $X(k)$ contains two summations over $N/2$-points, each of these summations is not an $N/2$-point DFT, since W_N^{nk} rather than $W_{N/2}^{nk}$ appears in each of the sums.

or $$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{nk} + W_N^{(N/2)k} \sum_{n=0}^{\frac{N}{2}-1} x\left(n + \frac{N}{2}\right) W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) W_N^{nk} + (-1)^k\, x\left(n + \frac{N}{2}\right) W_N^{nk} \right]$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^k x\left(n + \frac{N}{2}\right)\right] W_N^{nk}$$

Let us split $X(k)$ into even and odd numbered samples.

For even values of k, the $X(k)$ can be written as

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^{2k} x\left(n + \frac{N}{2}\right)\right] W_N^{2nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + x\left(n + \frac{N}{2}\right)\right] W_{N/2}^{kn}; \quad \text{for } k = 0, 1, 2, \ldots, \left(\frac{N}{2} - 1\right)$$

For odd values of k, the $X(k)$ can be written as

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^{2k+1} x\left(n + \frac{N}{2}\right) \right] W_N^{(2k+1)n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) - x\left(n + \frac{N}{2}\right) \right] W_N^n W_{N/2}^{kn}; \quad \text{for } k = 0, 1, 2, \ldots, \left(\frac{N}{2} - 1\right)$$

The above equations for $X(2k)$ and $X(2k + 1)$ can be recognized as $N/2$-point DFTs. $X(2k)$ is the DFT of the sum of first half and last half of the input sequence, i.e. of $\{x(n) + x(n + N/2)\}$ and $X(2k + 1)$ is the DFT of the product W_N^n with the difference of first half and last half of the input, i.e. of $\{x(n) - x(n + N/2)\}W_N^n$.

If we define new time domain sequences, $u_1(n)$ and $u_2(n)$ consisting of $N/2$-samples, such that

$$u_1(n) = x(n) + x\left(n + \frac{N}{2}\right); \qquad \text{for } n = 0, 1, 2, \ldots, \frac{N}{2} - 1$$

and $$u_2(n) = \left[x(n) - x\left(n + \frac{N}{2}\right) \right] W_N^n; \qquad \text{for } n = 0, 1, 2, \ldots, \frac{N}{2} - 1$$

then the DFTs $U_1(k) = X(2k)$ and $U_2(k) = X(2k + 1)$ can be computed by first forming the sequences $u_1(n)$ and $u_2(n)$, then computing the $N/2$-point DFTs of these two sequences to obtain the even numbered output points and odd numbered output points respectively. The procedure suggested above is illustrated in Figure 7.9 for the case of an 8-point sequence.

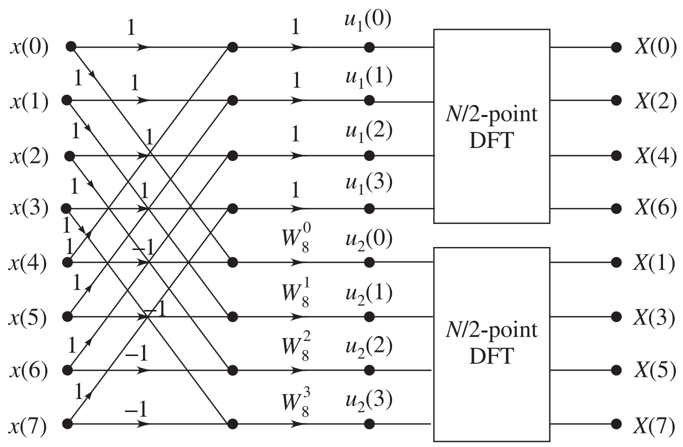


Figure 7.9 Flow graph of the DIF decomposition of an N-point DFT computation into two $N/2$-point DFT computations $N = 8$.

Now each of the $N/2$-point frequency domain sequences, $U_1(k)$ and $U_2(k)$ can be decimated into two numbers of $N/4$-point sequences and four numbers of new $N/4$-point sequences can be obtained from them.

Let the new sequences be $v_{11}(n)$, $v_{12}(n)$, $v_{21}(n)$, $v_{22}(n)$. On similar lines as discussed above, we can get

$$v_{11}(n) = u_1(n) + u_1(n+2); \quad \text{for } n = 0, 1, 2, \ldots, \frac{N}{4} - 1$$

$$v_{12}(n) = [u_1(n) - u_1(n+2)]W_{N/2}^n; \quad \text{for } n = 0, 1, 2, \ldots, \frac{N}{4} - 1$$

and

$$v_{21}(n) = u_2(n) + u_2(n+2); \quad \text{for } n = 0, 1, 2, \ldots, \frac{N}{4} - 1$$

$$v_{22}(n) = [u_2(n) - u_2(n+2)]W_{N/2}^n; \quad \text{for } n = 0, 1, 2, \ldots, \frac{N}{4} - 1$$

This second stage computation for $N = 8$ is shown in Figure 7.10.

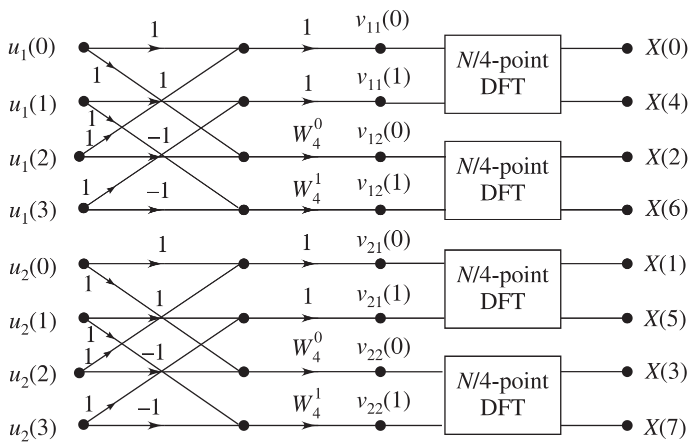


Figure 7.10 Signal flow graph or butterfly diagram for second stage of computation of 8-point radix-2 DIF FIT.

This process is continued till we get only 2-point sequences. The DFT of those 2-point sequences is the DFT of $x(n)$, i.e. $X(k)$ in bit reversed order.

The third stage of computation for $N = 8$ is shown in Figure 7.11.

The entire process of decimation involves m stages of decimation where $m = \log_2 N$. The computation of the N-point DFT via the DIF FFT algorithm requires $(N/2) \log_2 N$ complex multiplications and $(N-1) \log_2 N$ complex additions (i.e. total number of computations remains same in both DIF and DIT).

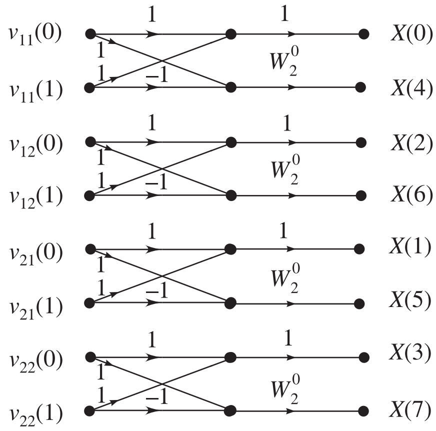


Figure 7.11 Third stage of computation of DFT of an 8-point sequence by radix-2 DIF FIT algorithm.

Observing the basic calculations, each stage involves $N/2$ butterflies of the type shown in Figure 7.12.

The butterfly computation involves the following operations:

(i) In each computation two complex numbers a and b are considered.
(ii) The sum of the two complex numbers is computed which forms a new complex number A.
(iii) Subtract the complex number b from a to get the term $(a - b)$. The difference term $(a - b)$ is multiplied with the phase factor or twiddle factor W_N^n to form a new complex number B.

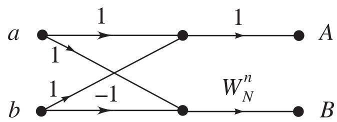


Figure 7.12 Basic butterfly diagram for DIF FIT.

The signal flow graph or butterfly diagram of all the three stages together is shown in Figure 7.13.

7.6 THE 8-POINT DFT USING RADIX-2 DIF FFT

The DIF computations for an 8-sample sequence are given below in detail.

Let $x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$ be the given 8-sample sequence.

First stage of computation

In the first stage of computation, two numbers of 4-point sequences $u_1(n)$ and $u_2(n)$ are obtained form the given 8-point sequence $x(n)$ as shown below.

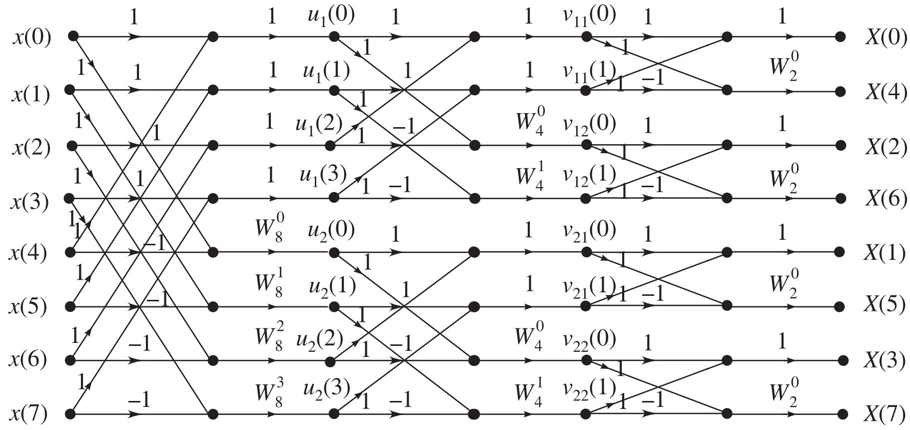


Figure 7.13 Signal flow graph or butterfly diagram for the 8-point radix-2 DIF FFT algorithm.

$$u_1(n) = x(n) + x\left(n + \frac{N}{2}\right)$$

$$= x(n) + x(n + 4), \quad \text{for } n = 0, 1, 2, 3$$

$\therefore$
$$u_1(0) = x(0) + x(4)$$
$$u_1(1) = x(1) + x(5)$$
$$u_1(2) = x(2) + x(6)$$
$$u_1(3) = x(3) + x(7)$$

and
$$u_2(n) = \left[x(n) - x\left(n + \frac{N}{2}\right)\right] W_N^n$$

$$= [x(n) - x(n + 4)] W_N^n$$

$\therefore$
$$u_2(0) = [x(0) - x(4)] W_8^0$$
$$u_2(1) = [x(1) - x(5)] W_8^1$$
$$u_2(2) = [x(2) - x(6)] W_8^2$$
$$u_2(3) = [x(3) - x(7)] W_8^3$$

The samples of the sequences $u_1(n)$ and $u_2(n)$ are obtained by the butterfly operation shown in Figure 7.14(a).

Second stage of computation

In the second stage of computation, four numbers of 2-point sequences $v_{11}(n)$, $v_{12}(n)$ and $v_{21}(n)$, $v_{22}(n)$ are obtained form the two 4-point sequences $u_1(n)$ and $u_2(n)$ obtained in stage one as follows:

$$v_{11}(n) = u_1(n) + u_1\left(n + \frac{N}{4}\right)$$

$$= u_1(n) + u_1(n+2), \qquad \text{for } n = 0, 1$$

$\therefore$
$$v_{11}(0) = u_1(0) + u_1(2)$$
$$v_{11}(1) = u_1(1) + u_1(3)$$

and
$$v_{12}(n) = \left[u_1(n) - u_1\left(n + \frac{N}{4}\right)\right] W_{N/2}^n$$

$$= [u_1(n) - u_1(n+2)] W_4^n$$

$\therefore$
$$v_{12}(0) = [u_1(0) - u_1(2)]\, W_4^0$$
$$v_{12}(1) = [u_1(1) - u_1(3)]\, W_4^1$$

Also, we have

$$v_{21}(n) = u_2(n) + u_2\left(n + \frac{N}{4}\right)$$

$$= u_2(n) + u_2(n+2), \qquad \text{for } \; n = 0, 1$$

$\therefore$
$$v_{21}(0) = u_2(0) + u_2(2)$$
$$v_{21}(1) = u_2(1) + u_2(3)$$

and
$$v_{22}(n) = \left[u_2(n) - u_2\left(n + \frac{N}{4}\right)\right] W_{N/2}^n$$

$$= [u_2(n) - u_2(n+2)] W_4^n$$

$\therefore$
$$v_{22}(0) = [u_2(0) - u_2(2)]\, W_4^0$$
$$v_{22}(1) = [u_2(1) - u_2(3)]\, W_4^1$$

The samples of the sequences $v_{11}(n)$, $v_{12}(n)$ and $v_{21}(n)$, $v_{22}(n)$ are obtained by the butterfly operation shown in Figure 7.14(b).

Third stage of computation

In the third stage of computation, the 2-point DFTs of the 2-point sequences obtained in the second stage are computed as follows:

$$\text{DFT } [v_{11}(n)] = V_{11}(k) = \sum_{n=0}^{1} v_{11}(n) W_2^{kn}, \qquad \text{for } k = 0, 1$$

$\therefore$
$$V_{11}(0) = \sum_{n=0}^{1} v_{11}(n) W_2^0 = v_{11}(0) + v_{11}(1)$$

$$V_{11}(1) = \sum_{n=0}^{1} v_{11}(n)W_2^n = v_{11}(0)W_2^0 + v_{11}(1)W_2^1 = [v_{11}(0) - v_{11}(1)]W_2^0$$

$$\text{DFT } [v_{12}(n)] = V_{12}(k) = \sum_{n=0}^{1} v_{12}(n)W_2^{nk}, \quad \text{for } k = 0, 1$$

$\therefore$

$$V_{12}(0) = \sum_{n=0}^{1} v_{12}(n)W_2^0 = v_{12}(0) + v_{12}(1)$$

$$V_{12}(1) = \sum_{n=0}^{1} v_{12}(n)W_2^n = v_{12}(0)W_2^0 + v_{12}(1)W_2^1 = [v_{12}(0) - v_{12}(1)]W_2^0$$

Similarly

$$V_{21}(0) = v_{21}(0) + v_{21}(1)$$
$$V_{21}(1) = [v_{21}(0) - v_{21}(1)]\,W_2^0$$

and

$$V_{22}(0) = v_{22}(0) + v_{22}(1)$$
$$V_{22}(1) = [v_{22}(0) - v_{22}(1)]\,W_2^0$$

The computation of 2-point DFTs is done by the butterfly operation shown in Figure 7.14(c).

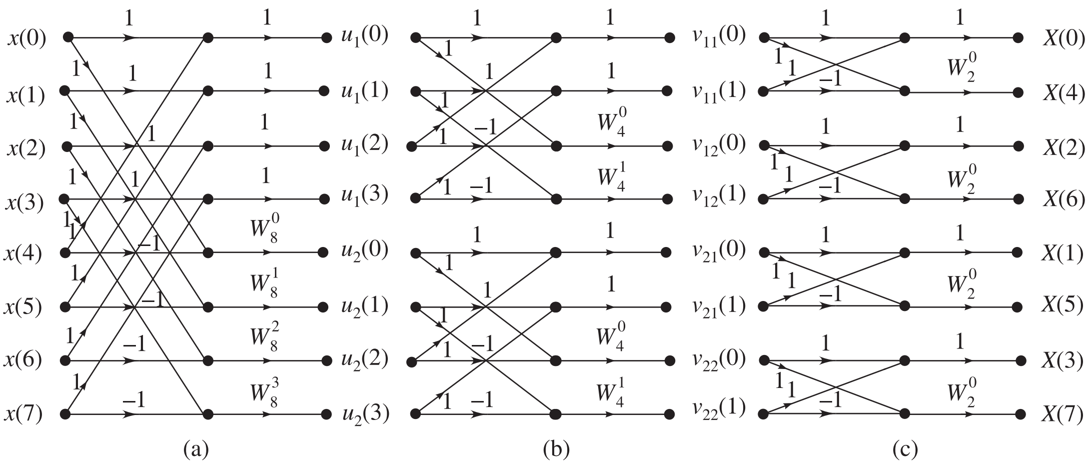


Figure 7.14 (a)–(c) The first, second and third stages of computation of 8-point DFT by Radix-2 DIF FFT.

Combining the 3 stages of computation

The final output $V_{ij}(k)$ gives the $X(k)$. The relation can be obtained as given below:

$X(2k) = U_1(k)$; $k = 0, 1, 2, 3$	$X(2k + 1) = U_2(k)$; $k = 0, 1, 2, 3$
$\therefore X(0) = U_1(0)$	$\therefore X(1) = U_2(0)$
$X(2) = U_1(1)$	$X(3) = U_2(1)$
$X(4) = U_1(2)$	$X(5) = U_3(2)$
$X(6) = U_1(3)$	$X(7) = U_3(3)$
$U_1(2k) = V_{11}(k)$; $k = 0, 1$	$U_1(2k + 1) = V_{12}(k)$; $k = 0, 1$
$\therefore U_1(0) = V_{11}(0)$	$\therefore U_1(1) = V_{12}(0)$
$U_1(2) = V_{11}(1)$	$U_1(3) = V_{12}(1)$
$U_2(2k) = V_{21}(k)$; $k = 0, 1$	$U_2(2k + 1) = V_{22}(k)$; $k = 0, 1$
$\therefore U_2(0) = V_{21}(0)$	$\therefore U_2(1) = V_{22}(0)$
$U_2(2) = V_{21}(1)$	$U_2(3) = V_{22}(1)$

From the above relation, we get

$$X(0) = U_1(0) = V_{11}(0)$$
$$X(4) = U_1(2) = V_{11}(1)$$
$$X(2) = U_1(1) = V_{12}(0)$$
$$X(6) = U_1(3) = V_{12}(1)$$
$$X(1) = U_2(0) = V_{21}(0)$$
$$X(5) = U_2(2) = V_{21}(1)$$
$$X(3) = U_2(1) = V_{22}(0)$$
$$X(7) = U_2(3) = V_{22}(1)$$

From the above, we observe that the output is in bit reversed order. If the input is in normal order, the output is in bit reversed order and the reverse is also true.

The combined signal flow graph or butterfly diagram for 8-point radix-2, DIF FFT algorithm is shown in Figure 7.13.

Comparison of DIT (Decimation-in-time) and DIF (Decimation-in-frequency) algorithms

Difference between DIT and DIF

1. In DIT, the input is bit reversed while the output is in normal order. For DIF, the reverse is true, i.e. the input is in normal order, while the output is bit reversed. However, both DIT and DIF can go from normal to shuffled data or vice versa.
2. Considering the butterfly diagram, in DIT, the complex multiplication takes place before the add subtract operation, while in DIF, the complex multiplication takes place after the add subtract operation.

Similarities

1. Both algorithms require the same number of operations to compute DFT.
2. Both algorithms require bit reversal at some place during computation.

7.6.1 Computation of IDFT through FFT

The IDFT of an N-point sequence $\{X(k)\}$; $k = 0, 1, ..., N-1$ is defined as

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}nk} = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-nk}$$

Taking the conjugate of the above equation for $x(n)$, we get

$$x^*(n) = \left[\frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-nk}\right]^* = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k) W_N^{nk}$$

Taking the conjugate of the above equation for $x^*(n)$, we get

$$x(n) = \frac{1}{N}\left[\sum_{k=0}^{N-1} X^*(k)\, W_N^{nk}\right]^*$$

The term inside the square brackets in the above equation for $x(n)$ is same as the DFT computation of a sequence $X^*(k)$ and may be computed using any FFT algorithm. So we can say that the IDFT of $X(k)$ can be obtained by finding the DFT of $X^*(k)$, taking the conjugate of that DFT and dividing by N. Hence, to compute the IDFT of $X(k)$ the following procedure can be followed

1. Take conjugate of $X(k)$, i.e. determine $X^*(k)$.
2. Compute the N-point DFT of $X^*(k)$ using radix-2 FFT.
3. Take conjugate of the output sequence of FFT.
4. Divide the sequence obtained in step-3 by N.

The resultant sequence is $x(n)$.

Thus, a single FFT algorithm serves the evaluation of both direct and inverse DFTs.

EXAMPLE 7.1 Draw the butterfly line diagram for 8-point FFT calculation and briefly explain. Use decimation-in-time algorithm.

Solution: The butterfly line diagram for 8-point DIT FFT algorithm is shown in Figure 7.15. For 8-point DIT FFT, the input sequence $x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$, must be fed in bit reversed order, i.e. as $x_r(n) = \{x(0), x(4), x(2), x(6), x(1), x(5), x(3), x(7)\}$. Since $N = 2^m = 2^3$, the 8-point DFT computation using radix-2 FFT involves 3 stages of computation, each stage involving 4 butterflies. The output $X(k)$ will be in normal order. In the first stage, four 2-point DFTs are computed. In the second stage they are combined into two 4-point DFTs. In the third stage, the two 4-point DFTs are combined into one 8-point DFT. The 8-point FFT calculation requires $8 \log_2 8 = 24$ complex additions and $(8/2) \log_2 8 = 12$ complex multiplications.

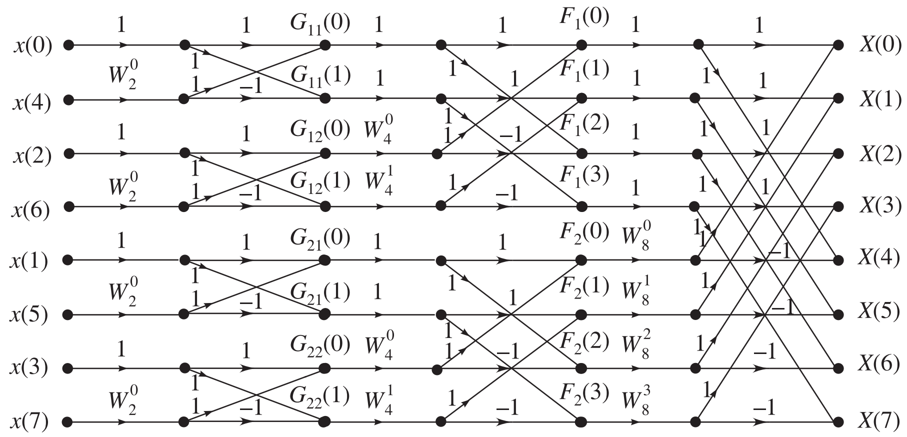


Figure 7.15 Butterfly line diagram for 8-point DIT FFT algorithm for $N = 8$.

EXAMPLE 7.2 Implement the decimation-in-frequency FFT algorithm of N-point DFT where $N = 8$. Also explain the steps involved in this algorithm.

Solution: The 8-point radix-2 DIF FFT algorithm involves 3 stages of computation. The input to the first stage is the input time sequence $x(n)$ in normal order. The output of first stage is the input to the second stage and the output of second stage is the input to the third stage. The output of third stage is the 8-point DFT in bit reversed order.

In DIF algorithm, the frequency domain sequence $X(k)$ is decimated. In this algorithm, the N-point time domain sequence is converted to two numbers of $N/2$-point sequences. Then each $N/2$-point sequence is converted to two numbers of $N/4$-point sequences. Thus, we get 4 numbers of $N/4$, i.e. 2-point sequences. Finally, the 2-point DFT of each 2-point sequence is computed. The 2-point DFTs of $N/2$ number of 2-point sequences will give N-samples which is the N-point DFT of the time domain sequence. The implementation of the 8-point radix-2 DIF FFT algorithm is shown in Figure 7.16.

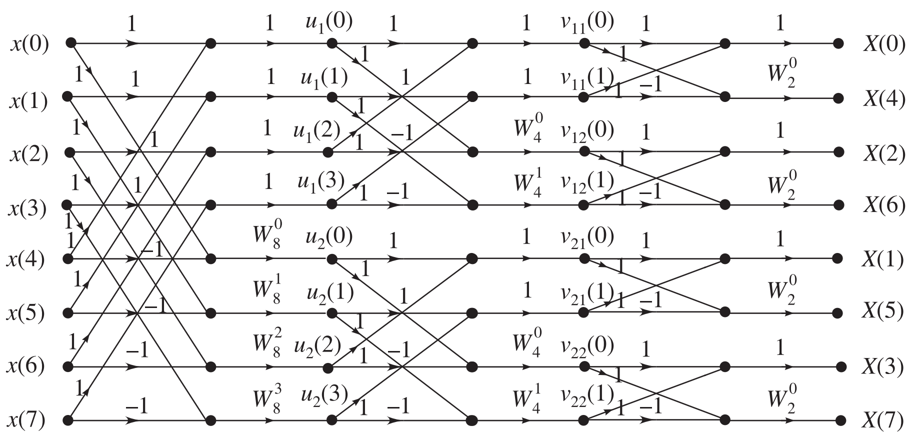


Figure 7.16 Butterfly line diagram for 8-point radix-2 DIF FFT algorithm.

EXAMPLE 7.3 (a) Implement the decimation-in-time FFT algorithm for N = 16, (b) In the above question how many non-trivial multiplications are required?

Solution: (a) The butterfly line diagram showing the implementation of 16-point radix-2 DIT FFT algorithm for N = 16 is shown in Figure 7.17. For DIT FFT, the given 16-point sequence $x(n)$ is to be fed in bit reversed order as

$$x_r(n) = \{x(0), x(8), x(4), x(12), x(2), x(10), x(6), x(14), x(1), x(9), x(5), x(13), x(3), x(11), x(7), x(15)\}$$

Since $N = 16 = 2^4$, the radix-2 DIT FFT computation involves 4 stages. The output is in normal order.

(b) The number of non-trivial multiplications required in the implementation of DIT FFT for $N = 16 = 2^4$ is $N/2 \log_2 N = 16/2 \log_2 16 = 8 \log_2 2^4 = 8 \times 4 = 32$. The number of complex additions are $N \log_2 N = 16 \log_2 16 = 16 \log_2 2^4 = 16 \times 4 = 64$.

Number of multiplications required by direct computation = $N^2 = 16^2 = 256$

Number of complex additions required by direct computation = $N(N-1) = 16 \times 15 = 240$.

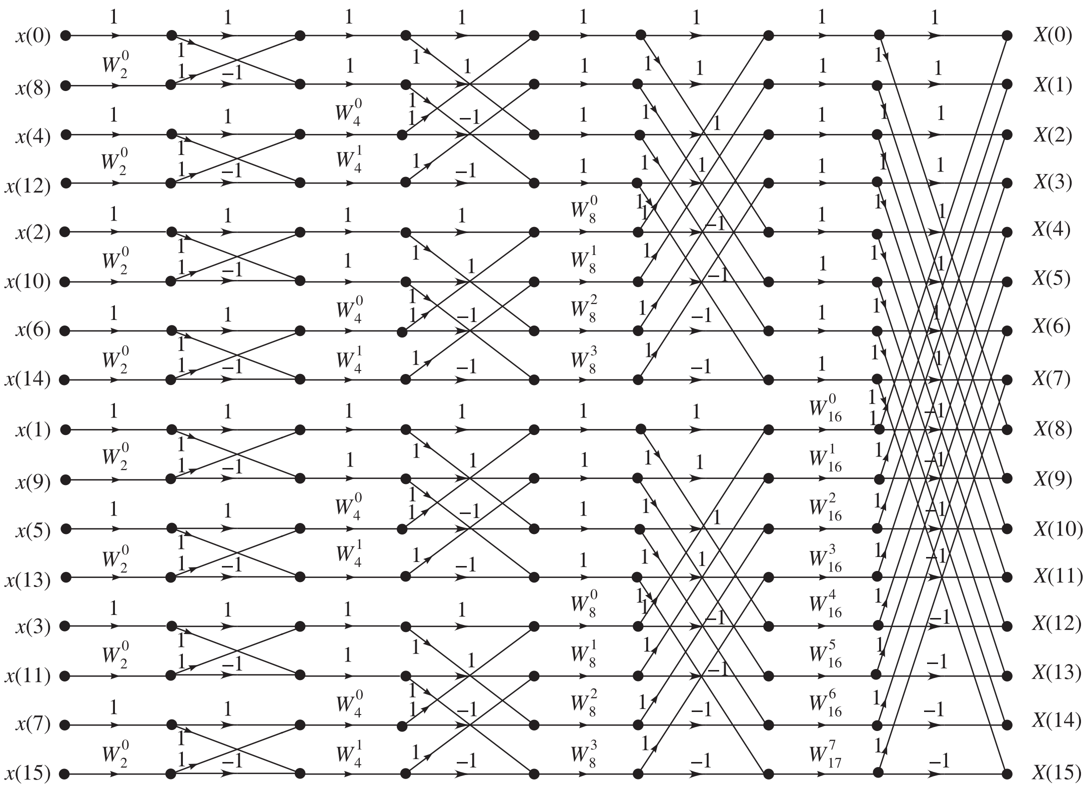


Figure 7.17 Butterfly line diagram for DIT FFT algorithm for N = 16.

EXAMPLE 7.4 What is FFT? Calculate the number of multiplications needed in the calculation of DFT using FFT algorithm with 32-point sequence.

Solution: The FFT, i.e. Fast Fourier transform is a method (or algorithm) for computing the DFT with reduced number of calculations. The computational efficiency is achieved by adopting a divide and conquer approach. This approach is based on the decomposition of an N-point DFT into successively smaller DFTs. This basic approach leads to a family of efficient computational algorithms known as FFT algorithms. Basically there are two FFT algorithms. (i) DIT FFT algorithm and (ii) DIF FFT algorithm. If the length of the sequence $N = 2^m$, 2 indicates the radix and m indicates the number of stages in the computation. In radix-2 FFT, the N-point sequence is decimated into two $N/2$-point sequences, each $N/2$-point sequence is decimated into two $N/4$-point sequences and so on till we get two point sequences. The DFTs of two point sequences are computed and DFTs of two 2-point sequences are combined into DFT of one 4-point sequence, DFTs of two 4-point sequences are combined into DFT of one 8-point sequence and so on till we get the N-point DFT.

The number of multiplications needed in the computation of DFT using FFT algorithm with $N = 32$-point sequence is $= \dfrac{N}{2}\log_2 N = \dfrac{32}{2}\log_2 2^5 = 80$.

The number of complex additions $= N\log_2 N = 32\log_2 32 = 32\log_2 2^5 = 160$

EXAMPLE 7.5 Explain the inverse FFT algorithm to compute inverse DFT of a 8-point DFT. Draw the flow graph for the same.

Solution: The IDFT of an 8-point sequence $\{X(k),\ k = 0, 1, 2, ..., 7\}$ is defined as

$$x(n) = \frac{1}{8}\sum_{k=0}^{7} X(k)\, W_8^{-nk},\quad n = 0, 1, 2, \ldots, 7$$

Taking the conjugate of the above equation for $x(n)$, we have

$$x^*(n) = \frac{1}{8}\left[\sum_{k=0}^{7} X^*(k)\, W_8^{nk}\right]$$

Taking the conjugate of the above equation for $x^*(n)$ we have

$$x(n) = \frac{1}{8}\left[\sum_{k=0}^{7} X^*(k)\, W_8^{nk}\right]^*$$

The term inside the square brackets in the RHS of the above expression for $x(n)$ is the 8-point DFT of $X^*(k)$. Hence, in order to compute the IDFT of $X(k)$ the following procedure can be followed:

1. Given $X(k)$, take conjugate of $X(k)$ i.e. determine $X^*(k)$.
2. Compute the DFT of $X^*(k)$ using radix-2 DIT or DIF FFT, [This gives $8x^*(n)$]

3. Take conjugate of output sequence of FFT. This gives $8x(n)$.
4. Divide the sequence obtained in step 3 by 8. The resultant sequence is $x(n)$.

The flow graph for computation of N = 8-point IDFT using DIT FFT algorithm is shown in Figure 7.18.

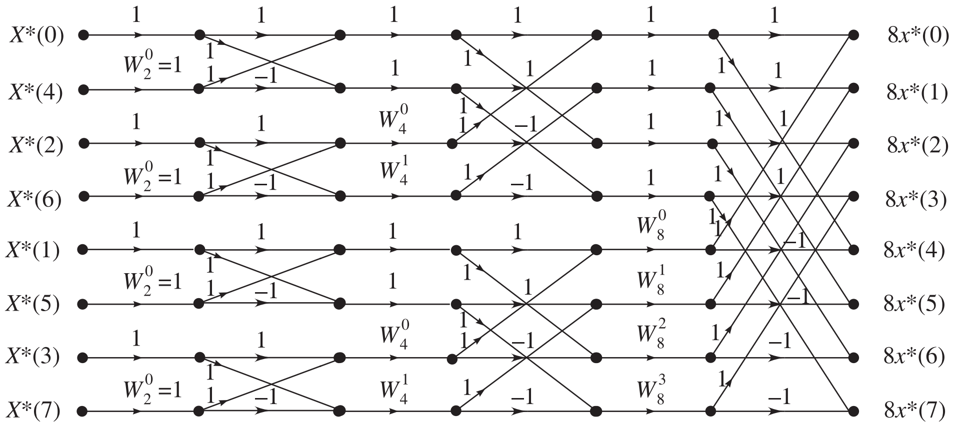


Figure 7.18 Computation of 8-point DFT of $X^*(k)$ by radix-2, DIT FFT.

From Figure 7.18, we get the 8-point DFT of $X^*(k)$ by DIT FFT as

$$8x^*(n) = \{8x^*(0), 8x^*(1), 8x^*(2), 8x^*(3), 8x^*(4), 8x^*(5), 8x^*(6), 8x^*(7)\}$$

$$\therefore \qquad x(n) = \frac{1}{8}\{8x^*(0), 8x^*(1), 8x^*(2), 8x^*(3), 8x^*(4), 8x^*(5), 8x^*(6), 8x^*(7)\}^*$$

EXAMPLE 7.6 Find the 4-point DFT of the sequence $x(n) = \{2, 1, 4, 3\}$ by
(a) DIT FFT algorithm (b) DIF FFT algorithm. Also plot the magnitude and phase plot.

Solution: (a) To compute the 4-point DFT by DIT FFT algorithm, first the given sequence $x(n) = \{x(0), x(1), x(2), x(3)\}$ is to be written in bit reversed order as $x_r(n) = \{x(0), x(2), x(1), x(3)\}$. The output will be in normal order. The given $x(n)$ in bit reversed order is $x_r(n) = \{2, 4, 1, 3\}$. The 4-point DFT of $x(n)$ using DIT FFT algorithm is computed as shown in Figure 7.19.

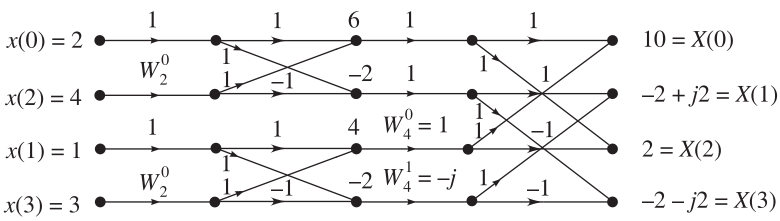


Figure 7.19 4-point DFT by DIT FFT.

From Figure 7.19, the 4-point DFT of $x(n)$ by radix-2, DIT FFT algorithm is

$$X(k) = \{10, -2 + j2, 2, -2, -j2\}$$

(b) To compute the DFT by DIF FFT, the input sequence is to be in normal order and the output sequence will be in bit reversed order. Figure 7.20 shows the computation of 4-point DFT of $x(n)$ by radix-2 DIF FFT algorithm.

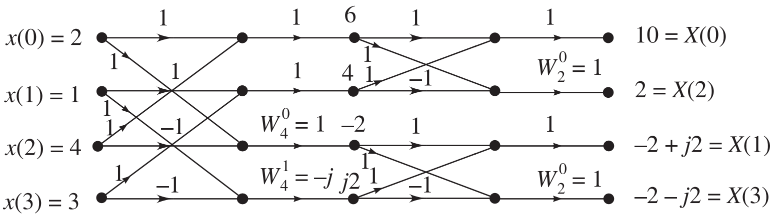


Figure 7.20 4-point DFT by DIF FFT.

From Figure 7.20, the 4-point DFT of $x(n)$ is $X(k) = \{10, -2 + j2, 2, -2 - j2\}$.

To draw the magnitude and phase plot, we have

$$X(0) = 10 \quad \therefore \left|X(0)\right| = 10 \text{ and } \angle X(0) = 0° = 0 \text{ rad}$$

$$X(1) = -2 + j2 \quad \therefore \left|X(1)\right| = \sqrt{2^2 + 2^2} = 2.828 \text{ and } \angle X(1) = 135° = 2.356 \text{ rad}$$

$$X(2) = 2 \quad \therefore \left|X(2)\right| = 2 \text{ and } \angle X(2) = 0° = 0 \text{ rad}$$

$$X(3) = -2 - j2 \quad \therefore \left|X(3)\right| = \sqrt{2^2 + 2^2} = 2.828 \text{ and } \angle X(3) = -135° = -2.356 \text{ rad}$$

The magnitude and phase spectrum are shown in Figure 7.21.

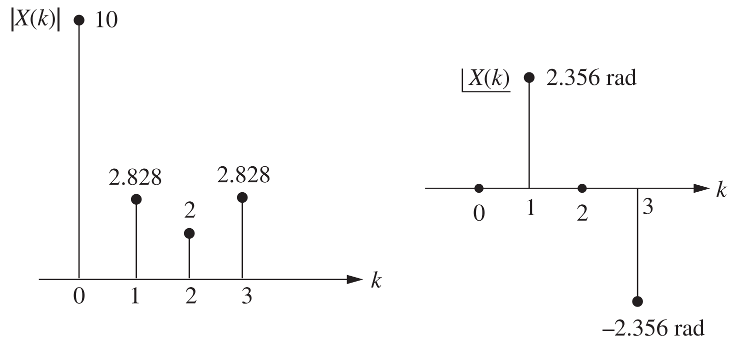


Figure 7.21 (a) Magnitude spectrum (b) Phase spectrum.

EXAMPLE 7.7 Compute the circular convolution of the two sequences $x_1(n) = \{1, 2, 0, 1\}$ and $x_2(n) = \{2, 2, 1, 1\}$ using DFT approach.

Solution: To compute the circular convolution of the two sequences $x_1(n)$ and $x_2(n)$, i.e. to compute $x_1(n) \oplus x_2(n)$ using DFT approach, first the DFTs of the two sequences, i.e. $X_1(k)$

and $X_2(k)$ are to be determined independently, then their product $Y(k) = X_1(k)X_2(k)$ is to be determined and then the IDFT of $Y(k)$, i.e. $y(n)$ is to be determined. Here DFTs and IDFT are performed via FFT. For DIT FFT, the input sequence is in bit reversed order and output sequence is in normal order. The computation of 4-point DFTs of $x_1(n)$ and $x_2(n)$ using radix-2 DIT FFT algorithm is shown in Figures 7.22(a) and (b).

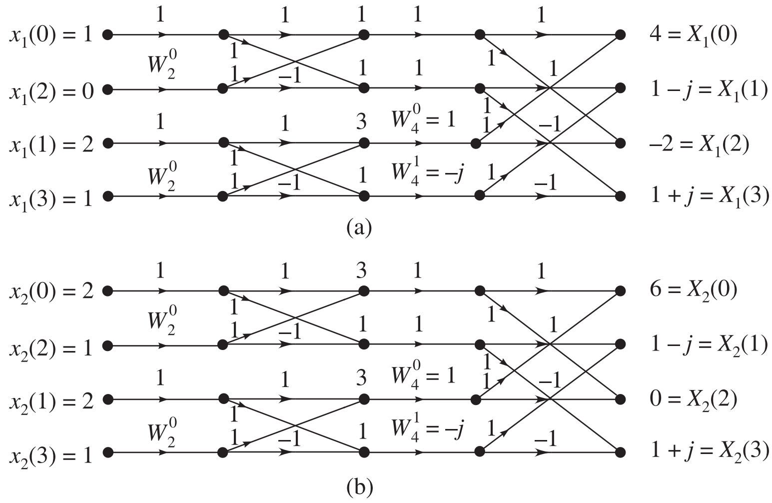


Figure 7.22 Computation of (a) DFT of $x_1(n)$; (b) DFT of $x_2(n)$ by DIT FFT.

From Figures 7.22(a) and (b), we have

$$X_1(k) = \{4,\ 1 - j,\ -2,\ 1 + j\} \text{ and } X_2(k) = \{6,\ 1 - j,\ 0,\ 1 + j\}$$

$\therefore$ The product sequence $X(k) = X_1(k)X_2(k)$

$$\therefore \qquad X(k) = \{4,\ 1 - j, -2,\ 1 + j\}\,\{6,\ 1 - j,\ 0,\ 1 + j\} = \{24, -j2,\ 0, j2\}$$

$$\therefore \qquad X^*(k) = \{24, j2, 0,\ -j2\} \text{ in normal order}$$

$$X^*(k) \text{ in bit reverse order is } X_r^*(k) = \{24, 0, j2,\ -j2\}$$

The computation of 4-point DFT of $X^*(k)$ using radix-2 DIT FFT algorithm is shown in Figure 7.23. For DIT FFT, the input is in bit reversed order and the output is in normal order.

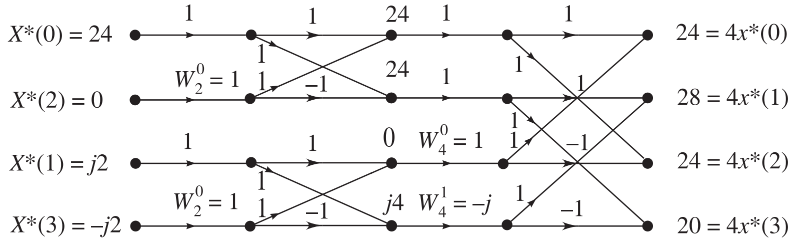


Figure 7.23 Computation of 4-point DFT of $X^*(k)$ by radix-2 DIT FFT.

From Figure 7.23, we have $4x^*(n) = \{24, 28, 24, 20\}$

$$\therefore \qquad x^*(n) = \frac{1}{4}\{24, 28, 24, 20\} = \{6, 7, 6, 5\}$$

$$x(n) = \{6, 7, 6, 5\}^* = \{6, 7, 6, 5\}$$

$$\therefore \qquad [x_1(n) = \{1, 2, 0, 1\}] \oplus [x_2(n) = \{2, 2, 1, 1\}] = [x(n) = \{6, 7, 6, 5\}]$$

EXAMPLE 7.8 Compute the DFT of the square wave sequence

$$x(n) = \begin{cases} 1, & 0 \le n \le \left(\dfrac{N}{2} - 1\right) \\ -1, & \dfrac{N}{2} \le n \le N \end{cases}$$

where N is even.

Solution: It is given that N is even, but the value of N is not given. Let us take $N = 4$.

$$\therefore \qquad x(n) = \{1, 1, -1, -1\}$$

Let us calculate $X(k)$ using 4-point radix-2 DIT FFT algorithm as shown in Figure 7.24. For DIT FFT, the input is in bit reversed order and output is in normal order. $x(n)$ in bit reversed order is $x_r(n) = \{1, -1, 1, -1\}$.

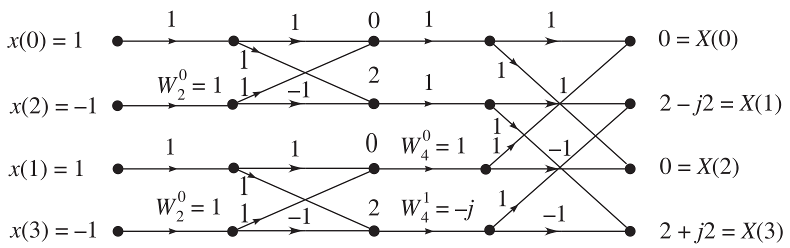


Figure 7.24 4-point DFT of x(n) by radix-2 DIT FFT.

From Figure 7.24, we have $X(k) = \{0, 2 - j2, 0, 2 + j2\}$.

EXAMPLE 7.9 In an LTI system, the input $x(n) = \{2, 2, 2\}$ and the impulse response $h(n) = \{-2, -2\}$. Determine the response of LTI system by radix-2, DIT FFT.

Solution: The response $y(n)$ of LTI system is given by the linear convolution of input $x(n)$ and impulse response $h(n)$.

$$\therefore \qquad \text{Response or output } y(n) = x(n) * h(n)$$

The DFT (or FFT) supports only circular convolution. Hence, to get the result of linear convolution from circular convolution, the sequences $x(n)$ and $h(n)$ should be converted to the size of $y(n)$ by appending with zeros and circular convolution of $x(n)$ and $h(n)$ is performed.

The length of $x(n)$ is 3 and the length of $h(n)$ is 2. Hence, the length of $y(n)$ is $3 + 2 - 1 = 4$. Therefore, the given sequences $x(n)$ and $h(n)$ are converted into 4-point sequences by appending zeros.

$$\therefore \qquad x(n) = \{2, 2, 2, 0\} \text{ and } h(n) = \{-2, -2, 0, 0\}$$

Now the response $y(n)$ is given by $y(n) = x(n) \oplus h(n)$

Let $\qquad \text{DFT}\{x(n)\} = X(k)$, $\text{DFT}\{h(n)\} = H(k)$, and $\text{DFT}\{y(n)\} = Y(k)$

By convolution theorem of DFT, we get

$$\text{DFT}\{x(n) \oplus h(n)\} = X(k)H(k)$$

$$\therefore \qquad y(n) = \text{IDFT}\{X(k)H(k)\} = \text{IDFT}\{Y(k)\}$$

The various steps in computing $y(n)$ are

Step 1: Determine $X(k)$ using radix-2 DIT FFT algorithm
Step 2: Determine $H(k)$ using radix-2 DIT FFT algorithm
Step 3: Determine the product $X(k)H(k)$
Step 4: Take IDFT of the product $X(k)H(k)$ using radix-2 DIT FFT algorithm.

1. The 4-point DFT of $x(n)$, i.e. $X(k)$ is determined using radix-2, DIT FFT algorithm as shown in Figure 7.25. For DIT FFT, the input is in bit reversed order and output is in normal order.

$$x(n) = \{2, 2, 2, 0\};\ x_r(n) = \{2, 2, 2, 0\}$$

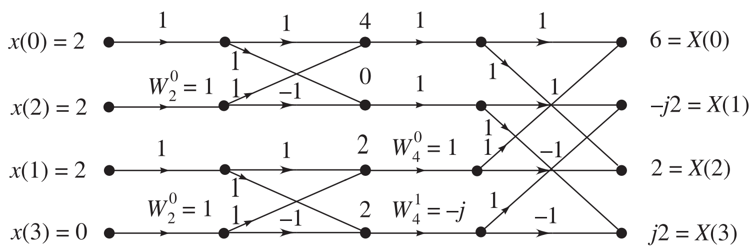


Figure 7.25 Computation of 4-point DFT of x(n) by radix-2, DIT FFT.

From Figure 7.25, $X(k) = \{6, -j2, 2, j2\}$.

2. The 4-point DFT of $h(n)$, i.e. $H(k)$ using 4-point DIT FFT is determined as shown in Figure 7.26.

$$h(n) = \{-2, -2, 0, 0\};\ h_r(n) = \{-2, 0, -2, 0\}$$

From Figure 7.26, $H(k) = \{-4, -2 + j2, 0, -2 - j2\}$

3. $Y(k) = X(k)H(k) = \{6, -j2, 2, j2\}\ \{-4, -2 + j2, 0, -2 - j2\}$
$= \{-24, 4 + j4, 0, 4 - j4\}$

4. $Y^*(k) = \{-24, 4 - j4, 0, 4 + j4\};\ Y_r^*(k) = \{-24, 0, 4 - j4, 4 + j4\}$

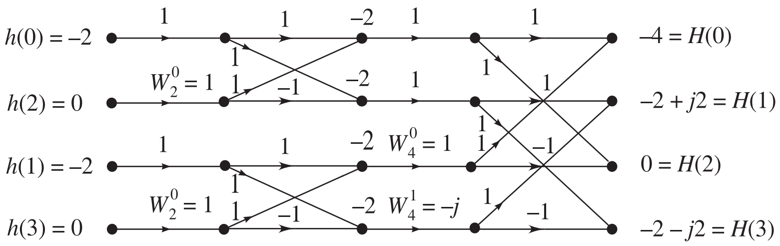


Figure 7.26 Computation of 4-point DFT of $h(n)$ by radix-2, DIT FFT.

The 4-point DFT of $Y^*(k)$ using radix-2, DIT FFT is computed as shown in Figure 7.27.

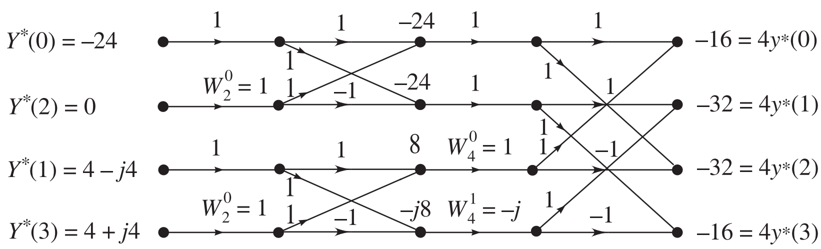


Figure 7.27 Computation of 4-point DFT of $Y^*(k)$ by radix-2, DIT FFT.

From Figure 7.27, we get $4y^*(n) = \{-16, -32, -32, -16\}$.

$$\text{Therefore,}\quad y^*(n) = \frac{1}{4}\{-16, -32, -32, -16\} = \{-4, -8, -8, -4\}$$

$$y(n) = \{-4, -8, -8, -4\}^* = \{-4, -8, -8, -4\}$$

EXAMPLE 7.10 Find the response of LTI system with impulse response $h(n) = \{0.5, 1\}$ when the input sequence $x(n) = \{1, 0.5, 0\}$ is applied to it by radix-2 DIT FFT.

Solution: The response $y(n)$ of LTI system is given by linear convolution of input sequence $x(n)$ and impulse response $h(n)$, i.e. $y(n) = x(n) * h(n)$.

The length of $x(n) = 3$ and the length of $h(n) = 2$. So the length of output sequence $y(n)$ is $3 + 2 + (-1) = 4$.

Since the DFT supports only circular convolution, to get the result of linear convolution from circular convolution, the sequences $x(n)$ and $h(n)$ are to be converted to the size of $y(n)$[i.e., $N = 4$] by appending with zeros and the circular convolution of $x(n)$ and $h(n)$ is performed.

$x(n)$ and $h(n)$ as 4-point sequences are $x(n) = \{1, 0.5, 0, 0\}$ and $h(n) = \{0.5, 1, 0, 0\}$.

Now to find $y(n)$ by FFT, we have to find $X(k)$ and $H(k)$ by 4-point radix-2 DIT FFT, get $Y(k) = X(k)H(k)$, find DFT of $Y^*(k)$, take the conjugate of that and divide by 4. The entire procedure is as given below:

Step 1: Determination of $X(k)$

Given $x(n) = \{1, 0.5, 0, 0\}$, $x(n)$ in bit reversed order is $x_r(n) = \{1, 0, 0.5, 0\}$.

The computation of 4-point DFT of $x(n)$, $X(k)$ by radix-2 DIT FFT algorithm is shown in Figure 7.28.

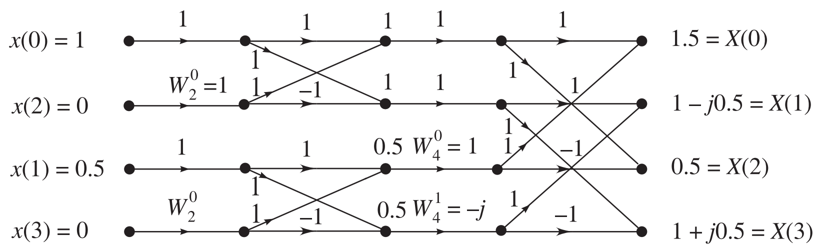


Figure 7.28 Computation of 4-point DFT of $x(n)$ by DIT FFT.

From Figure 7.28, the 4-point DFT of $x(n)$ is $X(k) = \{1.5,\ 1 - j0.5,\ 0.5,\ 1 + j0.5\}$.

Step 2: Computation of $H(k)$

Given $h(n) = \{0.5, 1, 0, 0\}$, $h(n)$ in bit reverse order is $h_r(n) = \{0.5, 0, 1, 0)$.

The computation of 4-point DFT of $h(n)$, i.e. $H(k)$ using radix-2, DIT FFT algorithm is shown in Figure 7.29.

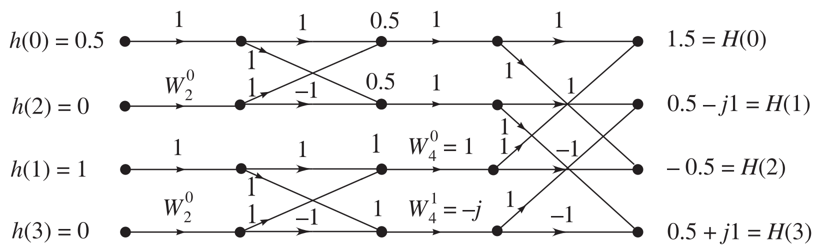


Figure 7.29 Computation of 4-point DFT of $h(n)$ by DIT FFT.

From Figure 7.29, the 4-point DFT of $h(n)$ is $H(k) = \{1.5,\ 0.5 - j1,\ -0.5,\ 0.5 + j1\}$.

Step 3: Computation of $Y(k)$

$$Y(k) = X(k)H(k) = \{1.5, 1 - j0.5, 0.5, 1 + j0.5\}\{1.5, 0.5 - j1, -0.5, 0.5 + j1\}$$
$$= \{2.25,\ -j1.25, -0.25, j1.25\}$$

$$\therefore \qquad Y^*(k) = \{2.25, j1.25,\ -0.25,\ -j1.25\}$$

Step 4: $Y^*(k)$ in bit reverse order is $Y_r^*(k) = \{2.25,\ -0.25, j1.25,\ -j1.25\}$

Computation of 4-point DFT of $Y^*(k)$ using radix-2 DIT FFT is shown in Figure 7.30.

From Figure 7.30, we get $4y^*(n) = \{2, 5, 2, 0\}$.

$$\therefore \qquad y^*(n) = \frac{1}{4}\{2, 5, 2, 0\} = \{0.5, 1.25, 0.5, 0\}$$

$$\therefore \qquad y(n) = \{0.5, 1.25, 0.5, 0\}^* = \{0.5, 1.25, 0.5, 0\}$$

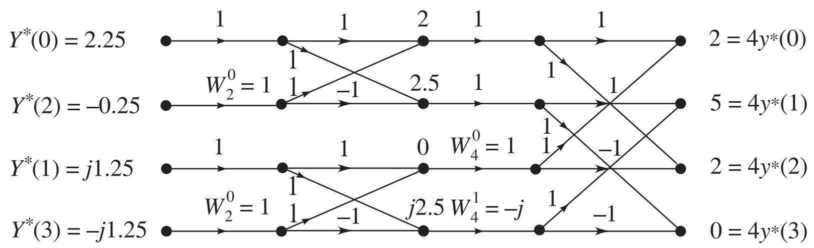


Figure 7.30 Computation of 4-point DFT of $Y^*(k)$ by DIT FFT.

EXAMPLE 7.11 Compute the DFT of the sequence $x(n) = \{1, 0, 0, 0, 0, 0, 0, 0\}$ (a) directly, (b) by FFT.

Solution: (a) Direct computation of DFT

The given sequence is $x(n) = \{1, 0, 0, 0, 0, 0, 0, 0\}$. We have to compute 8-point DFT. So $N = 8$.

$$\text{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}nk} = \sum_{n=0}^{N-1} x(n)\, W_N^{nk} = \sum_{n=0}^{7} x(n)\, W_8^{nk}$$

$$= x(0)W_8^0 + x(1)W_8^1 + x(2)W_8^2 + x(3)W_8^3 + x(4)W_8^4 + x(5)W_8^5 + x(6)W_8^6 + x(7)W_8^7$$

$$= (1)(1) + (0)(W_8^1) + (0)W_8^2 + (0)W_8^3 + (0)W_8^4 + (0)W_8^5 + (0)W_8^6 + (0)W_8^7 = 1$$

$\therefore$ $X(k) = 1$ for all k

$\therefore$ $X(0) = 1,\ X(1) = 1,\ X(2) = 1,\ X(3) = 1,\ X(4) = 1,\ X(5) = 1,\ X(6) = 1,\ X(7) = 1$

$\therefore$ $X(k) = \{1, 1, 1, 1, 1, 1, 1, 1\}$

(b) Computation by FFT. Here $N = 8 = 2^3$

The computation of 8-point DFT of $x(n) = \{1, 0, 0, 0, 0, 0, 0, 0\}$ by radix-2 DIT FFT algorithm is shown in Figure 7.31. $x(n)$ in bit reverse order is

$$x_r(n) = \{x(0), x(4), x(2), x(6), x(1), x(5), x(3), x(7)\}$$
$$= \{1, 0, 0, 0, 0, 0, 0, 0\}$$

For DIT FFT input is in bit reversed order and output is in normal order.

From Figure 7.31, the 8-point DFT of the given $x(n)$ is $X(k) = \{1, 1, 1, 1, 1, 1, 1, 1\}$

EXAMPLE 7.12 An 8-point sequence is given by $x(n) = \{2, 2, 2, 2, 1, 1, 1, 1\}$.

Compute the 8-point DFT of $x(n)$ by

(a) Radix-2 DIT FFT algorithm
(b) Radix-2 DIF FFT algorithm

Also sketch the magnitude and phase spectrum.

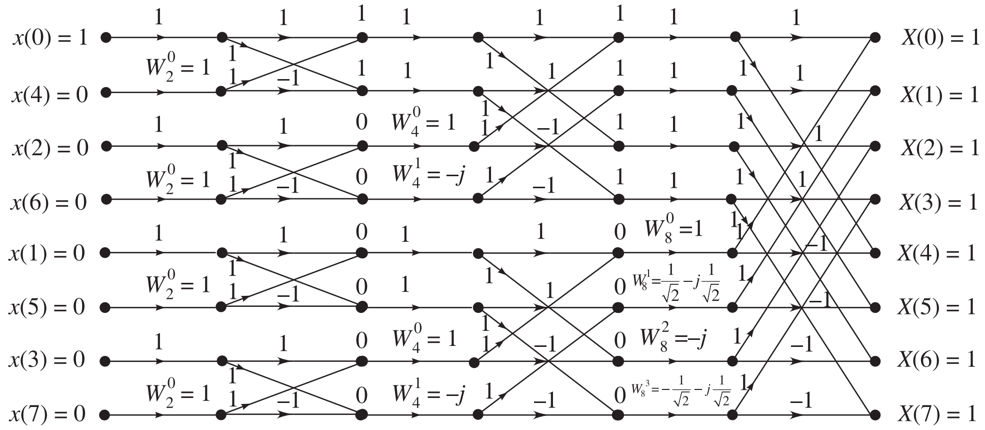


Figure 7.31 8-point DFT by DIT FFT.

Solution: (a) 8-point DFT by Radix-2 DIT FFT algorithm

The given sequence is $x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$

$$= \{2, 2, 2, 2, 1, 1, 1, 1\}$$

The given sequence in bit reversed order is

$$x_r(n) = \{x(0), x(4), x(2), x(6), x(1), x(5), x(3), x(7)\}$$

$$= \{2, 1, 2, 1, 2, 1, 2, 1\}$$

For DIT FFT, the input is in bit reversed order and the output is in normal order. The computation of 8-point DFT of $x(n)$, i.e. $X(k)$ by Radix-2 DIT FFT algorithm is shown in Figure 7.32.

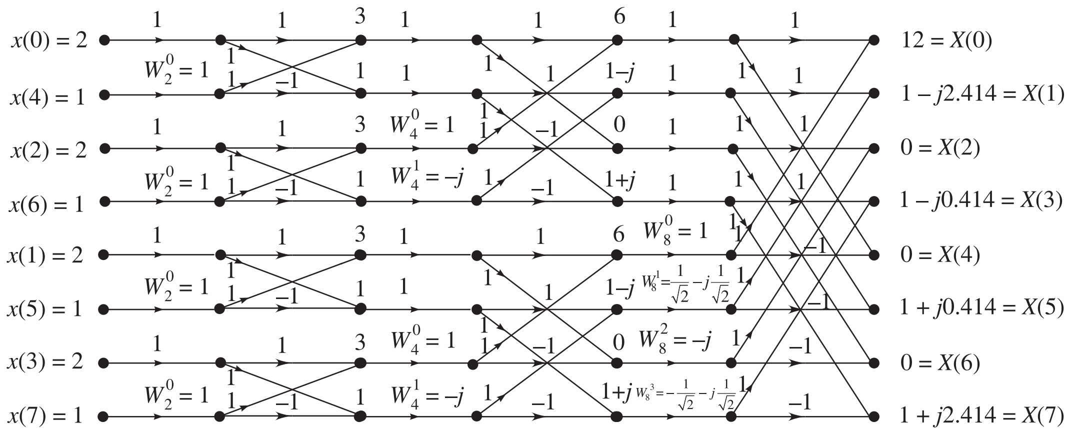


Figure 7.32 Computation of 8-point DFT of $x(n)$, i.e. $X(k)$ by DIT FFT.

From Figure 7.32, we get the 8-point DFT of $x(n)$ as

$$X(k) = \{12, 1 - j2.414, 0, 1 - j0.414, 0, 1 + j0.414, 0, 1 + j2.414\}$$

(b) 8-point DFT by radix-2 DIF FFT algorithm
For DIF FFT, the input is in normal order and the output is in bit reversed order. The computation of DFT by radix-2 DIF FFT algorithm is shown in Figure 7.33.

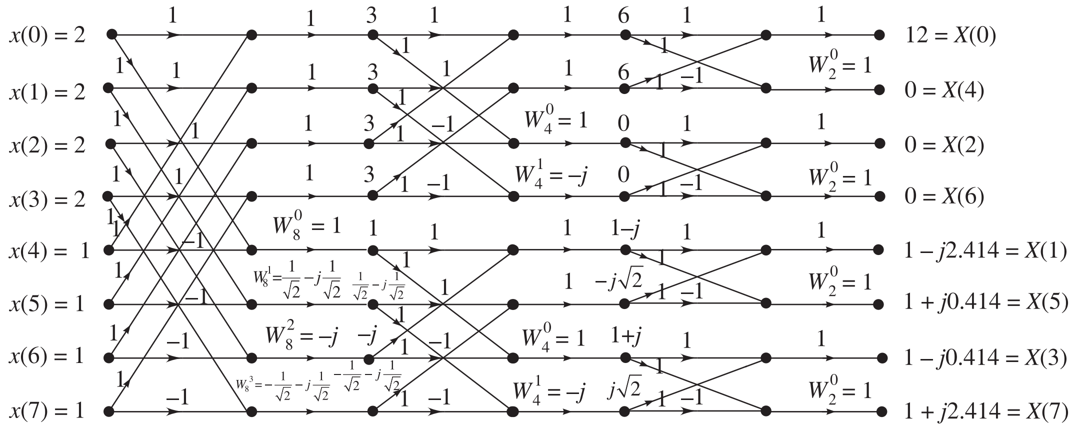


Figure 7.33 Computation of 8-point DFT of $x(n)$ by radix-2 DIF FFT algorithm.

From Figure 7.33, we observe that the 8-point DFT in bit reversed order is

$$X_r(k) = \{X(0), X(4), X(2), X(6), X(1), X(5), X(3), X(7)\}$$
$$= \{12, 0, 0, 0, 1 - j2.414, 1 + j0.414, 1 - j0.414, 1 + j2.414\}$$

$\therefore$ The 8-point DFT in normal order is

$$X(k) = \{X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)\}$$
$$= \{12, 1 - j2.414, 0, 1 - j0.414, 0, 1 + j0.414, 0, 1 + j2.414\}$$

Magnitude and Phase Spectrum

Each element of the sequence $X(k)$ is a complex number and they are expressed in rectangular coordinates. If they are converted to polar coordinates, then the magnitude and phase of each element can be obtained.

The magnitude spectrum is the plot of the magnitude of each sample of $X(k)$ as a function of k. The phase spectrum is the plot of phase of each sample of $X(k)$ as a function of k. When N-point DFT is performed on a sequence $x(n)$ then the DFT sequence $X(k)$ will have a periodicity of N. Hence, in this example, the magnitude and phase spectrum will have a periodicity of 8 as shown below.

$$X(k) = \{12, 1 - j2.414, 0, 1 - j0.414, 0, 1 + j0.414, 0, 1 + j2.414\}$$

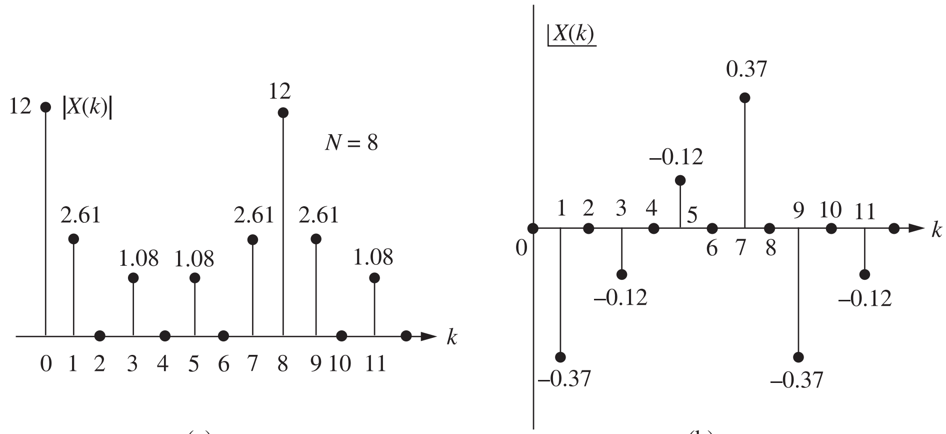
$$= \{12\angle 0°, 2.61\angle -67°, 0\angle 0°, 1.08\angle -22°, 0\angle 0°, 1.08\angle 22°, 0\angle 0°, 2.61\angle 67°\}$$

$$= \{12\angle 0, 2.61\angle -0.37\pi, 0\angle 0°, 1.08\angle -0.12\pi, 0\angle 0°, 1.08\angle 0.12\pi, 0\angle 0°, 2.61\angle 0.37\pi\}$$

$$\therefore \qquad |X(k)| = \{12, 2.61, 0, 1.08, 0, 1.08, 0, 2.61\}$$

$$\angle X(k) = \{0, -0.37\pi, 0, -0.12\pi, 0, 0.12\pi, 0, 0.37\pi\}$$

The magnitude and phase spectrum are shown in Figures 7.34(a) and (b).



(a) (b)

Figure 7.34 (a) Magnitude spectrum, (b) Phase spectrum.

EXAMPLE 7.13 Find the 8-point DFT by radix-2 DIT FFT algorithm.

$$x(n) = \{2, 1, 2, 1, 2, 1, 2, 1\}$$

Solution: The given sequence is $x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$

$$= \{2, 1, 2, 1, 2, 1, 2, 1\}$$

For DIT FFT computation, the input sequence must be in bit reversed order and the output sequence will be in normal order.

$x(n)$ in bit reverse order is

$$x_r(n) = \{x(0), x(4), x(2), x(6), x(1), x(5), x(3), x(7)\}$$

$$= \{2, 2, 2, 2, 1, 1, 1, 1\}$$

The computation of 8-point DFT of $x(n)$ by radix-2 DIT FFT algorithm is shown in Figure 7.35.

From Figure 7.35, we get the 8-point DFT of $x(n)$ as

$$X(k) = \{12, 0, 0, 0, 4, 0, 0, 0\}$$

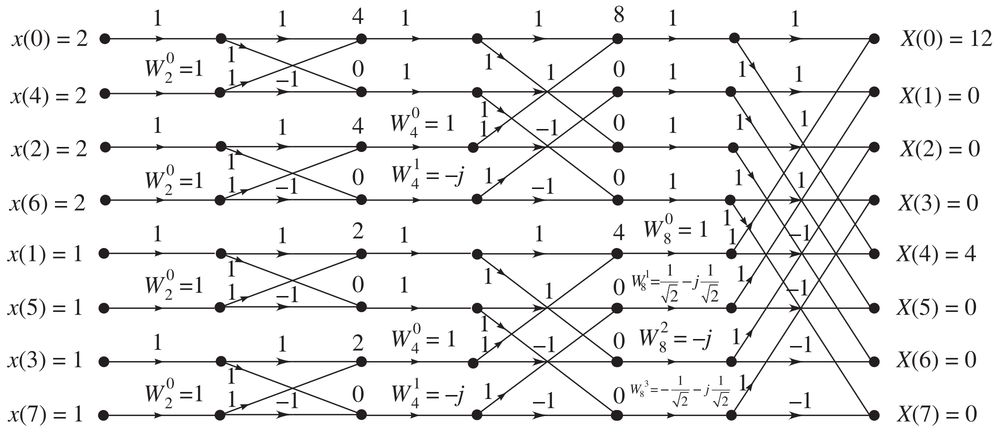


Figure 7.35 Computation of 8-point DFT of $x(n)$ by radix-2, DIT FFT.

EXAMPLE 7.14 Compute the DFT for the sequence $x(n) = \{1, 1, 1, 1, 1, 1, 1, 1\}$.

Solution: The given sequence is $x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$

$$= \{1, 1, 1, 1, 1, 1, 1, 1\}$$

The computation of 8-point DFT of $x(n)$, i.e. $X(k)$ by radix-2, DIT FFT algorithm is shown in Figure 7.36.

The given sequence in bit reversed order is

$$x_r(n) = \{x(0), x(4), x(2), x(6), x(1), x(5), x(3), x(7)\}$$
$$= \{1, 1, 1, 1, 1, 1, 1, 1\}$$

For DIT FFT, the input is in bit reversed order and output is in normal order.

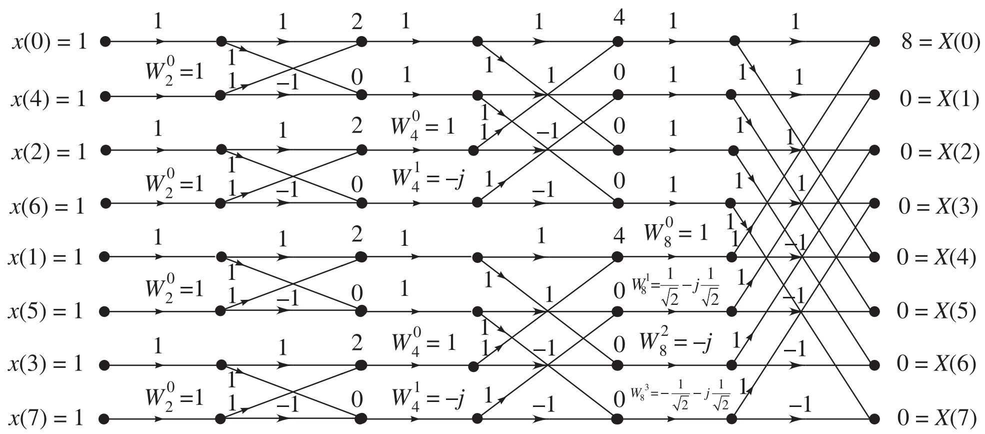


Figure 7.36 Computation of 8-point DFT of $x(n)$ by radix-2, DIT FFT.

From Figure 7.36, we get the 8-point DFT of $x(n)$ as $X(k) = \{8, 0, 0, 0, 0, 0, 0, 0\}$.

EXAMPLE 7.15 Given a sequence $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$, determine $X(k)$ using DIT FFT algorithm.

Solution: The given sequence is $x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$

$$= \{1, 2, 3, 4, 4, 3, 2, 1\}$$

The computation of 8-point DFT of $x(n)$, i.e. $X(k)$ by radix-2, DIT FFT algorithm is shown in Figure 7.37. For DIT FFT, the input is in bit reversed order and the output is in normal order.

The given sequence in bit reverse order is

$$x_r(n) = \{x(0), x(4), x(2), x(6), x(1), x(5), x(3), x(7)\} = \{1, 4, 3, 2, 2, 3, 4, 1\}$$

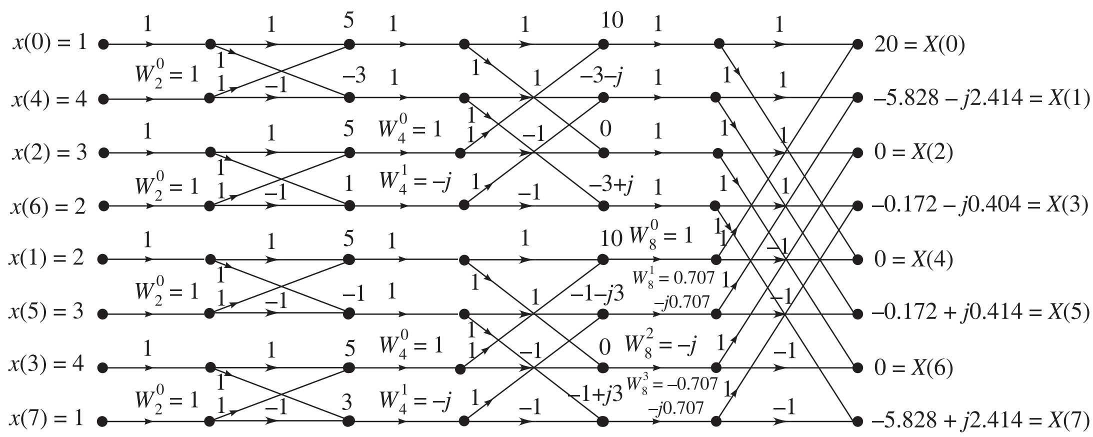


Figure 7.37 Computation of 8-point DFT of $x(n)$ by radix-2, DIT FFT.

From Figure 7.37, we get the 8-point DFT of $x(n)$ as

$$X(k) = \{20, -5.828 - j2.414, 0, -0.172 - j0.414, 0, -0.172 + j0.414, 0, -5.828 + j2.414\}$$

EXAMPLE 7.16 Given a sequence $x(n) = \{0, 1, 2, 3, 4, 5, 6, 7\}$, determine $X(k)$ using DIT FFT algorithm.

Solution: The given sequence is $x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$

$$= \{0, 1, 2, 3, 4, 5, 6, 7\}$$

The computation of 8-point DFT of $x(n)$, i.e. $X(k)$ by radix-2, DIT FFT algorithm is shown in Figure 7.38. For DIT FFT, the input is in bit reversed order and output is in normal order.

The given sequence in bit reverse order is

$$x_r(n) = \{x(0), x(4), x(2), x(6), x(1), x(5), x(3), x(7)\}$$

$$= \{0, 4, 2, 6, 1, 5, 3, 7\}$$

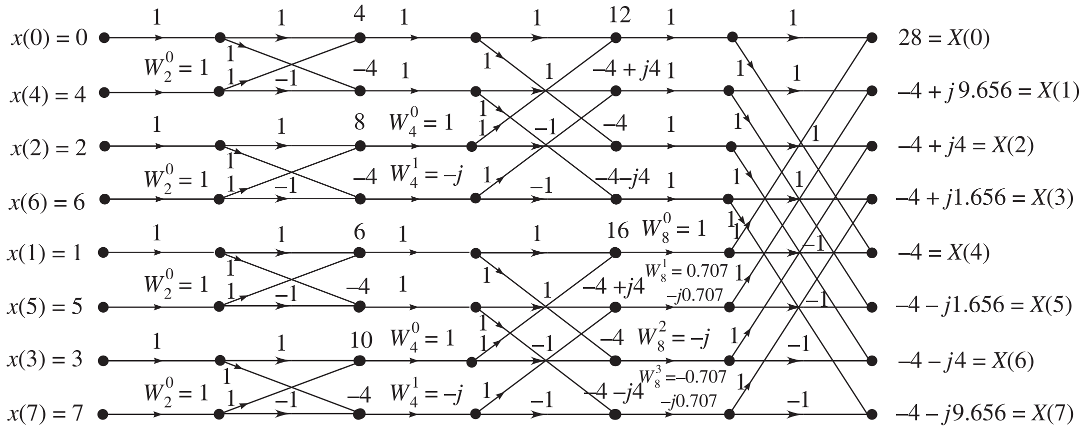


Figure 7.38 Computation of 8-point DFT of x(n) by radix-2, DIT FFT.

From Figure 7.38, we get the 8-point DFT of $x(n)$ as

$$X(k) = \{28, -4 + j9.656,\ -4 + j4,\ -4 + j1.656,\ -4,\ -4 - j1.656, -4 - j4, -4\ - j9.656\}$$

EXAMPLE 7.17 Determine the response of LTI system when the input sequence $x(n) = \{-1, 1, 2, 1, -1\}$ by radix-2 DIT FFT. The impulse response of the system is $h(n) = \{-1, 1, -1, 1\}$.

Solution: The response $y(n)$ is given by linear convolution of $x(n)$ and $h(n)$, but FFT supports only circular convolution and also for circular convolution the length of sequences should be same. Since $x(n)$ is of length 5 and $h(n)$ is of length 4, $y(n)$ will be of length $(5 + 4 - 1 = 8)$. So convert $x(n)$ and $h(n)$ to a length of $8(N_1 + N_2 - 1 = 5 + 4 - 1)$ by appending zeros.

$\therefore \quad x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\} = \{-1, 1, 2, 1, -1, 0, 0, 0\}$

and $\quad h(n) = \{h(0), h(1), h(2), h(3), h(4), h(5), h(6), h(7)\} = \{-1, 1, -1, 1, 0, 0, 0, 0\}$

To compute $y(n)$ by FFT, find DFT of $x(n)$, i.e., $X(k)$, find DFT of $h(n)$, i.e., $H(k)$, multiply them to find $Y(k) = X(k)H(k)$ and then take IDFT of $Y(k)$.

1. *Computation of* $X(k)$

$x(n)$ in bit reversed order is

$$x_r(n) = \{x(0), x(4), x(2), x(6), x(1), x(5), x(3), x(7)\} = \{-1,\ -1, 2, 0, 1, 0, 1, 0\}$$

The 8-point DFT of $x(n)$, i.e., $X(k)$ is computed by radix-2 DIT FFT as shown in Figure 7.39. For DIT FFT, the input is in bit reversed order and output is in normal order.

From Figure 7.39, we get the 8-point DFT of $x(n)$ as

$$X(k) = \{2,\ -j(2 + \sqrt{2}), -4,\ j(2 - \sqrt{2}), -2,\ j(-2 + \sqrt{2}), -4, j(2 + \sqrt{2})\}$$

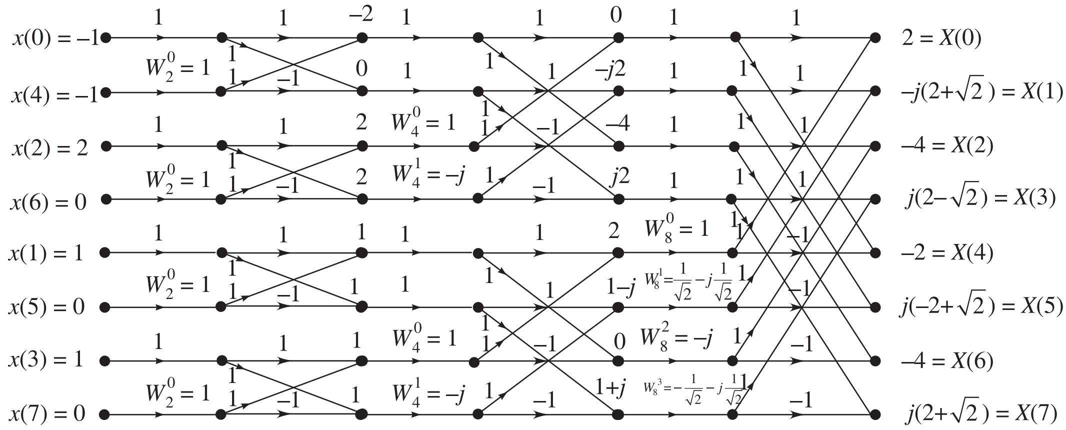


Figure 7.39 Computation of 8-point DFT of *x*(*n*) by radix-2 DIT FFT.

2. *Computation of H*(*k*)
h(*n*) in bit reverse order is

$$h_r(n) = \{h(0), h(4), h(2), h(6), h(1), h(5), h(3), h(7)\} = \{-1, 0, -1, 0, 1, 0, 1, 0\}$$

The 8-point DFT of *h*(*n*), i.e., *H*(*k*) is computed by radix-2 DIT FFT algorithm as shown in Figure 7.40. For DIT FFT, the input is in bit reversed order and output is in normal order.

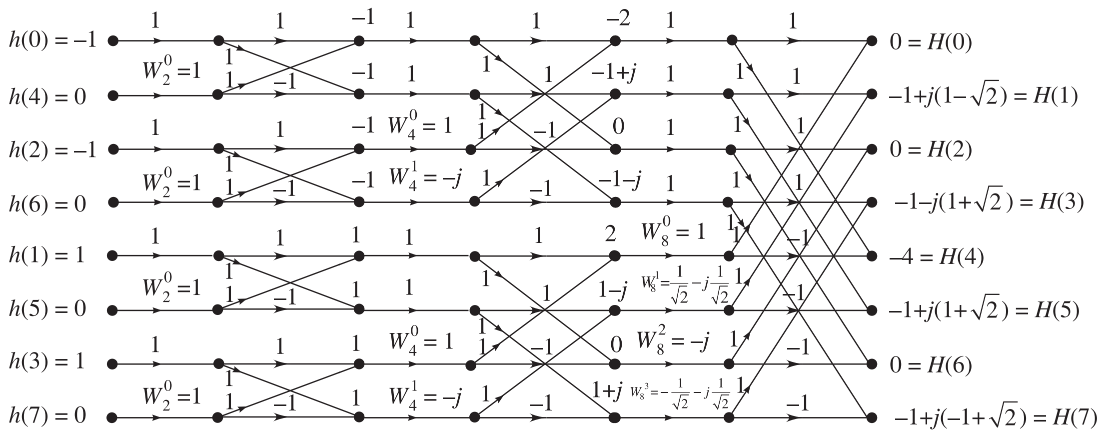


Figure 7.40 Computation of 8-point DFT of *h*(*n*) by radix-2 DIT FFT.

From Figure 7.40, we get the 8-point DFT of *h*(*n*) as

$$H(k) = \{0, -1 + j(1-\sqrt{2}), 0, \; -1 - j(1+\sqrt{2}), -4, \; -1 + j(1+\sqrt{2}), 0, -1 + j(-1+\sqrt{2})\}$$

3. *Determination of Y*(*k*)

$$Y(k) = X(k)H(k) = \{2, -j(2+\sqrt{2}), -4, j(2-\sqrt{2}), -2, j(-2+\sqrt{2}), -4, j(2+\sqrt{2})\}$$

$$\times \{0, -1+j(1-\sqrt{2}), 0, -1-j(1+\sqrt{2}), -4, -1+j(1+\sqrt{2}), 0, -1+j(-1+\sqrt{2})\}$$

$$= \{0, -\sqrt{2}+j(2+\sqrt{2}), 0, \sqrt{2}-j(2-\sqrt{2}), 8, \sqrt{2}+j(2-\sqrt{2}), 0, -\sqrt{2}-j(2+\sqrt{2})\}$$

4. *Computation of IDFT of Y*(*k*)
For the obtained $Y(k)$, we have

$$Y^*(k) = \{0, -\sqrt{2}-j(2+\sqrt{2}), 0, \sqrt{2}+j(2-\sqrt{2}), 8, \sqrt{2}-j(2-\sqrt{2}), 0, -\sqrt{2}+j(2+\sqrt{2})\}$$

$Y^*(k)$ in bit reverse order is

$$Y_r^*(k) = \{0, 8, 0, 0, -\sqrt{2}-j(2+\sqrt{2}), \sqrt{2}-j(2-\sqrt{2}), \sqrt{2}+j(2-\sqrt{2}), -\sqrt{2}+j(2+\sqrt{2})\}$$

The 8-point radix-2 DIT FFT is computed as shown in Figure 7.41. For DIT FFT, the input is in bit reversed order and output is in normal order.

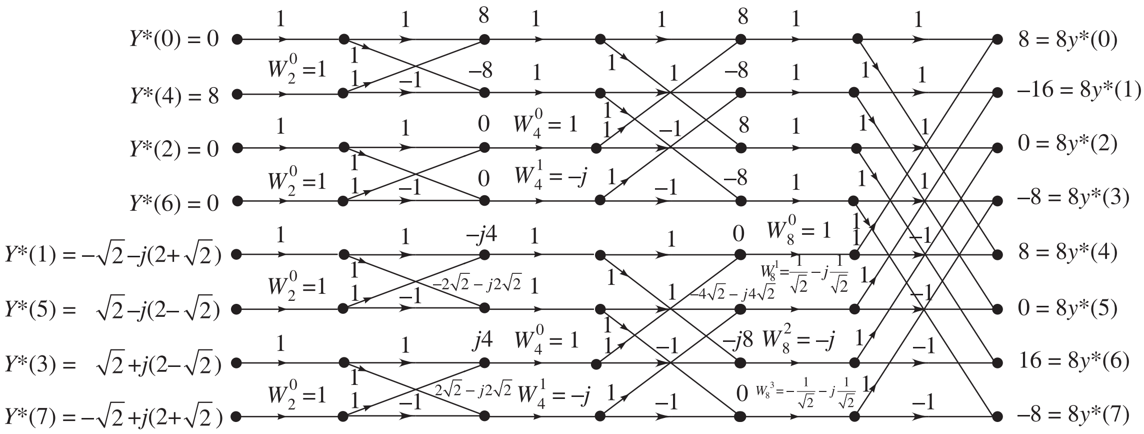


Figure 7.41 Computation of 8-point DFT of $Y^*(k)$ by radix-2, DIT FFT.

From Figure 7.41, we get $y^*(n) = \dfrac{1}{8}\{8, -16, 0, -8, 8, 0, 16, -8\}$

$$= \{1, -2, 0, -1, 1, 0, 2, -1\}$$

$\therefore$ $$y(n) = \{y^*(n)\}^* = \{1, -2, 0, -1, 1, 0, 2, -1\}$$

EXAMPLE 7.18 Find the IDFT of the sequence

$$X(k) = \{4, 1-j2.414, 0, 1-j0.414, 0, 1+j0.414, 0, 1+j2.414\}$$

using DIF algorithm.

Solution: The IDFT $x(n)$ of the given 8-point sequence $X(k)$ can be obtained by finding $X^*(k)$, the conjugate of $X(k)$, finding the 8-point DFT of $X^*(k)$, using DIF algorithm to get

$8x^*(n)$, taking the conjugate of that to get $8x(n)$ and then dividing the result by 8 to get $x(n)$. For DIF algorithm, input $X^*(k)$ must be in normal order. The output will be in bit reversed order for the given $X(k)$.

$$X^*(k) = \{4, 1 + j2.414, 0, 1 + j0.414, 0, 1 - j0.414, 0, 1 - j2.414\}$$

The DFT of $X^*(k)$ using radix-2, DIF FFT algorithm is computed as shown in Figure 7.42.

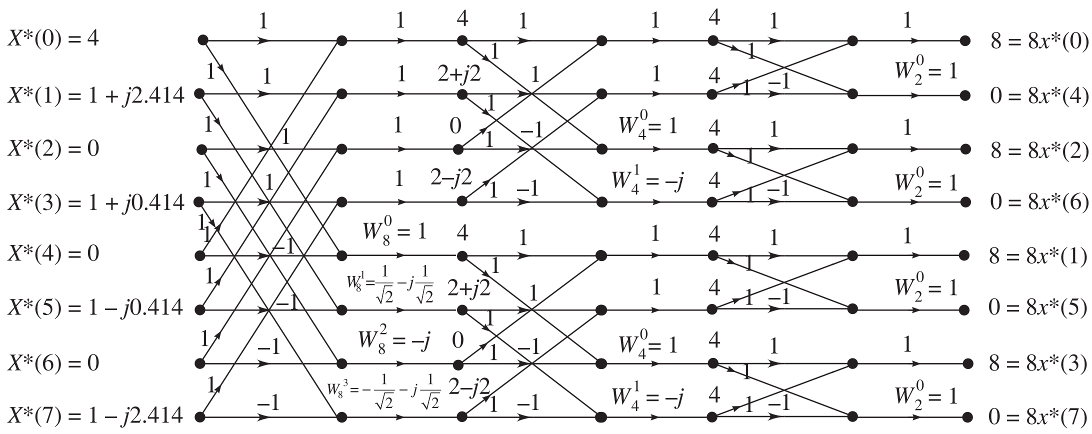


Figure 7.42 Computation of 8-point DFT of $X^*(k)$ by radix-2 DIF FFT.

From the DIF FFT algorithm of Figure 7.42, we get

$$8x_r^*(n) = \{8, 0, 8, 0, 8, 0, 8, 0\}$$

$$\therefore \qquad 8x_r(n) = \{8, 0, 8, 0, 8, 0, 8, 0\}^* = \{8, 0, 8, 0, 8, 0, 8, 0\}$$

$$\therefore \qquad x(n) = \frac{1}{8}\{8, 8, 8, 8, 0, 0, 0, 0\} = \{1, 1, 1, 1, 0, 0, 0, 0\}$$

EXAMPLE 7.19 Compute the IDFT of the sequence

$$X(k) = \{7, -0.707 - j0.707, -j, 0.707 - j0.707, 1, 0.707 + j0.707, j, -0.707 + j0.707\}$$

using DIT algorithm.

Solution: The IDFT $x(n)$ of the given sequence $X(k)$ can be obtained by finding $X^*(k)$, the conjugate of $X(k)$, finding the 8-point DFT of $X^*(k)$ using radix-2 DIT FFT algorithm to get $8x^*(n)$, taking the conjugate of that to get $8x(n)$ and then dividing by 8 to get $x(n)$. For DIT FFT, the input $X^*(k)$ must be in bit reverse order. The output $8x^*(n)$ will be in normal order. For the given $X(k)$.

$$X^*(k) = \{7, -0.707 + j0.707, j, 0.707 + j0.707, 1, 0.707 - j0.707, -j, -0.707 - j0.707\}$$

$X^*(k)$ in bit reverse order is

$$X_r^*(k) = \{7, 1, j, -j, -0.707 + j0.707, 0.707 - j0.707, 0.707 + j0.707, -0.707 - j0.707\}$$

The 8-point DFT of $X^*(k)$ using radix-2, DIT FFT algorithm is computed as shown in Figure 7.43.

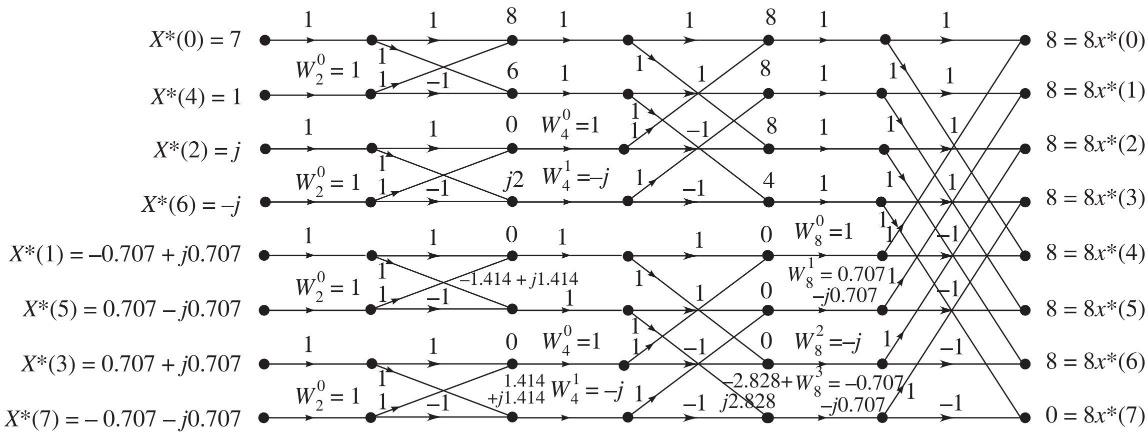


Figure 7.43 Computation of 8-point DFT of $X^*(k)$ by radix-2, DIT FFT.

From the DIT FFT algorithm of Figure 7.43, we have

$$8x^*(n) = \{8, 8, 8, 8, 8, 8, 8, 0\}$$

$$\therefore \quad 8x(n) = \{8, 8, 8, 8, 8, 8, 8, 0\}$$

$$\therefore \quad x(n) = \{1, 1, 1, 1, 1, 1, 1, 0\}$$

EXAMPLE 7.20 Compute the IDFT of the square wave sequence $X(k) = \{12, 0, 0, 0, 4, 0, 0, 0\}$ using DIF algorithm.

Solution: The IDFT $x(n)$ of the given sequence $X(k)$ can be obtained by finding $X^*(k)$, the conjugate of $X(k)$, finding the 8-point DFT of $X^*(k)$ using DIF algorithm to get $8x^*(n)$ taking the conjugate of that to get $8x(n)$ and then dividing the result by 8 to get $x(n)$. For DIF algorithm, the input $X^*(k)$ must be in normal order and the output $8x^*(n)$ will be in bit reversed order.

For the given $X(k)$

$$X^*(k) = \{12, 0, 0, 0, 4, 0, 0, 0\}$$

The 8-point DFT of $X^*(k)$ using radix-2, DIF FFT algorithm is computed as shown in Figure 7.44.

From Figure 7.44, we have

$$8x_r^*(n) = \{16, 16, 16, 16, 8, 8, 8, 8\}$$

$$\therefore \quad 8x_r(n) = \{16, 16, 16, 16, 8, 8, 8, 8\}^* = \{16, 16, 16, 16, 8, 8, 8, 8\}$$

$$\therefore \quad x(n) = \frac{1}{8}\{16, 8, 16, 8, 16, 8, 16, 8\} = \{2, 1, 2, 1, 2, 1, 2, 1\}$$

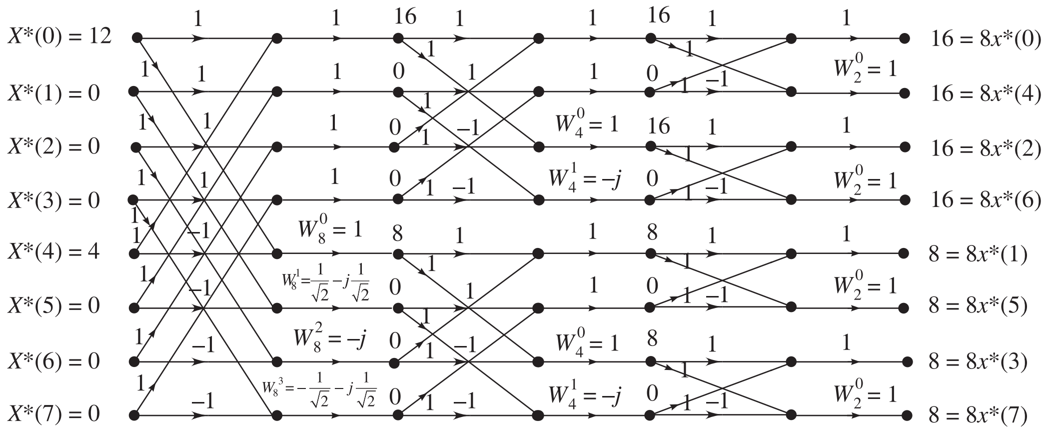


Figure 7.44 Computation of 8-point DFT of $X^*(k)$ by radix-2 DIF FFT.

EXAMPLE 7.21 Find the IDFT of the following

(a) $X(k) = \{1, 1 - j2, -1, 1 + j2\}$ (b) $X(k) = \{1, 0, 1, 0\}$ (c) $X(4) = \{3, 2 + j, 1, 2 - j\}$

Solution: (a) Given $X(k) = \{1, 1 - j2, -1, 1 + j2\}$

Let us find the IDFT of $X(k)$ by radix-2, DIT FFT as shown in Figure 7.45.

$$X^*(k) = \{1, 1 + j2, -1, 1 - j2\}; \; X_r^*(k) = \{1, -1, 1 + j2, 1 - j2\}$$

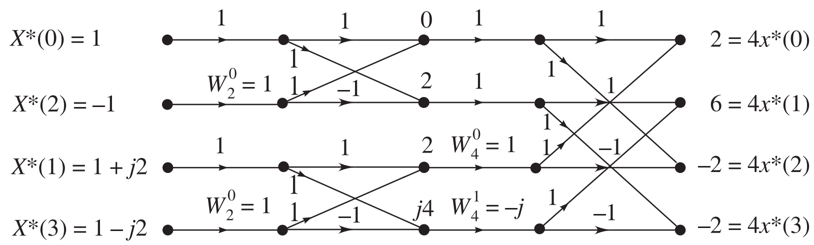


Figure 7.45 Computation of 4-point DFT of $X^*(k)$ by radix-2 DIT FFT.

From Figure 7.45, we have

$$4x^*(n) = \{2, 6, -2, -2\}$$

$$\therefore \qquad x(n) = \frac{1}{4}\{2, 6, -2, -2\}^* = \{0.5, 1.5, -0.5, -0.5\}$$

(b) Given $X(k) = \{1, 0, 1, 0\}$

Let us find the IDFT of the given 4-point $X(k)$ by radix-2, DIF FFT as shown in Figure 7.46.

$$X^*(k) = \{1, 0, 1, 0\}$$

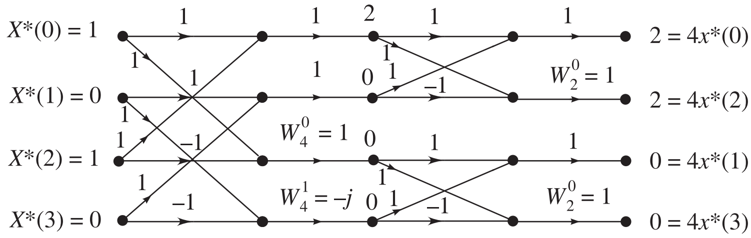


Figure 7.46 Computation of 4-point DFT of $X^*(k)$ using radix-2 DIF FFT.

From Figure 7.46, we have

$$4x_r^*(n) = \{2, 2, 0, 0\}$$

$$\therefore \qquad x(n) = \frac{1}{4}[2, 0, 2, 0]^* = \{0.5, 0, 0.5, 0\}$$

(c) Given $X(k) = \{3, 2 + j, 1, 2 - j\}$

Let us find the IDFT of the given 4-point $X(k)$ by radix-2, DIF FFT as shown in Figure 7.47.

$$X^*(k) = \{3, 2 - j, 1, 2 + j)$$

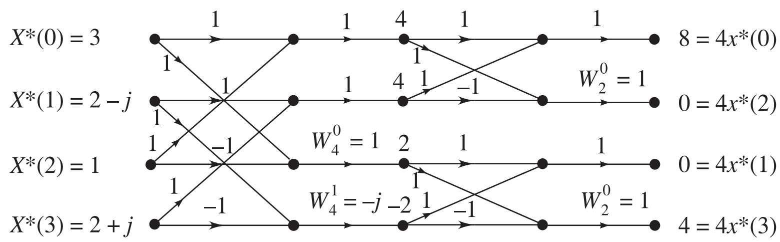


Figure 7.47 Computation of 4-point DFT of $X^*(k)$ by radix-2, DIF FFT.

From Figure 7.47, we have

$$4x_r^*(n) = \{8, 0, 0, 4\}$$

$$\therefore \qquad x(n) = \frac{1}{4}[8, 0, 0, 4]^* = \{2, 0, 0, 1\}$$

7.7 FFT ALGORITHMS FOR *N* A COMPOSITE NUMBER

Till now we have discussed DIT and DIF FFT algorithms for the important special case of N a power of 2, i.e. $N = 2^m$. They are called radix-2 FFTs. When N is a power of 2, the decomposition leads to a highly efficient computational algorithm. Furthermore, all the required computations are butterfly computations that correspond essentially to two-point DFTs. For this reason, the power-of-2 algorithms are particularly simple to implement. So in

some cases, even if N is not a power of 2, it is made a power of 2 by simply augmenting with zeros. However, in some cases it may not be possible to choose N to be a power of 2. So we have to consider composite radix FFT.

A composite or mixed radix FFT is used when N is a composite number which has more than one prime factor; for example $N = 6$ or 10 or 12. For these cases also, efficient DIT and DIF algorithms can be developed. Let us consider DIT FFT decomposition for N a composite number.

If $$N = p_1 p_2 \ldots p_m = p_1 N_1$$

where $N_1 = p_2 p_3 \ldots p_m$, the input sequence $x(n)$ can be separated into p_1 subsequences of N_1 samples each. Then the DFT can be written as

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk}$$

$$= \sum_{n=0}^{N_1-1} x(np_1)\, W_N^{np_1 k} + \sum_{n=0}^{N_1-1} x(np_1+1)\, W_N^{(np_1+1)k} + \cdots + \sum_{n=0}^{N_1-1} x(np_1+p_1-1)\, W_N^{(np_1+p_1-1)k}$$

7.7.1 Radix-3 FFT

When N is assumed a power of 3, i.e. $N = 3^m$, it becomes radix-3 FFT. For example when $N = 9 = 3^2$, the given sequence $x(n)$ can be decimated into three sequences of length $N/3$, the first sequence consisting of samples $x(0)$, $x(3)$, $x(6)$, the second sequence consisting of samples $x(1)$, $x(4)$, $x(7)$ and the third sequence consisting of samples $x(2)$, $x(5)$, $x(7)$.

For evaluating the DFT for $N = 9$, radix-3 DIT and DIF algorithms can be developed starting from fundamentals.

7.7.2 Radix-4 FFT

When N is a assumed a power of 4, i.e. $N = 4^m$, it becomes radix-4 FFT. For example, when $N = 16 = 4^2$, the given of sequence $x(n)$ can be decimated into four sequences of length $N/4$, the first sequence consisting of samples $x(0)$, $x(4)$, $x(8)$, $x(12)$ the second sequence consisting of samples $x(1)$, $x(5)$, $x(9)$, $x(13)$ and the third sequence consisting of samples $x(2)$, $x(6)$, $x(10)$, $x(14)$ and the fourth sequence consisting of samples $x(3)$, $x(7)$, $x(11)$, $x(15)$.

For evaluating the DFT for $N = 16$, radix-4 DIT and DIF algorithms can be developed starting from fundamentals.

EXAMPLE 7.22 Develop a radix-3 DIT FFT algorithm for evaluating the DFT for $N = 9$.

Solution: Given $N = 9 = 3 \times 3 = p_1 N_1$. Therefore, $p_1 = 3$ and $N_1 = 3$. So the N-point sequence is decimated into 3 sequences of length 3 samples. Therefore, the expression for DFT is

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk} = \sum_{n=0}^{N_1-1} x(p_1 n)\, W_N^{p_1 nk} + \sum_{n=0}^{N_1-1} x(p_1 n+1)\, W_N^{(p_1 n+1)k} + \sum_{n=0}^{N_1-1} x(p_1 n+2)\, W_N^{(p_1 n+2)k}$$

$$= \sum_{n=0}^{2} x(3n)\, W_9^{3nk} + \sum_{n=0}^{2} x(3n+1)\, W_9^{(3n+1)k} + \sum_{n=0}^{2} x(3n+2)\, W_9^{(3n+2)k}$$

$$= X_1(k) + W_9^k\, X_2(k) + W_9^{2k}\, X_3(k)$$

where,

$$X_1(k) = \sum_{n=0}^{2} x(3n)\, W_9^{3nk} = x(0) + x(3)\, W_9^{3k} + x(6)\, W_9^{6k}$$

$$X_2(k) = \sum_{n=0}^{2} x(3n+1)\, W_9^{3nk} = x(1) + x(4)\, W_9^{3k} + x(7)\, W_9^{6k}$$

$$X_3(k) = \sum_{n=0}^{2} x(3n+2)\, W_9^{3nk} = x(2) + x(5)\, W_9^{3k} + x(8)\, W_9^{6k}$$

$\therefore$

$$X_1(0) = x(0) + x(3)\, W_9^0 + x(6)\, W_9^0$$

$$X_1(1) = x(0) + x(3)\, W_9^3 + x(6)\, W_9^6$$

$$X_1(2) = x(0) + x(3)\, W_9^6 + x(6)\, W_9^{12}$$

and

$$X_2(0) = x(1) + x(4)\, W_9^0 + x(7)\, W_9^0$$

$$X_2(1) = x(1) + x(4)\, W_9^3 + x(7)\, W_9^6$$

$$X_2(2) = x(1) + x(4)\, W_9^6 + x(7)\, W_9^{12}$$

and

$$X_3(0) = x(2) + x(5)\, W_9^0 + x(8)\, W_9^0$$

$$X_3(1) = x(2) + x(5)\, W_9^3 + x(8)\, W_9^6$$

$$X_3(2) = x(2) + x(5)\, W_9^6 + x(8)\, W_9^{12}$$

Therefore,

$$X(0) = X_1(0) + W_9^0\, X_2(0) + W_9^0\, X_3(0)$$

$$X(1) = X_1(1) + W_9^1\, X_2(1) + W_9^2\, X_3(1)$$

$$X(2) = X_1(2) + W_9^2\, X_2(2) + W_9^4\, X_3(2)$$

$$X(3) = X_1(3) + W_9^3\, X_2(3) + W_9^6\, X_3(3)$$

$$= X_1(0) + W_9^3 X_2(0) + W_9^6 X_3(0)$$

$$X(4) = X_1(1) + W_9^4\, X_2(1) + W_9^8\, X_3(1)$$

$$X(5) = X_1(2) + W_9^5\, X_2(2) + W_9^{10}\, X_3(2)$$

$$X(6) = X_1(6) + W_9^6\, X_2(6) + W_9^{12}\, X_3(6)$$
$$= X_1(0) + W_9^6\, X_2(0) + W_9^{12}\, X_3(0)$$
$$X(7) = X_1(1) + W_9^7\, X_2(1) + W_9^{14}\, X_3(1)$$
$$X(8) = X_1(2) + W_9^8\, X_2(2) + W_9^{16}\, X_3(2)$$

Figure 7.48 shows the DIT radix-3, FFT flow diagram for $N = 9$. The input sequence is in bit reversed order and the output sequence is in normal order.

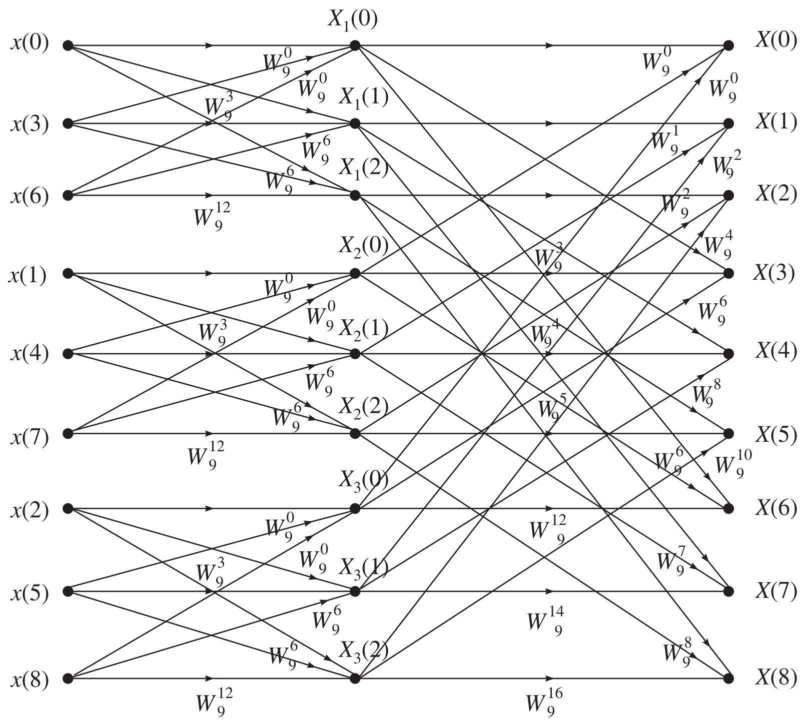


Figure 7.48 Radix-3 DIT FFT flow diagram for $N = 9$.

EXAMPLE 7.23 Develop a radix-3 DIF FFT algorithm for evaluating the DFT for $N = 9$.

Solution: Here $N = p_1 N_1 = 9 = 3 \times 3$

So to develop radix-3 DIF FFT algorithm, we have to generate three subsequences each of length 3 samples from the given 9 sample sequence $x(n)$. Starting from fundamentals, we have

$$X(k) = \sum_{n=0}^{2} x(n)\, W_9^{nk} + \sum_{n=0}^{2} x(n+3)\, W_9^{(n+3)k} + \sum_{n=0}^{2} x(n+6)\, W_9^{(n+6)k}$$

$$= \sum_{n=0}^{2} \left[x(n) + x(n+3)\, W_9^{3k} + x(n+6)\, W_9^{6k} \right] W_9^{nk}$$

$$\therefore \qquad X(3k) = \sum_{n=0}^{2} [x(n) + x(n+3) + x(n+6)] W_9^{3nk} = \sum_{n=0}^{2} f(n) W_9^{3nk} \qquad [\text{Since } W_N^{Nk} = 1]$$

where $f(n) = x(n) + x(n + 3) + x(n + 6)$

$$X(3k+1) = \sum_{n=0}^{2} [x(n) + x(n+3) W_9^3 + x(n+6) W_9^6] W_9^n W_9^{3nk} = \sum_{n=0}^{2} g(n) W_9^n W_9^{3nk}$$

where $g(n) = x(n) + x(n+3) W_9^3 + x(n+6) W_9^6$

$$X(3k+2) = \sum_{n=0}^{2} [x(n) + x(n+3) W_9^6 + x(n+6) W_9^3] W_9^{2n} W_9^{3nk} = \sum_{n=0}^{2} h(n) W_9^{2n} W_9^{3nk}$$

where $h(n) = x(n) + x(n+3) W_9^6 + x(n+6) W_9^3$

$$\therefore \qquad f(0) = x(0) + x(3) + x(6)$$
$$f(1) = x(1) + x(4) + x(7)$$
$$f(2) = x(2) + x(5) + x(8)$$
$$g(0) = x(0) + x(3)\, W_9^3 + x(6) W_9^6$$
$$g(1) = x(1) + x(4)\, W_9^3 + x(7) W_9^6$$
$$g(2) = x(2) + x(5)\, W_9^3 + x(8) W_9^6$$
$$h(0) = x(0) + x(3)\, W_9^6 + x(6) W_9^3$$
$$h(1) = x(1) + x(4)\, W_9^6 + x(7) W_9^3$$
$$h(2) = x(2) + x(5)\, W_9^6 + x(8) W_9^3$$

$$\therefore \qquad X(0) = f(0) + f(1) + f(2)$$
$$X(3) = f(0) + f(1) W_9^3 + f(2) W_9^6$$
$$X(6) = f(0) + f(1) W_9^6 + f(2) W_9^3$$
$$X(1) = g(0) + g(1) W_9^1 + g(2) W_9^2$$
$$X(4) = g(0) + g(1) W_9^1 W_9^3 + g(2) W_9^2 W_9^6$$
$$X(7) = g(0) + g(1) W_9^1 W_9^6 + g(2) W_9^2 W_9^3$$
$$X(2) = h(0) + h(1) W_9^2 + h(2) W_9^4$$
$$X(5) = h(0) + h(1) W_9^2 W_9^3 + h(2) W_9^4 W_9^6$$
$$X(8) = h(0) + h(1) W_9^2 W_9^6 + h(2) W_9^4 W_9^3$$

Figure 7.49 shows the radix-3 DIF FFT flow diagram for decomposing the DFT for $N = 3 \times 3$.

EXAMPLE 7.24 Develop DIT FFT algorithms for decomposing the DFT for $N = 6$ and draw the flow diagrams for (a) $N = 2 \times 3$ and (b) $N = 3 \times 2$. Also by using FFT algorithm developed in part (a) and (b) evaluate the DFT values for $x(n) = \{1, 1, 1, 2, 2, 2\}$.

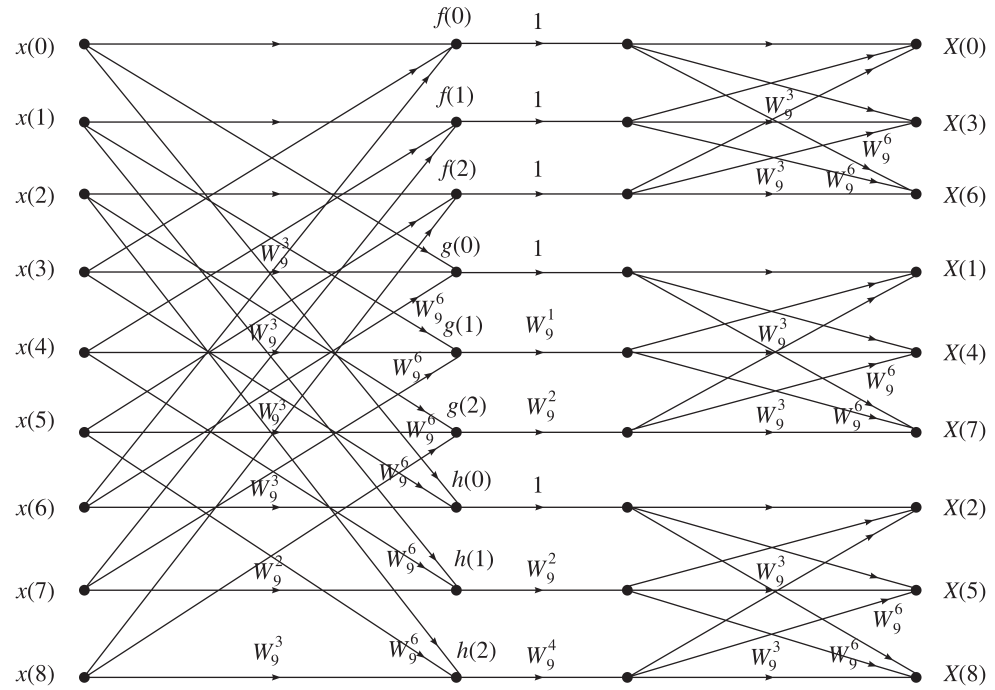


Figure 7.49 A radix-3 DIF FFT flow diagram for decomposing the DFT for $N = 9 = 3 \times 3$.

Solution:

(a) For $N = 6 = 2 \times 3 = p_1N_1$, for DIT FFT, the given sequence $x(n)$ is decomposed into two sequences each of 3 samples.

$$X(k) = \sum_{n=0}^{2} x(2n)W_6^{2nk} + \sum_{n=0}^{2} x(2n+1)W_6^{(2n+1)k}$$

$$= \sum_{n=0}^{2} x(2n)W_6^{2nk} + W_6^k \sum_{n=0}^{2} x(2n+1)W_6^{2nk}$$

$$= X_1(k) + W_6^k X_2(k), \qquad k = 0, 1, 2, 3, 4, 5$$

$$W_6^0 = e^{-j\frac{2\pi}{6}0} = 1$$

$$W_6^1 = e^{-j\frac{2\pi}{6}1} = \cos\frac{\pi}{3} - j\sin\frac{\pi}{3} = 0.5 - j0.866$$

$$W_6^2 = e^{-j\frac{2\pi}{6}2} = -0.5 - j0.866$$

$$W_6^3 = e^{-j\frac{2\pi}{6}3} = -1$$

$$W_6^4 = -0.5 + j0.866$$

$$W_6^5 = 0.5 + j0.866$$

Also $$X_i(k+3) = X_i(k)$$

$$X_1(k) = \sum_{n=0}^{2} x(2n) W_6^{2nk} = x(0) + x(2)\, W_6^{2k} + x(4)\, W_6^{4k}$$

$\therefore$ $$X_1(0) = x(0) + x(2) + x(4)$$
$$= 1 + 1 + 2 = 4$$

$$X_1(1) = x(0) + x(2) W_6^2 + x(4) W_6^4$$
$$= 1 + 1(-0.5 - j0.866) + 2(-0.5 + j0.866) = -0.5 + j0.866$$

$$X_1(2) = x(0) + x(2) W_6^4 + x(4) W_6^8 = x(0) + x(2) W_6^4 + x(4) W_6^2$$
$$= 1 + 1(-0.5 + j0.866) + 2(-0.5 - j0.866) = -0.5 - j0.866$$

$$X_2(k) = \sum_{n=0}^{2} x(2n+1) W_6^{2nk} = x(1) + x(3) W_6^{2k} + x(5) W_6^{4k}$$

$\therefore$ $$X_2(0) = x(1) + x(3) + x(5)$$
$$= 1 + 2 + 2 = 5$$

$$X_2(1) = x(1) + x(3) W_6^2 + x(5) W_6^4$$
$$= 1 + 2(-0.5 - j0.866) + 2(-0.5 + j0.866) = -1$$

$$X_2(2) = x(1) + x(3) W_6^4 + x(5) W_6^8 = x(1) + x(3) W_6^4 + x(5) W_6^2$$
$$= 1 + 2(-0.5 + j0.866) + 2(-0.5 - j0.866) = -1$$

Since $X(k) = X_1(k) + W_6^k X_2(\mathrm{k})$, we have

$$X(0) = X_1(0) + X_2(0)$$
$$= 4 + 5 = 9$$

$$X(1) = X_1(1) + W_6^1 X_2(1)$$
$$= (-0.5 + j0.866) + (0.5 - j0.866)(-1) = -1 + j1.732$$

$$X(2) = X_1(2) + W_6^2 X_2(2)$$
$$= (-0.5 - j0.866) + (-0.5 - j0.866)(-1) = 0$$

$$X(3) = X_1(3) + W_6^3 X_2(3) = X_1(0) + W_6^3 X_2(0)$$
$$= 4 + (-1)5 = -1$$

$$X(4) = X_1(4) + W_6^4\, X_2(4) = X_1(1) + W_6^4\, X_2(1)$$
$$= (-0.5 + j0.866) + (-0.5 + j0.866)(-1) = 0$$
$$X(5) = X_1(5) + W_6^5\, X_2(5) = X_1(2) + W_6^5\, X_2(2)$$
$$= (-0.5 - j0.866) + (0.5 + j0.866)\,(-1) = -1 - j1.732$$

Figure 7.50 shows the decimation-in-time FFT flow diagram for $n = 2 \times 3$.

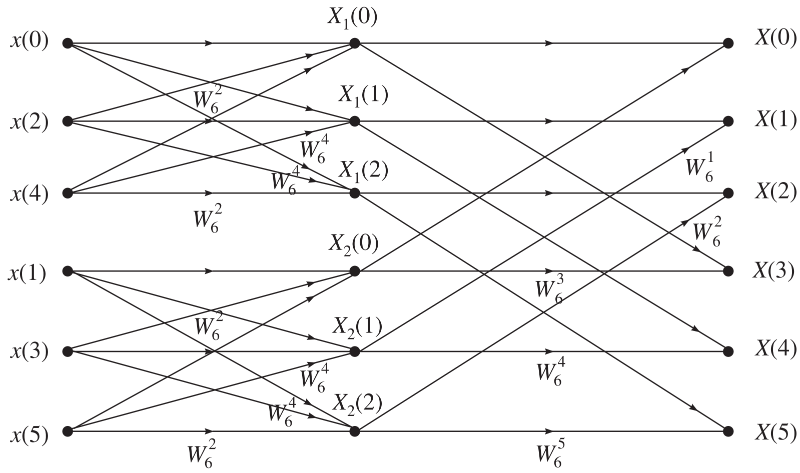


Figure 7.50 DIT FFT flow diagram for $N = 6 = 2 \times 3$.

$\therefore$ The DFT $X(k) = \{9,\ -1 + j1.732,\ 0,\ -1,\ 0, -1 - j1.732\}$

(b) For $N = 3 \times 2 = p_1 N_1$, we have $p_1 = 3$ and $N_1 = 2$, So the given sequence $x(n)$ is to be split into 3 sequences each of two samples. So we have

$$X(k) = \sum_{n=0}^{5} x(n)\, W_6^{nk} = \sum_{n=0}^{1} x(3n)\, W_6^{3nk} + \sum_{n=0}^{1} x(3n+1)\, W_6^{(3n+1)k} + \sum_{n=0}^{1} x(3n+2)\, W_6^{(3n+2)k}$$

$$= \sum_{n=0}^{1} x(3n)\, W_6^{3nk} + W_6^k \sum_{n=0}^{1} x(3n+1)\, W_6^{3nk} + W_6^{2k} \sum_{n=0}^{1} x(3n+2)\, W_6^{3nk}$$

$$= X_1(k) + W_6^k X_2(k) + W_6^{2k} X_3(k)$$

where $X_1(k) = \sum_{n=0}^{1} x(3n)\, W_6^{3nk} = x(0) + x(3)\, W_6^{3k}$

$$X_2(k) = \sum_{n=0}^{1} x(3n+1)\, W_6^{3nk} = x(1) + x(4)\, W_6^{3k}$$

$$X_3(k) = \sum_{n=0}^{1} x(3n+2)\, W_6^{3nk} = x(2) + x(5)\, W_6^{3k}$$

Also $X_i(k + 2) = X_i(k)$

$\therefore$
$$X_1(0) = x(0) + x(3)W_6^0$$
$$= 1 + 2(1) = 3$$
$$X_1(1) = x(0) + x(3)W_6^3$$
$$= 1 + 2(-1) = -1$$
$$X_2(0) = x(1) + x(4)W_6^0$$
$$= 1 + 2(1) = 3$$
$$X_2(1) = x(1) + x(4)W_6^3$$
$$= 1 + 2(-1) = -1$$
$$X_3(0) = x(2) + x(5)W_6^0$$
$$= 1 + 2(1) = 3$$
$$X_3(1) = x(2) + x(5)W_6^3$$
$$= 1 + 2(-1) = -1$$

$$X(k) = X_1(k) + W_6^k X_2(k) + W_6^{2k} X_3(k)$$

$\therefore$
$$X(0) = X_1(0) + W_6^0 X_2(0) + W_6^0 X_3(0)$$
$$= 3 + (1)(3) + 1(3) = 9$$
$$X(1) = X_1(1) + W_6^1 X_2(1) + W_6^2 X_3(1)$$
$$= -1 + (0.5 - j0.866)(-1) + (-0.5 - j0.866)(-1) = -1 + j1.732$$
$$X(2) = X_1(2) + W_6^2 X_2(2) + W_6^4 X_3(2) = X_1(0) + W_6^2 X_2(0) + W_6^4 X_3(0)$$
$$= 3 + (-0.5 - j0.866)(3) + (-0.5 + j0.866)(3) = 0$$
$$X(3) = X_1(3) + W_6^3 X_2(3) + W_6^6 X_3(3) = X_1(1) + W_6^3 X_2(1) + W_6^6 X_3(1)$$
$$= -1 + (-1)(-1) + 1(-1) = -1$$
$$X(4) = X_1(4) + W_6^4 X_2(4) + W_6^8 X_3(4) = X_1(0) + W_6^4 X_2(0) + W_6^2 X_3(0)$$
$$= 3 + (-0.5 + j0.866)3 + (-0.5 - j0.866)3 = 0$$
$$X(5) = X_1(1) + W_6^5 X_2(1) + W_6^4 X_3(1)$$
$$= -1 + (0.5 + j0.866)(-1) + (-0.5 + j0.866)(-1) = -1 - j1.732$$

Therefore, the DFT is $X(k) = \{9, -1 + j1.732, 0, -1, 0, -1 - j1.732\}$

Figure 7.51 shows the decimation-in-time FFT flow diagram for $N = 6 = 3 \times 2$.

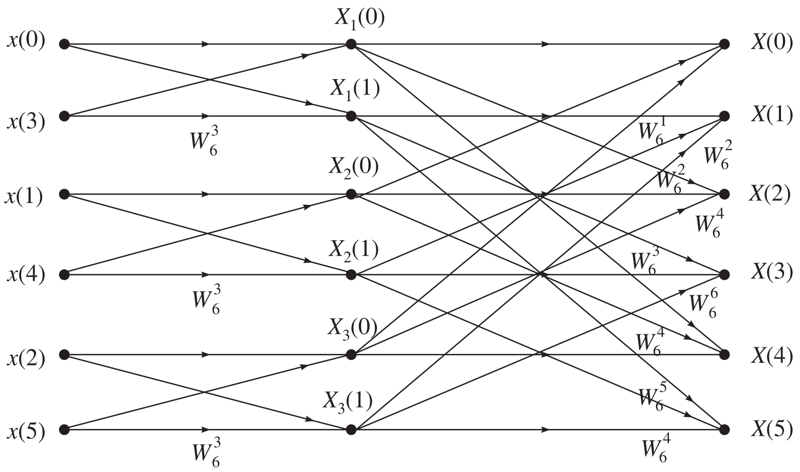


Figure 7.51 DIT FFT flow diagram for $N = 6 = 3 \times 2$.

EXAMPLE 7.25 Develop a DIF FFT algorithm for decomposing the DFT for $N = 6$ and draw the flow diagrams for (a) $N = 3 \times 2$ and (b) $N = 2 \times 3$.

Solution:

(a) For DIF FFT, for $N = 3 \times 2$, we have to generate two sequences of length 3 samples each and find the DFT. So we have

$$X(k) = \sum_{n=0}^{5} x(n) W_6^{nk} = \sum_{n=0}^{2} x(n) W_6^{nk} + \sum_{n=3}^{5} x(n) W_6^{nk}$$

i.e.

$$X(k) = \sum_{n=0}^{2} x(n) W_6^{nk} + \sum_{n=0}^{2} x(n+3) W_6^{(n+3)k}$$

$$= \sum_{n=0}^{2} \left[x(n) + x(n+3) W_6^{3k} \right] W_6^{nk}$$

$$X(2k) = \sum_{n=0}^{2} \left[x(n) + x(n+3) W_6^{6k} \right] W_6^{2nk}$$

$$= \sum_{n=0}^{2} [x(n) + x(n+3)] W_6^{2nk}$$

$$X(2k+1) = \sum_{n=0}^{2} \left[x(n) + x(n+3) W_6^{3(2k+1)} \right] W_6^{(2k+1)n}$$

$$= \sum_{n=0}^{2} [x(n) - x(n+3)] W_6^n W_6^{2kn}$$

Let the generated sequences be $g(n)$ and $h(n)$, where

$$g(n) = x(n) + x(n + 3), \qquad h(n) = x(n) - x(n + 3)$$

$$\therefore \qquad X(2k) = \sum_{n=0}^{2} g(n) W_6^{2nk} \quad \text{and} \quad X(2k+1) = \sum_{n=0}^{2} h(n) W_6^{n} W_6^{2nk}$$

$$\therefore \qquad g(0) = x(0) + x(3), \qquad g(1) = x(1) + x(4), \qquad g(2) = x(2) + x(5)$$

$$h(0) = x(0) - x(3), \qquad h(1) = x(1) - x(4), \qquad h(2) = x(2) - x(5)$$

$$X(0) = g(0) + g(1) + g(2)$$

$$X(2) = g(0) + g(1) W_6^2 + g(2) W_6^4$$

$$X(4) = g(0) + g(1) W_6^4 + g(2) W_6^8$$

$$X(1) = h(0) + h(1) W_6^1 + h(2) W_6^2$$

$$X(3) = h(0) + h(1) W_6^1 W_6^2 + h(2) W_6^2 W_6^4$$

$$X(5) = h(0) + h(1) W_6^1 W_6^4 + h(2) W_6^2 W_6^8$$

Figure 7.52 shows the DIF FFT flow diagram for decomposing the DFT for $N = 6$. Its input is in normal order and output is in bit reversed order.

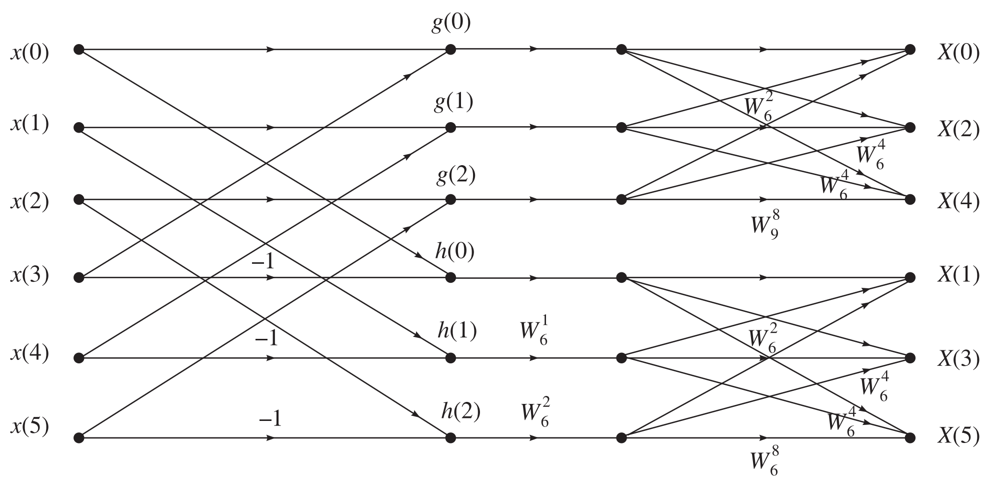


Figure 7.52 DIF FFT flow diagram for decomposing the DFT for $N = 6 = 3 \times 2$.

(b) For DIF FFT, for $N = 2 \times 3$ we have to generate 3 sequences of length 2 samples each and find the DFT. So we have

$$X(k) = \sum_{n=0}^{5} x(n) W_6^{nk} = \sum_{n=0}^{1} x(n) W_6^{nk} + \sum_{n=2}^{3} x(n) W_6^{nk} + \sum_{n=4}^{5} x(n) W_6^{nk}$$

$$= \sum_{n=0}^{1} \left[x(n) + x(n+2) W_6^{2k} + x(n+4) W_6^{4k} \right] W_6^{nk}$$

$$\therefore \quad X(3k) = \sum_{n=0}^{1} \left[x(n) + x(n+2) W_6^{6k} + x(n+4) W_6^{12k} \right] W_6^{3nk}$$

$$= \sum_{n=0}^{1} \left[x(n) + x(n+2) + x(n+4) \right] W_6^{3nk}$$

$$X(3k+1) = \sum_{n=0}^{1} \left[x(n) + x(n+2) W_6^{2(3k+1)} + x(n+4) W_6^{4(3k+1)} \right] W_6^{n(3k+1)}$$

$$= \sum_{n=0}^{1} \left[x(n) + x(n+2) W_6^{2} + x(n+4) W_6^{4} \right] W_6^{n}\, W_6^{3nk}$$

$$X(3k+2) = \sum_{n=0}^{1} \left[x(n) + x(n+2) W_6^{2(3k+2)} + x(n+4) W_6^{4(3k+2)} \right] W_6^{n(3k+2)}$$

$$= \sum_{n=0}^{1} \left[x(n) + x(n+2) W_6^{4} + x(n+4) W_6^{2} \right] W_6^{2n}\, W_6^{3nk}$$

Let the generated two sample sequences be $f(n)$, $g(n)$ and $h(n)$ where

$$f(n) = x(n) + x(n+2) + x(n+4)$$

$$\therefore \quad f(0) = x(0) + x(2) + x(4), \quad f(1) = x(1) + x(3) + x(5)$$

$$g(n) = x(n) + x(n+2) W_6^2 + x(n+4) W_6^4$$

$$\therefore \quad g(0) = x(0) + x(2) W_6^2 + x(4) W_6^4, \quad g(1) = x(1) + x(3) W_6^2 + x(5) W_6^4$$

$$h(n) = x(n) + x(n+2) W_6^4 + x(n+4) W_6^2$$

$$h(0) = x(0) + x(2) W_6^4 + x(4) W_6^2, \quad h(1) = x(1) + x(3) W_6^4 + x(5) W_6^2$$

Therefore, we have the DFT sequence

$$X(0) = f(0) + f(1) W_6^0$$

$$X(1) = g(0) + g(1) W_6^1$$

$$X(2) = h(0) + h(1) W_6^2$$

$$X(3) = f(0) + f(1)W_6^0 W_6^3$$

$$X(4) = g(0) + g(1)W_6^1 W_6^3$$

$$X(5) = h(0) + h(1)W_6^2 W_6^3$$

Figure 7.53 shows the DIF FFT flow diagram for decomposing the DFT for N = 6.

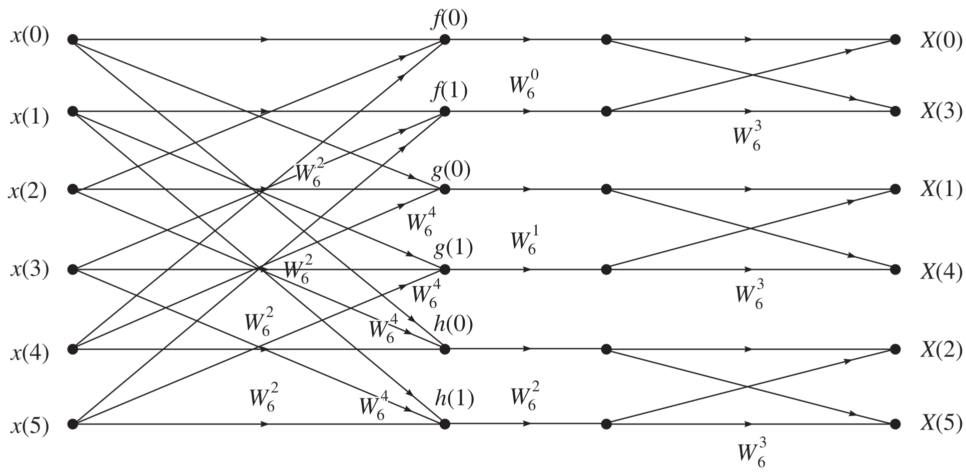


Figure 7.53 DIF FFT flow diagram for decomposing the DFT for $N = 6 = 2 \times 3$.

SHORT QUESTIONS WITH ANSWERS

1. What is the importance of DFT?
Ans. The importance of DFT is it plays an important role in the analysis, the design, and the implementation of DSP algorithms and systems.

2. What is the DFT of a single number A?
Ans. The DFT of a single number A is the number A itself.

3. What is the DFT of a 2 sample sequence $x(n) = \{A, B\}$?
Ans. The DFT of a 2 sample sequence $x(n) = \{A, B\}$ is $X(k) = \{A + B, A - B\}$.

4. What is FFT?
Ans. The FFT [Fast Fourier Transform] is a method or algorithm for computing the DFT with reduced number of calculations. The computational efficiency is achieved by employing divide and conquer approach. This is based on the decomposition of an N-point DFT into successively smaller DFTs.

5. Compare the number of multiplications required to compute the DFT of a 64-point sequence using direct computation and that using FFT?
Ans. The direct computation of DFT of a 64-point sequence requires $64^2 = 4096$ complex multiplications, whereas using FFT it requires only $(64/2) \log_2 64 = 192$ complex multiplications.

6. How many complex multiplications and complex additions does the direct computation of the DFT of an N-point sequence require?
Ans. The direct computation of the DFT of an N-point sequence requires N^2 complex multiplications and $N(N - 1)$ complex additions.

7. How many real multiplications and real additions does the direct computation of the DFT of an N-point sequence require?
Ans. The direct computation of the DFT of an N-point sequence requires $N(4N) = 4N^2$ real multiplications and $N(4N - 2)$ real additions.

8. What is the need for FFT algorithm?
Ans. The direct computation of N-point DFT involves N^2 complex multiplications and $N(N - 1)$ complex additions. So to reduce the computational complexity, FFT algorithm is required. The DFT is needed for spectrum analysis and filtering operation on the signals using the digital computers.

9. How FFT improves the speed of computation?
Ans. The computation of DFT by FFT is based on exploiting the special properties of the twiddle factor W_N^{kn}.

Symmetry property $W_N^{k+N/2} = -W_N^k$
Periodicity property $W_N^{k+N} = W_N^k$

Using the symmetry and periodicity properties, some terms can be grouped and some calculations can be avoided, thus reducing the computations and increasing the speed.

10. What is the Twiddle factor W_N?
Ans. The twiddle factor W_N is a complex-valued phase factor which is an Nth root of unity and is expressed by $W_N = e^{-j\frac{2\pi}{N}}$.

11. What is the fundamental principle on which FFT is based?
Ans. The fundamental principle on which the FFT algorithms are based is that of decomposing the computation of the DFT of a sequence of length N into successively smaller DFTs.

12. Why is FFT called so?
Ans. FFT is called so because using this algorithm DFT is computed in a faster way. This is achieved by utilizing the symmetry and periodicity properties of W_N^k.

13. Calculate the percentage saving in calculations in a 256-point radix-2 FFT, when compared to direct DFT?
Ans. *Direct computation of DFT*
Number of complex multiplications = $N^2 = (256)^2 = 65{,}536$.
Number of complex additions = $N(N - 1) = 256(256 - 1) = 65{,}280$

Radix-2 FFT

Number of complex multiplications $= \dfrac{N}{2}\log_2 N = \dfrac{256}{2}\log_2 256 = 128 \times \log_2 2^8$
$= 128 \times 8 = 1024.$

Number of complex additions $= N\log_2 N = 256\ \log_2 256 = 256 \times \log_2 2^8$

$$= 256 \times 8 = 2048.$$

Percentage saving

Percentage saving in multiplications

$$= 100 - \frac{\text{Number of multiplications in radix-2 FFT}}{\text{Number of multiplications in direct FFT}} \times 100$$

$$= 100 - \frac{1024}{65536} \times 100 = 98.43\%$$

Percentage saving in additions $= 100 - \dfrac{\text{Number of additions in radix-2 FFT}}{\text{Number of additions in direct FFT}} \times 100$

$$= 100 - \frac{2048}{65280} \times 100 = 96.86\%$$

14. What are the two basic classes of FFT algorithms?

Ans. The two basic classes of FFT algorithms are:

(i) decimation-in-time (DIT) FFT algorithm

(ii) decimation-in-frequency (DIF) FFT algorithm

15. Why the name decimation-in-time?

Ans. The decimation-in-time derives its name from the fact that in the process of arranging the computation into smaller transformations, the sequence $x(n)$ is decomposed into successively smaller subsequences.

16. State the computational requirements of FFT?

Ans. The computation of N-point DFT by FFT requires $\frac{N}{2}\log_2 N$ complex multiplications and $N\log_2 N$ complex additions.

17. What is the importance of radix-2?

Ans. When N is a power of 2 (i.e. radix-2), the decomposition leads to a highly efficient computational algorithm. Furthermore, all the required computations are butterfly computations that correspond essentially to 2-point DFTs. For this reason, the power of 2 algorithms are particularly simple to implement and often, in applications, it is advantageous to always deal with sequences whose length is a power of 2.

18. Why the name decimation in frequency?

Ans. The decimation-in-frequency derives its name because in this method, the sequence of DFT coefficients $X(k)$ is decomposed into smaller subsequences.

19. What do you mean by radix-2?

Ans. Radix-2 implies that the number of samples $N = 2^m$, where m is the number of stages.

20. How many complex multiplications and complex additions are required to be performed for N-point DIT or DIF FFT computation?
Ans. N-point DIT or DIF FFT computation requires $N/2 \log_2 N$ complex multiplications and $N \log_2 N$ complex additions.

21. What is radix-2 FFT?
Ans. The radix-2 FFT is an efficient algorithm for computing N-point DFT of an N-point sequence. In radix-2 FFT, the N-point sequence is decimated into 2-point sequences and the 2-point DFT for each decimated sequence is computed. From the results of 2-point DFTs, the 4-point DFTs are computed. From the results of 4-point DFTs, the 8-point DFTs are computed and so on until we get N-point DFT.

22. Which FFT algorithm procedure is the lowest possible level of DFT decomposition?
Ans. The radix-2 FFT algorithm procedure is the lowest possible level of DFT decomposition.

23. How many computations are required for the first stage of DIT FFT computation for N-point DFT?
Ans. The first stage of computation of N-point DFT by radix-2 DIT FFT requires $N + (N^2/2)$ complex multiplications and complex additions.

24. For DIT FFT, what is the order of the input and output sequences?
Ans. For DIT FFT, the input is fed in bit reversed order and the output is obtained in normal order.

25. For N-point DFT computation by DIT or DIF FFT, how many butterflies are there per stage?
Ans. For N-point radix-2, DIT or DIF FFT, $N/2$ butterflies will be there per stage.

26. How many stages are there for radix-2, DIT or DIF FFT for N sample sequence?
Ans. For N-point DFT computation using radix-2 DIT or DIF FFT, $\log_2 N$ stages of computations are required.

27. For DIF FFT, what is the order of the input and output sequences?
Ans. For DIF FFT, the input is fed in normal order and output is obtained in bit reversed order.

28. What is radix-2 DIF FFT?
Ans. The radix-2 DIF [Decimation In Frequency] FFT is an efficient algorithm for computing DFT. In this algorithm the given N-point time domain sequence is converted to two numbers of $N/2$-point sequences. Then each $N/2$-point sequence is converted to two numbers of $N/4$-point sequences and so on, till we obtain $N/2$ number of 2-point sequences. Now the 2-point DFTs of $N/2$ numbers of 2-point sequences will give N samples which is the N-point DFT of the time domain sequence. Here, the equations for forming $N/2$-point sequences, $N/4$-point sequences, etc., are obtained by decimation of frequency domain sequences. Hence this method is called DIF.

29. What are the phase factors involved in the third stage of computation in the 8-point radix-2 DIT FFT?

Ans. The phase factors involved in the third stage of computation in 8-point radix-2 DIT FFT are W_8^0, W_8^1, W_8^2 and W_8^3.

30. What is radix-2 DIT FFT?

Ans. The radix-2 DIT (Decimation In Time) FFT is an efficient algorithm for computing DFT. In radix-2 DIT FFT, the N-point time domain sequence is decimated into $N/2$ numbers of 2-point sequences. The 2-point DFTs of those $N/2$ numbers of 2-point sequences are computed. The results of 2-point DFTs are used to compute 4-point DFTs. Two numbers of 2-point DFTs are combined to get a 4-point DFT. The results of 4-point DFTs are used to compute 8-point DFTs. Two numbers of 4-point DFTs are combined to get an 8-point DFT. This process is continued till we get the N-point DFT.

31. What are the phase factors involved in the first stage of computation in the 8-point radix-2, DIF FFT?

Ans. The phase factors involved in the first stage of computation in 8-point radix-2 DIF FFT are W_8^0, W_8^1, W_8^2 and W_8^3.

32. Compare the radix-2 DIT and DIF FFTs.

Radix-2 DIT FFT	*Radix-2 DIF FFT*
1. The time domain sequence is decimated.	1. The frequency domain sequence is decimated.
2. The input is in bit reverse order and output is in normal order.	2. The input is in normal order and output is in bit reverse order.
3. In each stage of computation, the phase factors are multiplied before the add subtract operations.	3. In each stage of computation, the phase factors are multiplied after the add subtract operations.
4. The value of N should be expressed such that $N = 2^m$ and this algorithm consists of m stages of computation.	4. The value of N should be expressed such that $N = 2^m$ and this algorithm consists of m stages of computation.
5. Total number of arithmetic operations are $N \log_2 N$ complex additions and $(N/2) \log_2 N$ complex multiplications.	5. Total number of arithmetic operations are $N \log_2 N$ complex additions and $(N/2) \log_2 N$ complex multiplications.

33. How will you compute IDFT using radix-2 FFT algorithm?

Ans. Let $x(n)$ be an N-point sequence and $X(k)$ be the N-point DFT of $x(n)$. The IDFT of $X(k)$, i.e. $x(n)$ can be computed using radix-2 FFT algorithm using the equation

$$x(n) = \frac{1}{N}\left[\sum_{k=0}^{N-1} X^*(k) W_N^{-nk}\right]^*$$

The following steps are to be followed to determine the $x(n)$ using radix-2 FFT algorithm.

(i) Take conjugate of $X(k)$ (i.e. determine $X^*(k)$)

(ii) Compute the N-point DFT of $X^*(k)$ using radix-2 FFT.

(iii) Take conjugate of the output sequence from FFT.
(iv) Divide the sequence obtained in step (iii) by N. The resultant sequence is $x(n)$.

34. What is direct or slow convolution and fast convolution?
Ans. The response of an LTI system is given by the convolution of input and impulse response. The computation of the convolution using direct evaluation of DFT is called slow convolution because it involves very large number of calculations. The number of calculations in DFT computations can be reduced to a very large extent by FFT algorithms. Hence the computation of convolution by FFT algorithm is called fast convolution.

35. What is magnitude and phase spectrum?
Ans. Let $X(k)$ be the N-point DFT of the time domain sequence $x(n)$. The sequence $X(k)$ is a frequency domain sequence consisting of N-samples. The samples of $X(k)$ are complex numbers and so they have a magnitude and phase. The sequence obtained by taking magnitude of the samples is called the magnitude spectrum and the sequence obtained by taking the phase of the samples is called the phase spectrum.

36. Draw and explain the basic butterfly diagram or flow graph of radix-2 DIT FFT?
Ans. The basic butterfly diagram of radix-2, DIT FFT is shown in Figure 7.54. It performs the following operations.

Here a and b are input complex numbers and A and B are output complex numbers.
(i) Input complex number b is multiplied by the phase factor W_N^k.
(ii) The product bW_N^k is added to the input complex number a to form a new complex number A.
(iii) The product bW_N^k is subtracted from the input complex number a to form a new complex number B.

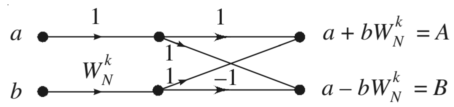


Figure 7.54 Answer of Q. 36.

37. Draw and explain the basic butterfly diagram or flow graph of radix-2 DIF FFT?
Ans. The basic butterfly diagram of radix-2 DIF FFT is shown in Figure 7.55. Here a and b are input complex numbers and A and B are output complex numbers.
(i) The sum of the complex numbers a and b is computed to form a new complex number A.
(ii) The complex number b is subtracted from a to get the difference $(a - b)$.
(iii) The difference $(a - b)$ is multiplied with phase factor W_N^k to form a new complex number B.

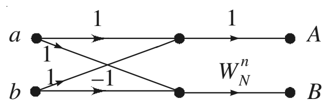


Figure 7.55 Answer of Q. 37.

38. What is a composite number?
Ans. A number which is the product of two or more integers is called a composite number. It has more than one prime factor.

39. What is composite radix FFT?
Ans. A composite or mixed radix FFT is a FFT computed when N is a composite number which has more than one prime factor.

REVIEW QUESTIONS

1. Explain radix-2 DIT FFT algorithm and draw the butterfly diagram for 8-point DIT FFT?
2. Explain the radix-2 DIF FFT algorithm and draw the butterfly diagram for 8-point DIF FFT?
3. Draw the butterfly diagram for 16-point (i) DIT FFT and (ii) DIF FFT.
4. Compare DIT and DIF FFT algorithms.
5. Draw and explain the basic butterfly diagrams for (i) DIT FFT and (ii) DIF FFT.
6. Write the procedure to compute IDFT using radix-2 FFT.
7. Develop DIT FFT algorithms for decomposing the DFT for $N = 12$ and draw the flow diagrams for (a) $N = 3 \times 4$ and (b) $N = 4 \times 3$.
8. Develop DIF FFT algorithms for decomposing the DFT for $N = 12$ and draw the flow diagrams for (a) $N = 3 \times 4$ and (b) $N = 4 \times 3$.
9. Develop a radix-4 DIT FFT algorithm for evaluating the DFT for $N = 16$.

FILL IN THE BLANKS

1. The DFT of a single number $x(n) = \{A\}$ is _______.
2. The DFT of a two sample sequence $x(n) = \{A, B\}$ is $X(k) =$ _______.
3. The direct computation of an N-point DFT requires _______ complex multiplications and _______ complex additions.
4. The direct computation of DFT requires _______ real multiplications and _______ real additions.
5. The computation of DFT by radix-2 FFT requires _______ complex multiplications and _______ complex additions.
6. The FFT may be defined as an _______ for computing DFT.
7. In FFT, the computational efficiency is achieved by adopting a _______ approach.
8. The basic FFT algorithms are _______ and _______.
9. FFT is a faster method of computation, because it exploits the _______ and _______ properties of the phase factor W_N.

10. In DFT computation using radix-2 FFT, the value of N should be such that ________.

11. When $N = r^m$, r is the ________ and m indicates the number of ________ of computation.

12. For DIT FFT, the input is in ________ order and the output is in ________ order.

13. For DIF FFT, the input is in ________ order and the output is in ________ order.

14. The IDFT is computed through FFT using the formula $x(n)$ = ________.

15. The computation of 32-point DFT by radix-2 DIT FFT involves ________ stages of computation.

16. The computation of 64-point DFT by radix-2 DIF FFT involves ________ stages of computation.

17. The signal flow graph for computing DFT by radix-2 FFT is also called the ________ diagram.

18. In radix-2 FFT, ________ butterflies per stage are required to represent the computational process.

19. The convolution by convolution sum formula is called ________ convolution.

20. The convolution by FFT is called ________ convolution.

21. In 16-point DFT by radix-2 FFT, there are ________ stages of computation with ________ butterflies per stage.

22. In DIT algorithm, the sequence ________ is decimated and in DIF algorithm the sequence ________ is decimated.

OBJECTIVE TYPE QUESTIONS

1. The number of complex multiplications involved in the direct computation of 8-point DFT is
(a) 8 (b) 64 (c) 16 (d) 56

2. The number of complex additions involved in the direct computation of 8-point DFT is
(a) 8 (b) 64 (c) 16 (d) 56

3. The DFT $X(k)$ of a 2 sample sequence $x(n) = \{4, 2\}$ is
(a) $\{6, 2\}$ (b) $\{4, 2\}$ (c) $\{8, 4\}$ (d) $\{2, 1\}$

4. The number of stages in the computation of 1024-point DFT by radix-2 FFT is
(a) 1024 (b) 32 (c) 8 (d) 10

5. The number of butterflies in each stage of computation of a 64-point radix-2 FFT is
(a) 64 (b) 32 (c) 16 (d) 8

6. The number of complex multiplications involved in the computation of 256-point DFT by radix-2 FFT is
(a) 256 (b) 1024 (c) 512 (d) 128

7. The number of complex additions involved in the computation of 256-point DFT by radix-2 FFT is
(a) 256 (b) 2048 (c) 1024 (d) 128

8. For the number of stages in the computation of DFT by radix-2 FFT to be 8, how many samples must $x(n)$ have
(a) 256 (b) 128 (c) 512 (d) 8

9. For radix-2 FFT, N must be a power of
(a) N (b) 4 (c) 2 (d) $N/2$

10. The IDFT of $X(k)$ is computed by the equation $x(n) =$

(a) $\frac{1}{N}\left[\sum_{k=0}^{N-1} X^*(k) W_N^{nk}\right]^*$ (b) $\frac{1}{N}\left[\sum_{k=0}^{N-1} X^*(k) W_N^{-nk}\right]^*$

(c) $N\left[\sum_{k=0}^{N-1} X^*(k) W_N^{nk}\right]^*$ (d) $N\left[\sum_{k=0}^{N-1} X^*(k) W_N^{-nk}\right]^*$

PROBLEMS

1. Find the DFT of the following sequences by (a) DIT FFT (b) DIF FFT
(a) $x(n) = \{0.5, 1.5, -0.5, -0.5\}$ (b) $x(n) = \{0.5, 0, 0.5, 0\}$
(c) $x(n) = \{2, 0, 0, 1\}$ (d) $x(n) = \{1, 0, -1, 0\}$

2. Find the IDFT of the following sequences by (a) DIT FFT (b) DIF FFT
(a) $X(k) = \{10, -2 + j2, 2, -2 - j2\}$ (b) $X(k) = \{0, 2 - j2, 0, 2 + j2\}$
(c) $X(k) = \{6, -j2, 2, j2\}$ (d) $X(k) = \{-4, -2 + j2, 0, -2 - j2\}$

3. Compute the DFT of the following sequences by (a) DIT FFT (b) DIF FFT
(a) $x(n) = \{1, -1, -1, -1, 1, 1, 1, -1\}$ (b) $x(n) = \{0, 1, 2, 3, 0, 0, 0, 0\}$
(c) $x(n) = n + 1$, where $N = 8$ (d) $x(n) = \{1, 1, 1, 1, 0, 0, 0, 0\}$

4. Compute the IDFT of the following sequences by (a) DIT FFT (b) DIF FFT
(a) $X(k) = \{1, 1, 1, 1, 1, 1, 1, 1\}$ (b) $X(k) = \{12, 0, 2 - j2, 0, 0, 0, -2 + j2, 0\}$
(c) $X(k) = \{20, -5.828 - j2.414, 0, -0.172 - j0.414, 0, -0.172 + j0.414, 0, -5.828 + j2.414\}$
(d) $X(k) = \{28, -4 + j9.656, -4 + j4, -4 + j1.656, -4, -4 - j1.656, -4 - j4, -4 + j9.656\}$

5. Compute the circular convolution of the following sequences using radix-2 DIT FFT
(a) $x(n) = \{1, 0.5\}$, $h(n) = \{0.5, 1\}$
(b) $x_1(n) = \{1, 2, 1, 2\}$, $x_2(n) = \{4, 3, 2, 1\}$
(c) $x(n) = \{1, -1, 1, -1\}$, $h(n) = \{1, 2, 3, 4\}$
(d) $x(n) = \{1, 2, 0, 1\}$, $h(n) = \{2, 2, 1, 1\}$

6. Find the linear convolution of the following sequence, i.e. the response of the systems

(a) $x(n) = \{1, 0, 2\}$, $h(n) = \{1, 1\}$

(b) $x(n) = \{1, 2, 3\}$, $h(n) = \{1, -1\}$

(c) $x(n) = \{1, 2, 1, 2, 1\}$, $h(n) = \{1, -1, 1, -1\}$

(d) $x(n) = \{1, -2, 1, -1, 1\}$, $h(n) = \{1, 0, 1, 0\}$

7. For DIT FFT for $N = 6$, draw the flow diagrams for (a) $N = 2 \times 3$ and (b) $N = 3 \times 2$.
Also by using FFT developed in part (a) and (b) evaluate the DFT values for $x(n) = \{1, 0, 2, 2, 0, 2\}$.

8. For DIF FFT for $N = 6$, draw the flow diagrams for (a) $N = 2 \times 3$ and (b) $N = 3 \times 2$.
Also by using FFT developed in part (a) and (b) evaluate the DFT values for $x(n) = \{2, 0, -2, 1, 0, -1\}$.

MATLAB PROGRAMS

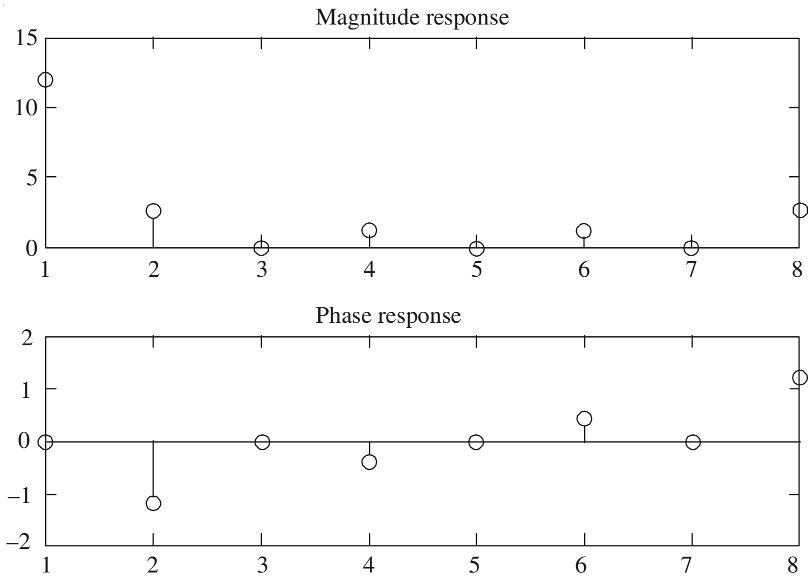
Program 7.1

% Calculation of the DFT of a given sequence using FFT

```
clc; clear all; close all;
x=[2 2 2 2 1 1 1 1];
y=fft(x);
disp('the fft of the input sequence')
disp(y)
y1=abs(y);
subplot(2,1,1),stem(y1);
title('magnitude response ')
y2=angle(y);
subplot(2,1,2),stem(y2);
title('phase response')
y3=ifft(y);
disp('the inverse fft is')
disp(y3)
```

Output:

```
the fft of the input sequence

  Columns 1 through 4

  12.0000            1.0000 – 2.4142i        0            1.0000 – 0.4142i

  Columns 5 through 8

       0            1.0000 + 0.4142i        0            1.0000 + 2.4142i

the inverse fft is

     2     2     2     2     1     1     1     1
```



Program 7.2

% Linear convolution using FFT

```
clc; clear all; close all;
x=[1 2];
h=[2 1];
x1=[x zeros(1,length(h)–1)];
h1=[h zeros(1,length(x)–1)];
X=fft(x1);
H=fft(h1);
y=X.*H;
y1=ifft(y);
disp('the linear convolution of the given sequence')
disp(y1)
```

Output:

```
the linear convolution of the given sequence

    2    5    2
```

Program 7.3

% Circular convolution using FFT

```
clc; clear all; close all;
x=[1 2 1 2];
h=[4 3 2 1];
X=fft(x);
H=fft(h);
y=X.*H;
y1=real(ifft(y));
disp('the circular convolution of the given sequence')
disp(y1)
```

Output:

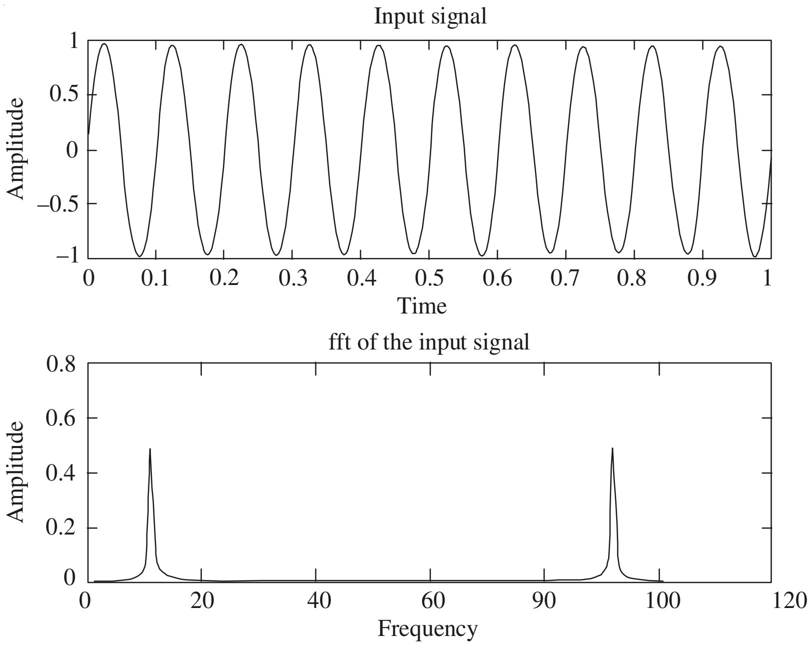
```
the circular convolution of the given sequence

    14    16    14    16
```

Program 7.4

% Plotting of DFT of sinusoidal wave

```
clc; clear all; close all;
t=0:0.01:1;
a=sin(2*pi*10*t)+sin(2*pi*100*t);
b=fft(a);
c=abs(b);
d=length(a);
e=c/d;
subplot(2,1,1),plot(t,a);
xlabel('time')
ylabel('amplitude')
title('input signal')
subplot(2,1,2),plot(e);
xlabel('frequency')
ylabel('amplitude')
title('fft of the input signal')
```

Output:


Input signal
Amplitude
1
0.5
0
–0.5
–1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Time
fft of the input signal
Amplitude
0.8
0.6
0.4
0.2
0
0
20
40
60
90
100
120
Frequency

8

Infinite-duration Impulse Response (IIR) Filters

8.1 INTRODUCTION

Filters are of two types—FIR and IIR. The type of filters which make use of feedback connection to get the desired filter implementation are known as recursive filters. Their impulse response is of infinite duration. So they are called IIR filters. The type of filters which do not employ any kind of feedback connection are known as non-recursive filters. Their impulse response is of finite duration. So they are called FIR filters. IIR filters are designed by considering all the infinite samples of the impulse response. The impulse response is obtained by taking inverse Fourier transform of ideal frequency response. There are several techniques available for the design of digital filters having an infinite duration unit impulse response. The popular methods for such filter design uses the technique of first designing the digital filter in analog domain and then transforming the analog filter into an equivalent digital filter because the analog filter design techniques are well developed. In this chapter, we discuss various methods of transforming an analog filter into a digital filter and methods of designing digital filters.

8.2 REQUIREMENTS FOR TRANSFORMATION

The system function describing an analog filter may be written as:

$$H_a(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

where $\{a_k\}$ and $\{b_k\}$ are filter coefficients. The impulse response of these filter coefficients is related to $H_a(s)$ by the Laplace transform

$$H_a(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st} dt$$

The analog filter having the rational system function $H_a(s)$ given above can also be described by the linear constant coefficient differential equation.

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

where $x(t)$ is the input signal and $y(t)$ is the output of the filter.

The above three equivalent characterizations of an analog filter leads to three alternative methods for transforming the analog filter into digital domain. The restriction on the design is that the filters should be realizable and stable.

For stability and causality of analog filter, the analog transfer function should satisfy the following requirements:

1. The $H_a(s)$ should be a rational function of s, and the coefficients of s should be real.
2. The poles should lie on the left half of s-plane.
3. The number of zeros should be less than or equal to the number of poles.

For stability and causality of digital filter, the digital transfer function should satisfy the following requirements:

1. The $H(z)$ should be a rational function of z and the coefficients of z should be real.
2. The poles should lie inside the unit circle in z-plane.
3. The number of zeros should be less than or equal to the number of poles.

We know that the analog filter with transfer function $H_a(s)$ is stable if all its poles lie in the left half of the s-plane. Consequently for the conversion technique to be effective, it should possess the following desirable properties:

1. The imaginary axis in the s-plane should map into the unit circle in the z-plane. Thus, there will be a direct relationship between the two frequency variables in the two domains.
2. The left half of the s-plane should map into the interior of the unit circle centred at the origin in z-plane. Thus, a stable analog filter will be converted to a stable digital filter.

The physically realizable and stable IIR filter cannot have a linear phase. For a filter to have a linear phase, the condition to be satisfied is $h(n) = h(N - 1 - n)$ where N is the length of the filter and the filter would have a mirror image pole outside the unit circle for every pole inside the unit circle. This results in an unstable filter. As a result, a causal and stable IIR filter cannot have linear phase. In the design of IIR filters, only the desired magnitude

response is specified and the phase response that is obtained from the design methodology is accepted.

The comparison of digital and analog filters is given in Table 8.1.

TABLE 8.1 Comparison of Digital and Analog Filters

Digital filter	*Analog filter*
1. It operates on digital samples (or sampled version) of the signal.	1. It operates on analog signals (or actual signals).
2. It is governed (or defined) by linear difference equations.	2. It is governed (or defined) by linear differential equations.
3. It consists of adders, multipliers, and delay elements implemented in digital logic (either in hardware or software or both).	3. It consists of electrical components like resistors, capacitors, and inductors.
4. In digital filters, the filter coefficients are designed to satisfy the desired frequency response.	4. In analog filters, the approximation problem is solved to satisfy the desired frequency response.

Advantages of digital filters

1. The values of resistors, capacitors and inductors used in analog filters change with temperature. Since the digital filters do not have these components, they have high thermal stability.
2. In digital filters, the precision of the filter depends on the length (or size) of the registers used to store the filter coefficients. Hence by increasing the register bit length (in hardware) the performance characteristics of the filter like accuracy, dynamic range, stability and frequency response tolerance, can be enhanced.
3. The digital filters are programmable. Hence the filter coefficients can be changed any time to implement adaptive features.
4. A single filter can be used to process multiple signals by using the techniques of multiplexing.

Disadvantages of digital filters

1. The bandwidth of the discrete signal is limited by the sampling frequency. The bandwidth of real discrete signal is half the sampling frequency.
2. The performance of the digital filter depends on the hardware (i.e., depends on the bit length of the registers in the hardware) used to implement the filter.

Important features of IIR filters

1. The physically realizable IIR filters do not have linear phase.
2. The IIR filter specifications include the desired characteristics for the magnitude response only.

8.3 DESIGN OF IIR FILTER BY APPROXIMATION OF DERIVATIVES

The analog filter having the rational system function $H(s)$ can also be described by the linear constant coefficient differential equation.

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

In this method of IIR filter design by approximation of derivatives, an analog filter is converted into a digital filter by approximating the above differential equation into an equivalent difference equation.

The backward difference formula is substituted for the derivative $\frac{dy(t)}{dt}$ at time $t = nT$. Thus,

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{y(nT) - y(n-1)T}{T}$$

or

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{y(n) - y(n-1)}{T}$$

where T is the sampling interval and $y(n) = y(nT)$.

The system function of an analog differentiator with an output $dy(t)/dt$ is $H(s) = s$, and the digital system which produces the output $[y(n) - y(n-1)]/T$ has the system function $H(z) = [1 - z^{-1}]/T$. Comparing these two, we can say that the frequency domain equivalent for the relationship $\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{y(n) - y(n-1)}{T}$ is:

$$s = \frac{1 - z^{-1}}{T}$$

Thus, this is the analog domain to digital domain transformation.

Also, the second derivative $\frac{d^2 y(t)}{dt^2}$ can be replaced by the second backward difference:

$$\left.\frac{d^2 y(t)}{dt^2}\right|_{t=nT} = \left.\frac{d}{dt}\left[\frac{dy(t)}{dt}\right]\right|_{t=nT}$$

$$= \frac{[y(nT) - y(nT-T)]/T - [y(nT-T) - y(nT-2T)]/T}{T}$$

$$= \frac{y(n) - 2y(n-1) + y(n-2)}{T^2}$$

The equivalent expression in frequency domain is:

$$s^2 = \frac{1 - 2z^{-1} + z^{-2}}{T^2}$$

or
$$s^2 = \left(\frac{1 - z^{-1}}{T}\right)^2$$

The *i*th derivative of function $y(t)$ results in the equivalent frequency domain relationship as:

$$s^i = \left(\frac{1 - z^{-1}}{T}\right)^i$$

As a result, the digital filter's system function $H(z)$ can be obtained from the analog filter's system function $H_a(s)$ by the method of approximation of the derivatives as:

$$H(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{T}}$$

The outcomes of the mapping of the z-plane from the s-plane are discussed below.

We have
$$s = \frac{1 - z^{-1}}{T}, \quad \text{i.e.} \quad z = \frac{1}{1 - sT}$$

Substituting $s = j\Omega$ in the expression for z, we have

$$z = \frac{1}{1 - j\Omega T}$$

$$= \frac{1}{1 + \Omega^2 T^2} + j\frac{\Omega T}{1 + \Omega^2 T^2}$$

Varying Ω from $-\infty$ to ∞ the corresponding locus of points in the z-plane is a circle with radius 1/2 and with centre at $z = 1/2$, as shown in Figure 8.1.

It can be observed that the mapping of the equation $s = (1 - z^{-1})/T$, takes the left half plane of s-domain into the corresponding points inside the circle of radius 0.5 and centre at $z = 0.5$. Also the right half of the s-plane is mapped outside the unit circle. Because of this, this mapping results in a stable analog filter transformed into a stable digital filter. However, since the location of poles in the z-domain are confined to smaller frequencies, this design method can be used only for transforming analog low-pass filters and band pass filters which are having smaller resonant frequencies. This means that neither a high-pass filter nor a band-reject filter can be realized using this technique.

The forward difference can be substituted for the derivative instead of the backward difference.

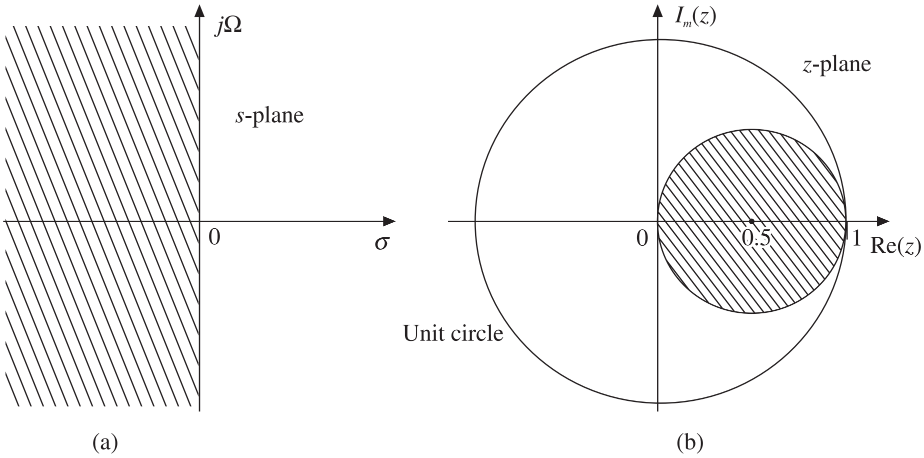


Figure 8.1 Mapping of *s*-plane into *z*-plane by the backward difference method.

This provides

$$\frac{dy(t)}{dt} = \frac{y(nT + T) - y(nT)}{T}$$

$$= \frac{y(n + 1) - y(n)}{T}$$

The transformation formula would be

$$s = \frac{z - 1}{T}$$

or

$$z = 1 + sT$$

The mapping of the equation $z = 1 + sT$ is shown in Figure 8.2. This results in a worse situation than the backward difference substitution for the derivative. When $s = j\Omega$, the mapping of these points in the *s*-domain results in a straight line in the *z*-domain with co-ordinates $(z_{\text{real}}, z_{\text{imag}}) = (1, \Omega T)$. As a result of this, stable analog filters do not always map into stable digital filters.

The limitations of the mapping methods discussed above can be overcome by using more complex substitution for the derivatives. An *N*th order difference is proposed for the derivative, as shown

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{1}{T}\sum_{k=1}^{N} a_k \frac{y(nT + kT) - y(nT - kT)}{T}$$

Here $\{a_k\}$ are a set of parameters selected so as to optimize the approximation. The transformation from the *s*-plane to the *z*-plane will be

$$s = \frac{1}{T}\sum_{k=1}^{N} a_k (z^k - z^{-k})$$

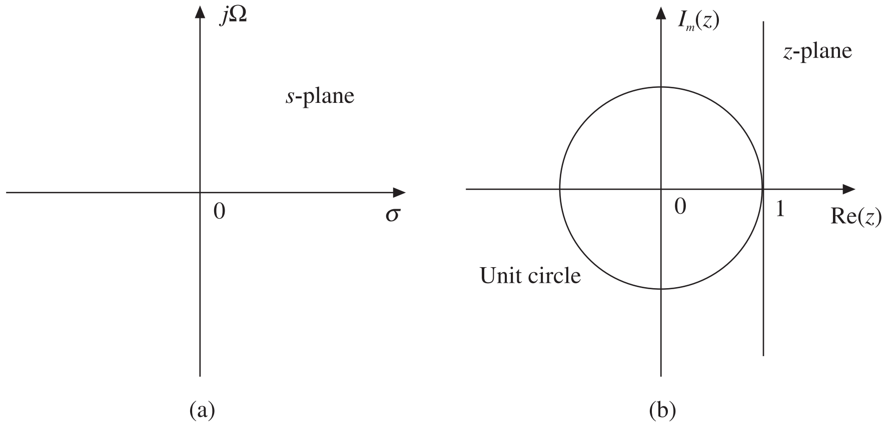


Figure 8.2 Mapping of s-plane into z-plane by the forward difference method.

Thus, if we choose proper values for $\{a_k\}$, then the $j\Omega$ axis can be mapped into the unit circle and the left half of the s-plane can be mapped into points inside the unit in the z-plane.

EXAMPLE 8.1 Convert the analog low-pass filter specified by

$$H_a(s) = \frac{2}{s+3}$$

into a digital filter making use of the backward difference for the derivative.

Solution: We know that the mapping formula for the backward difference for the derivative is given by

$$s = \frac{1-z^{-1}}{T}$$

For the given analog filter function $H_a(s) = \dfrac{2}{s+3}$, the corresponding digital filter function is:

$$H(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{T}} = \frac{2}{\dfrac{(1-z^{-1})}{T}+3}$$

$$= \frac{2T}{1-z^{-1}+3T}$$

If $T = 1$ s,
$$H(z) = \frac{2}{1-z^{-1}+3} = \frac{2}{4-z^{-1}}$$

EXAMPLE 8.2 Making use of the backward difference for the derivative, convert the analog filter function given below to a digital filter function.

$$H_a(s) = \frac{4}{s^2 + 9}$$

Solution: The mapping formula for the backward difference by the derivative is:

$$s = \frac{1 - z^{-1}}{T}$$

Therefore, for the given $H_a(s)$, the corresponding digital filter function is:

$$H(z) = H_a(s)\Big|_{s = \frac{1-z^{-1}}{T}} = \frac{4}{\left[\dfrac{1 - z^{-1}}{T}\right]^2 + 9}$$

$$= \frac{4T^2}{1 - 2z^{-1} + z^{-2} + 9T^2}$$

If $T = 1$ s, then

$$H(z) = \frac{4}{1 - 2z^{-1} + z^{-2} + 9} = \frac{4}{10 - 2z^{-1} + z^{-2}}$$

EXAMPLE 8.3 Convert the analog filter given below into a digital filter using the backward difference for the derivative:

$$H_a(s) = \frac{3}{(s + 0.5)^2 + 16}$$

Solution: For the given $H_a(s)$, the system function of the corresponding digital filter is:

$$H(z) = H_a(s)\Big|_{s = \frac{1-z^{-1}}{T}} = \frac{3}{(s + 0.5)^2 + 16}\Bigg|_{s = \frac{1-z^{-1}}{T}}$$

$$= \frac{3}{\left[\dfrac{1 - z^{-1}}{T} + 0.5\right]^2 + 16}$$

$$= \frac{3T^2}{[(1 + 0.5T) - z^{-1}]^2 + 16T^2}$$

$$= \frac{3T^2}{(1 + 0.5T)^2 + z^{-2} - 2(1 + 0.5T)z^{-1} + 16T^2}$$

If $T = 1$ s, then

$$H(z) = \frac{3}{2.25 + z^{-2} - 3z^{-1} + 16} = \frac{3}{18.25 - 3z^{-1} + z^{-2}}$$

8.4 DESIGN OF IIR FILTER BY IMPULSE INVARIANT TRANSFORMATION

In this technique, the desired impulse response of the digital filter is obtained by uniformly sampling the impulse response of the equivalent analog filter. The main idea behind this is to preserve the frequency response characteristics of the analog filter. For the digital filter to possess the frequency response characteristics of the corresponding analog filter, the sampling period T should be sufficiently small (or the sampling frequency should be sufficiently high) to minimize (or completely avoid) the effects of aliasing.

Let $h_a(t)$ = Impulse response of analog filter
T = Sampling period
$h(n)$ = Impulse response of digital filter

For impulse invariant transformation,

$$h(n) = h_a(t)|_{t=nT} = h_a(nT)$$

The Laplace transform of the analog filter impulse response $h_a(t)$ gives the transfer function of analog filter.

$$\therefore \qquad L[h_a(t)] = H_a(s)$$

The transformation technique can be well understood by first considering a simple distinct poles case for the analog filter's system function as shown below.

$$H_a(s) = \sum_{i=1}^{N} \frac{A_i}{s - p_i}$$

The impulse response $h_a(t)$ of the analog filter is obtained by taking the inverse Laplace transform of the system function $H_a(s)$.

$$\therefore \qquad h_a(t) = \mathrm{L}^{-1}[H_a(s)] = \sum_{i=1}^{N} A_i e^{p_i t} u_a(t)$$

where $u_a(t)$ is the unit step function in the continuous-time case.

The impulse response $h(n)$ of the equivalent digital filter is obtained by uniformly sampling $h_a(t)$, i.e.,

$$h(n) = h_a(nT) = \sum_{i=1}^{N} A_i e^{p_i nT} u_a(nT)$$

The system function of the digital system of above expression can be obtained by taking z-transform, i.e.

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

Using the above equation for $h(n)$, we have

$$H(z) = \sum_{n=0}^{\infty} \left[\sum_{i=1}^{N} A_i e^{p_i nT} u_a(nT) \right] z^{-n}$$

Interchanging the order of summation, we have

$$H(z) = \sum_{i=1}^{N} \left[\sum_{n=0}^{\infty} A_i e^{p_i nT} u_a(nT) \right] z^{-n}$$

$$= \sum_{i=1}^{N} \frac{A_i}{1 - e^{p_i T} z^{-1}}$$

Comparing the above expressions for $H_a(s)$ and $H(z)$, we can say that the impulse invariant transformation is accomplished by the mapping.

$$\frac{1}{s - p_i} \xrightarrow[\text{(is tranformed to)}]{} \frac{1}{1 - e^{p_i T} z^{-1}}$$

Relation between analog and digital poles

The above mapping shows that the analog pole at $s = p_i$ is mapped into a digital pole at $z = e^{p_c T}$. Therefore, the analog poles and the digital poles are related by the relation.

$$z = e^{sT}$$

The general characteristic of the mapping $z = e^{sT}$ can be obtained by substituting $s = \sigma + j\Omega$ and expressing the complex variable z in polar form as $z = re^{j\omega}$.

$\therefore$ $$re^{j\omega} = e^{(\sigma + j\Omega)T} = e^{\sigma T} e^{j\Omega T}$$

That means

$$|z| = r = e^{\sigma T}$$

and $$\angle z = \omega = \Omega T$$

So the relationship between analog frequency Ω and digital frequency ω is $\omega = \Omega T$ or $\Omega = \dfrac{\omega}{T}\cdot$

As a result of this, $\sigma < 0$ implies that $0 < r < 1$ and $\sigma > 0$ implies that $r > 1$ and $\sigma = 0$ implies that $r = 1$. Therefore, the left half of s-plane is mapped into the interior of the unit circle in the z-plane. The right half of the s-plane is mapped into the exterior of the unit circle in the z-plane. This is one of the desirable properties for stability. The $j\Omega$-axis is mapped into the unit circle in z-plane. However, the mapping of $j\Omega$-axis is not one-to-one.

The mapping $\omega = \Omega T$ implies that the strip of width $2\pi/T$ in the s-plane for values of s in the range $-\pi/T \le \Omega \le \pi/T$ maps into the corresponding values of $-\pi \le \omega \le \pi$, i.e., into the entire z-plane. Similarly, the strip of width $2\pi/T$ in the s-plane for values of s in the range $\pi/T \le \Omega \le 3\pi/T$ also maps into the interval $-\pi \le \omega \le \pi$, i.e., into the entire z-plane. Similarly, the strip of width $2\pi/T$ in the s-plane for values of s in the range $-\pi/T \le \Omega \le -3\pi/T$ also maps into the interval $-\pi \le \omega \le \pi$, i.e., into the entire z-plane. In general, any frequency interval $(2k - 1)\pi/T \le \Omega \le (2k + 1)\pi/T$, where k is an integer, will also map into the interval $-\pi \le \omega \le \pi$ in the z-plane, i.e., into the entire z-plane. Hence the mapping from the analog frequency Ω to the digital frequency ω by impulse invariant transformation is many-to-one which simply reflects the effects of aliasing due to sampling of the impulse response. Figure 8.3 illustrates the mapping from the s-plane to z-plane.

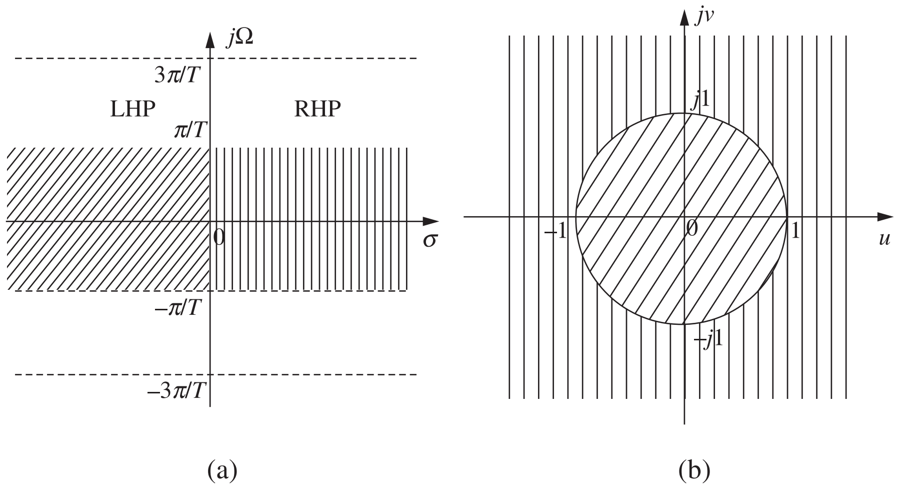


Figure 8.3 Mapping of (a) s-plane into (b) z-plane by impulse invariant transformation.

The stability of a filter (or system) is related to the location of the poles. For a stable analog filter the poles should lie on the left half of the s-plane. That means for a stable digital filter the poles should lie inside the unit circle in the z-plane.

Useful impulse invariant transformations

Some of the useful impulse invariant transformations are given below. The first one can be used when the analog real pole has a multiplicity of m. The second and third equations can be used when the analog poles are complex conjugate.

1. $$\frac{1}{(s+p_i)^m} \longrightarrow \frac{(-1)^{m-1}}{(m-1)!}\frac{d^{m-1}}{ds^{m-1}}\left(\frac{1}{1-e^{-sT}z^{-1}}\right); \quad s = p_i$$

2. $$\frac{s+a}{(s+a)^2+b^2} \longrightarrow \frac{1-e^{-aT}(\cos bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

3. $$\frac{b}{(s+a)^2+b^2} \longrightarrow \frac{e^{-aT}(\sin bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

EXAMPLE 8.4 For the analog transfer function

$$H_a(s) = \frac{2}{(s+1)(s+3)}$$

determine $H(z)$ if (a) $T = 1$ s and (b) $T = 0.5$ s using impulse invariant method.

Solution: Given, $H_a(s) = \dfrac{2}{(s+1)(s+3)}$

Using partial fractions, $H_a(s)$ can be expressed as:

$$H_a(s) = \frac{A}{s+1} + \frac{B}{s+3}$$

$$A = (s+1)\,H_a(s)\Big|_{s=-1} = \frac{2}{s+3}\bigg|_{s=-1} = 1$$

$$B = (s+3)\,H_a(s)\Big|_{s=-3} = \frac{2}{s+1}\bigg|_{s=-3} = -1$$

$$\therefore \qquad H_a(s) = \frac{1}{s+1} - \frac{1}{s+3} = \frac{1}{s-(-1)} - \frac{1}{s-(-3)}$$

By impulse invariant transformation, we know that

$$\frac{A_i}{s-p_i} \xrightarrow[\text{(is transformed to)}]{} \frac{A_i}{1-e^{p_iT}z^{-1}}$$

Here $H_a(s)$ has two poles and $p_1 = -1$ and $p_2 = -3$.

Therefore, the system function of the digital filter is:

$$H(z) = \frac{1}{1-e^{p_1T}z^{-1}} - \frac{1}{1-e^{p_2T}z^{-1}}$$

$$= \frac{1}{1-e^{-T}z^{-1}} - \frac{1}{1-e^{-3T}z^{-1}}$$

(a) When T = 1 s

$$H(z) = \frac{1}{1-e^{-1}z^{-1}} - \frac{1}{1-e^{-3}z^{-1}}$$

$$= \frac{1}{1-0.3678z^{-1}} - \frac{1}{1-0.0497z^{-1}}$$

$$= \frac{(1-0.0497z^{-1})-(1-0.3678z^{-1})}{(1-0.3678z^{-1})(1-0.0497z^{-1})}$$

$$= \frac{0.3181z^{-1}}{1-0.4175z^{-1}+0.0182z^{-2}}$$

(b) When T = 0.5 s

$$H(z) = \frac{1}{1-e^{-0.5}z^{-1}} - \frac{1}{1-e^{-3\times 0.5}z^{-1}}$$

$$= \frac{1}{1-0.606z^{-1}} - \frac{1}{1-0.223z^{-1}}$$

$$= \frac{(1-0.223z^{-1})-(1-0.606z^{-1})}{(1-0.606z^{-1})(1-0.223z^{-1})}$$

$$= \frac{0.383z^{-1}}{1-0.829z^{-1}+0.135z^{-2}}$$

EXAMPLE 8.5 Convert the analog filter with transfer function

$$H_a(s) = \frac{s+0.1}{(s+0.1)^2+9}$$

into a digital filter using the impulse invariant transformation.

Solution: Observe that the given system function of the analog filter is of the standard form $H_a(s) = \dfrac{s+a}{(s+a)^2+b^2}$, where we are given a = 0.1 and b = 3.

By the impulse invariant transformation, we know that

$$\frac{s+a}{(s+a)^2+b^2} \xrightarrow[\text{(is transformed to)}]{} \frac{1-e^{-aT}(\cos bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

Therefore, for the given $H_a(s)$, we can write the system function of the digital filter

$$H(z) = \frac{1 - e^{-0.1T}(\cos 3T)z^{-1}}{1 - 2e^{-0.1T}(\cos 3T)z^{-1} + e^{-2(0.1)T}z^{-2}}$$

Assuming $T = 1$ s, we have

$$H(z) = \frac{1 - e^{-0.1}(\cos 3)z^{-1}}{1 - 2e^{-0.1}(\cos 3)z^{-1} + e^{-0.2}z^{-2}}$$

$$= \frac{1 - 0.9048(-0.9899)\,z^{-1}}{1 - 2(0.9048)(-0.9899)z^{-1} + 0.8187z^{-2}}$$

$$= \frac{1 + 0.8956z^{-1}}{1 + 1.7913z^{-1} + 0.8187z^{-2}}$$

EXAMPLE 8.6 The system function of an analog filter is expressed as:

$$H_a(s) = \frac{s + 0.5}{(s + 0.5)^2 + 4}$$

Convert this analog filter into a digital filter using the impulse invariant transformation. Assume $T = 1$ s.

Solution: Observe that the given system function of the analog filter is of the standard form $H_a(s) = \dfrac{s + a}{(s + a)^2 + b^2}$, where we are given $a = 0.5$ and $b = 2$.

By the impulse invariant transformation, we know that

$$\frac{s + a}{(s + a)^2 + b^2} \xrightarrow[\text{(is transformed to)}]{} \frac{1 - e^{-aT}(\cos bT)z^{-1}}{1 - 2e^{-aT}(\cos bT)z^{-1} + e^{-2aT}z^{-2}}$$

Therefore, for the given $H_a(s)$, we can write the system function of the digital filter

$$H(z) = \frac{1 - e^{-0.5T}(\cos 2T)z^{-1}}{1 - 2e^{-0.5T}(\cos 2T)z^{-1} + e^{-2(0.5)T}z^{-2}}$$

Given $T = 1$s, we have

$$H(z) = \frac{1 - e^{-0.5}(\cos 2)z^{-1}}{1 - 2e^{-0.5}(\cos 2)z^{-1} + e^{-1}z^{-2}}$$

$$= \frac{1 - 0.606(-0.416)\,z^{-1}}{1 - 2(0.606)(-0.416)z^{-1} + 0.3678z^{-2}}$$

$$= \frac{1 + 0.252z^{-1}}{1 + 0.504z^{-1} + 0.3678z^{-2}}$$

EXAMPLE 8.7 The system function of an analog filter is expressed as:

$$H_a(s) = \frac{2}{s(s+2)}$$

Find the corresponding $H(z)$ using the impulse invariant method for a sampling frequency of 4 samples per second.

Solution: Given sampling rate = 4 samples/second

$\therefore$ Sampling period $T = \frac{1}{4} = 0.25$ s

Expressing the given $H_a(s)$ in terms of partial fractions, we have

$$H_a(s) = \frac{2}{s(s+2)} = \frac{1}{s} - \frac{1}{s+2} = \frac{1}{s-(0)} - \frac{1}{s-(-2)}$$

By the impulse invariant transformation, we know that

$$\frac{A}{s-p_i} \xrightarrow[\text{(is transformed to)}]{} \frac{A}{1-e^{p_iT}z^{-1}}$$

Here $H_a(s)$ has two poles and $p_1 = 0$ and $p_2 = -2$.

Therefore, the system function of the digital filter is:

$$\begin{aligned}
H(z) &= \frac{1}{1-e^{p_1T}z^{-1}} - \frac{1}{1-e^{p_2T}z^{-1}} \\
&= \frac{1}{1-e^{(0)T}z^{-1}} - \frac{1}{1-e^{(-2)T}z^{-1}} \\
&= \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-2(0.25)}z^{-1}} \\
&= \frac{1}{1-z^{-1}} - \frac{1}{1-0.606\,z^{-1}} \\
&= \frac{(1-0.606z^{-1})-(1-z^{-1})}{(1-z^{-1})(1-0.606z^{-1})} \\
&= \frac{0.394z^{-1}}{1-1.606z^{-1}+0.606z^{-2}}
\end{aligned}$$

EXAMPLE 8.8 Convert the analog filter with system transfer function

$$H_a(s) = \frac{2}{(s+0.4)^2+4}$$

into a digital filter using the impulse invariant transformation.

Solution: Observe that the given system function of the analog filter is of the standard form $H_a(s) = \dfrac{b}{(s+a)^2 + b^2}$, where we are given $a = 0.4$ and $b = 2$.

By the impulse invariant transformation, we know that

$$\frac{b}{(s+a)^2 + b^2} \xrightarrow[\text{(is transformed to)}]{} \frac{e^{-aT}(\sin bT)z^{-1}}{1 - 2e^{-aT}(\cos bT)z^{-1} + e^{-2aT}z^{-2}}$$

Therefore, for the given $H_a(s)$, we can write the digital filter function as:

$$H(z) = \frac{e^{-(0.4)T}(\sin 2T)z^{-1}}{1 - 2e^{-(0.4)T}(\cos 2T)z^{-1} + e^{-2(0.4)T}z^{-2}}$$

For $T = 1$ s,

$$H(z) = \frac{e^{-0.4}(\sin 2)z^{-1}}{1 - 2e^{-0.4}(\cos 2)z^{-1} + e^{-0.8}z^{-2}}$$

$$= \frac{0.909z^{-1}}{1 + 0.5578z^{-1} + 0.449z^{-2}}$$

EXAMPLE 8.9 Determine $H(z)$ using the impulse invariant technique for the analog system function

$$H_a(s) = \frac{1}{(s+1)(s^2+s+2)}$$

Solution: Using partial fractions, the given $H_a(s)$ can be written as

$$H_a(s) = \frac{1}{(s+1)(s^2+s+2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+s+2}$$

Therefore, we can write

$$A(s^2+s+2) + (Bs+C)(s+1) = 1$$

i.e.,
$$(A+B)s^2 + (A+B+C)s + (2A+C) = 1$$

Comparing the coefficients of s^2, s and the constants on either side of the above expression, we get

$$A + B = 0, \qquad \text{i.e., } B = -A$$

$$A + B + C = 0, \qquad \therefore\ C = 0$$

$$2A + C = 1, \qquad \therefore\ A = 0.5 \text{ and } B = -0.5$$

So the system response can be written as:

$$H_a(s) = \frac{0.5}{s+1} - \frac{0.5s}{s^2+s+2}$$

$$= \frac{0.5}{s+1} - 0.5\left(\frac{s}{(s+0.5)^2+(1.3228)^2}\right)$$

$$= \frac{0.5}{s+1} - 0.5\left[\frac{s+0.5}{(s+0.5)^2+(1.3228)^2} - \frac{0.5}{(s+0.5)^2+(1.3228)^2}\right]$$

$$= \frac{0.5}{s+1} - 0.5\left[\frac{s+0.5}{(s+0.5)^2+(1.3228)^2}\right] + \frac{0.25}{1.3228}\left[\frac{1.3228}{(s+0.5)^2+(1.3228)^2}\right]$$

$$= \frac{0.5}{s+1} - 0.5\frac{(s+0.5)}{(s+0.5)^2+(1.3228)^2} + 0.1889\frac{1.3228}{(s+0.5)^2+(1.3228)^2}$$

Using the impulse invariant transformation, this analog system function $H_a(s)$ can be transformed into digital system function as:

$$H(z) = \frac{0.5}{1-e^{-T}z^{-1}} - 0.5\left[\frac{1-e^{-0.5T}(\cos 1.3228T)z^{-1}}{1-2e^{-0.5T}(\cos 1.3228T)z^{-1}+e^{-2(0.5)T}z^{-2}}\right]$$

$$+ 0.1889\left[\frac{e^{-0.5T}(\sin 1.3228)z^{-1}}{1-2e^{-0.5T}(\cos 1.3228)z^{-1}+e^{-2(0.5)T}z^{-2}}\right]$$

Let $T = 1$ s, we have

$$H(z) = \frac{0.5}{1-0.3678z^{-1}} - 0.5\left[\frac{1-0.606(\cos 1.3228)z^{-1}}{1-1.213(\cos 1.3228)z^{-1}+0.3678z^{-2}}\right]$$

$$+0.1889\left[\frac{0.606(\sin 1.3228)z^{-1}}{1-1.213(\cos 1.3228)z^{-1}+0.3678z^{-2}}\right]$$

$$= \frac{0.5}{1-0.3678z^{-1}} - 0.5\left[\frac{1-0.1487z^{-1}}{1-0.2977z^{-1}+0.3678z^{-2}}\right]$$

$$+0.1889\left[\frac{0.5874z^{-1}}{1-0.2977z^{-1}+0.3678z^{-2}}\right]$$

$$= \frac{0.5}{1-0.3678z^{-1}} - \left[\frac{0.5-0.6109z^{-1}}{1-0.2977z^{-1}+0.3678z^{-2}}\right]$$

$$= \frac{0.646z^{-1}-0.0407z^{-2}}{1-0.6655z^{-1}+0.4773z^{-2}+0.1352z^{-3}}$$

8.5 DESIGN OF IIR FILTER BY THE BILINEAR TRANSFORMATION METHOD

In the previous sections, we have studied the IIR filter design using (a) approximation of derivatives method and (b) Impulse invariant transformation method. However the IIR filter design using these methods is appropriate only for the design of low-pass filters and band pass filters whose resonant frequencies are small. These techniques are not suitable for high-pass or band reject filters. The limitation is overcome in the mapping technique called the **bilinear transformation.** This transformation is a one-to-one mapping from the *s*-domain to the *z*-domain. That is, the bilinear transformation is a conformal mapping that transforms the imaginary axis of *s*-plane into the unit circle in the *z*-plane only once, thus avoiding aliasing of frequency components. In this mapping, all points in the left half of *s*-plane are mapped inside the unit circle in the *z*-plane, and all points in the right half of *s*-plane are mapped outside the unit circle in the *z*-plane. So the transformation of a stable analog filter results in a stable digital filter. The bilinear transformation can be obtained by using the trapezoidal formula for the numerical integration.

Let the system function of the analog filter be $H_a(s) = \dfrac{b}{s+a}$

The differential equation describing the above analog filter can be obtained as:

$$H_a(s) = \frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

or $$sY(s) + aY(s) = bX(s)$$

Taking inverse Laplace transform on both sides, we get

$$\frac{dy(t)}{dt} + a\,y(t) = bx(t)$$

Integrating the above equation between the limits $(nT - T)$ and nT, we have

$$\int_{nT-T}^{nT} \frac{dy(t)}{dt}\,dt + a\int_{nT-T}^{nT} y(t)\,dt = b\int_{nT-T}^{T} x(t)\,dt$$

The trapezoidal rule for numeric integration is expressed as:

$$\int_{nT-T}^{nT} a(t)\,dt = \frac{T}{2}[a(nT) + a(nT-T)]$$

Therefore, we get

$$y(nT) - y(nT-T) + a\frac{T}{2}y(nT) + a\frac{T}{2}y(nT-T) = b\frac{T}{2}x(nT) + b\frac{T}{2}x(nT-T)$$

Taking z-transform, we get

$$Y(z)[1 - z^{-1}] + a\frac{T}{2}[1 + z^{-1}]\,Y(z) = b\frac{T}{2}[1 + z^{-1}]X(z)$$

Therefore, the system function of the digital filter is:

$$\frac{Y(z)}{X(z)} = H(z) = \frac{b}{\dfrac{2}{T}\dfrac{1 - z^{-1}}{1 + z^{-1}} + a}$$

Comparing this with the analog filter system function $H_a(s)$ we get

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) = \frac{2}{T}\left(\frac{z - 1}{z + 1}\right)$$

Rearranging, we can get

$$z = \frac{1 + \dfrac{T}{2}s}{1 - \dfrac{T}{2}s}$$

This is the relation between analog and digital poles in bilinear transformation. So to convert an analog filter function into an equivalent digital filter function, just put

$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}} \quad \text{in } H_a(s)$$

The general characteristic of the mapping $z = e^{sT}$ may be obtained by putting $s = \sigma + j\Omega$ and expressing the complex variable z in the polar form as $z = re^{j\omega}$ in the above equation for s.

Thus,
$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) = \frac{2}{T}\left(\frac{re^{j\omega} - 1}{re^{j\omega} + 1}\right)$$

or
$$s = \frac{2}{T}\frac{(re^{j\omega} - 1)(re^{-j\omega} + 1)}{(re^{j\omega} + 1)(re^{-j\omega} + 1)} = \frac{2}{T}\left[\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega} + j\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}\right]$$

Since $s = \sigma + j\Omega$, we get

$$\sigma = \frac{2}{T}\left[\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega}\right]$$

and
$$\Omega = \frac{2}{T}\left[\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}\right]$$

From the above equation for σ, we observe that if $r < 1$ then $\sigma < 0$ and if $r > 1$, then $\sigma > 0$, and if $r = 1$, then $\sigma = 0$. Hence the left half of the s-plane maps into points inside the unit

circle in the z-plane, the right half of the s-plane maps into points outside the unit circle in the z-plane and the imaginary axis of s-plane maps into the unit circle in the z-plane. This transformation results in a stable digital system.

Relation between analog and digital frequencies

On the imaginary axis of s-plane $\sigma = 0$ and correspondingly in the z-plane $r = 1$.

$$\therefore \qquad \Omega = \frac{2}{T}\left(\frac{2\sin\omega}{1+1+2\cos\omega}\right) = \frac{2}{T}\left(\frac{\sin\omega}{1+\cos\omega}\right)$$

$$= \frac{2}{T}\left(\frac{2\sin\frac{\omega}{2}\cos\frac{\omega}{2}}{1+2\cos^2\omega/2-1}\right) = \frac{2}{T}\tan\frac{\omega}{2}$$

$\therefore$ The relation between analog and digital frequencies is:

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

or equivalently, we have $\omega = 2\tan^{-1}\dfrac{\Omega T}{2}$.

The above relation between analog and digital frequencies shows that the entire range in Ω is mapped only once into the range $-\pi \le \omega \le \pi$. The entire negative imaginary axis in the s-plane (from $\Omega = -\infty$ to 0) is mapped into the lower half of the unit circle in z-plane (from $\omega = -\pi$ to 0) and the entire positive imaginary axis in the s-plane (from $\Omega = \infty$ to 0) is mapped into the upper half of unit circle in z-plane (from $\omega = 0$ to $+\pi$).

But as seen in Figure 8.4, the mapping is non-linear and the lower frequencies in analog domain are expanded in the digital domain, whereas the higher frequencies are

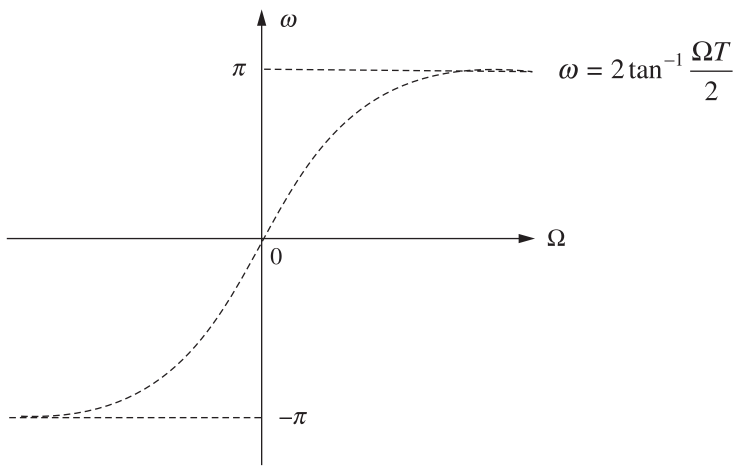


Figure 8.4 Mapping between Ω and ω in bilinear transformation.

compressed. This is due to the nonlinearity of the arctangent function and usually known as frequency warping.

The effect of warping on the magnitude response can be explained by considering an analog filter with a number of passbands as shown in Figure 8.5(a). The corresponding digital filter will have same number of passbands, but with disproportionate bandwidth, as shown in Figure 8.5(a).

In designing digital filter using bilinear transformation, the effect of warping on amplitude response can be eliminated by prewarping the analog filter. In this method, the specified digital frequencies are converted to analog equivalent using the equation $\Omega = \frac{2}{T}\tan\frac{\omega}{2}$. This analog frequencies are called prewarp frequencies. Using the prewarp frequencies, the analog filter transfer function is designed, and then it is transformed to digital filter transfer function.

This effect of warping on the phase response can be explained by considering an analog filter with linear phase response as shown in Figure 8.5(b). The phase response of corresponding digital filter will be nonlinear.

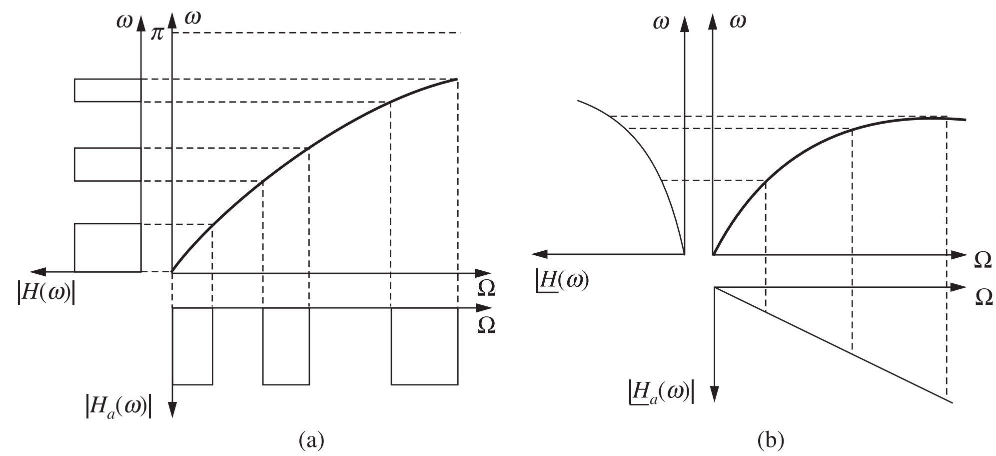


Figure 8.5 The warping effect on (a) magnitude response and (b) phase response.

From the earlier discussions, it can be stated that the bilinear transformation preserves the magnitude response of an analog filter only if the specification requires piecewise constant magnitude, but the phase response of the analog filter is not preserved. Therefore, the bilinear transformation can be used only to design digital filters with prescribed magnitude response with piecewise constant values. A linear phase analog filter cannot be transformed into a linear phase digital filter using the bilinear transformation.

EXAMPLE 8.10 Convert the following analog filter with transfer function

$$H_a(s) = \frac{s + 0.1}{(s + 0.1)^2 + 9}$$

into a digital IIR filter by using bilinear transformation. The digital IIR filter is having a resonant frequency of $\omega_r = \pi/2$.

Solution: From the transfer function, we observe that $\Omega_c = 3$. The sampling period T can be determined using the equation:

$$\Omega_c = \frac{2}{T} \tan \frac{\omega_r}{2}$$

$$\therefore \qquad T = \frac{2}{\Omega_c} \tan \frac{\omega_r}{2} = \frac{2}{3} \tan \frac{\pi/2}{2} = 0.6666 \text{ s}$$

Using the bilinear transformation, the digital filter system function is:

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = H_a(s)\Big|_{s=3\frac{1-z^{-1}}{1+z^{-1}}}$$

$$\therefore \qquad H(z) = \frac{s+0.1}{(s+0.1)^2+9}\Bigg|_{s=3\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{3\dfrac{1-z^{-1}}{1+z^{-1}} + 0.1}{\left[3\dfrac{1-z^{-1}}{1+z^{-1}} + 0.1\right]^2 + 9}$$

$$= \frac{\left[3(1-z^{-1}) + 0.1(1+z^{-1})\right][1+z^{-1}]}{\left[3(1-z^{-1}) + 0.1\,(1+z^{-1})\right]^2 + 9(1+z^{-1})^2}$$

$$= \frac{3.1 + 0.2z^{-1} - 2.9z^{-2}}{18.61 + 0.02z^{-1} + 17.41\,z^{-2}}$$

EXAMPLE 8.11 Convert the analog filter with system function $H_a(s) = \dfrac{s+0.5}{(s+0.5)^2+16}$ into a digital IIR filter using the bilinear transformation. The digital filter should have a resonant frequency of $\omega_r = \pi/2$.

Solution: From the system function, we observe that $\Omega_c = 4$. The sampling period T can be determined using the equation $\Omega = \dfrac{2}{T} \tan \dfrac{\omega}{2}$.

$$\therefore \qquad \Omega_c = \frac{2}{T}\tan\frac{\omega_r}{2}$$

i.e.
$$T = \frac{2}{\Omega_c}\tan\frac{\omega_r}{2} = \frac{2}{4}\tan\frac{\pi}{4} = 0.5 \text{ s}$$

Using the bilinear transformation, the digital filter system function is:

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = H(s)\Big|_{s=4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$H(z) = \frac{s+0.5}{(s+0.5)^2+16}\Bigg|_{s=4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{4\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)+0.5}{\left[4\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)+0.5\right]^2+16}$$

$$= \frac{\left[4(1-z^{-1})+0.5(1+z^{-1})\right][1+z^{-1}]}{\left[4(1-z^{-1})+0.5(1+z^{-1})\right]^2+16[1+z^{-1}]^2}$$

$$= \frac{4.5+z^{-1}-3.5z^{-2}}{36.25+0.5z^{-1}+28.25z^{-2}}$$

EXAMPLE 8.12 Apply the bilinear transformation to

$$H_a(s) = \frac{4}{(s+3)(s+4)}$$

with T = 0.5 s and find $H(z)$.

Solution: Given that $H_a(s) = \dfrac{4}{(s+3)(s+4)}$ and T = 0.5 s.

To obtain $H(z)$ using the bilinear transformation, replace s by $\dfrac{2}{T}\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)$ in $H_a(s)$

$$\therefore \qquad H(z) = \left.\frac{4}{(s+3)(s+4)}\right|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \left.\frac{4}{(s+3)(s+4)}\right|_{s=4\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{4}{\left[4\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)+3\right]\left[4\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)+4\right]}$$

$$= \frac{4}{\left[\dfrac{4-4z^{-1}+3+3z^{-1}}{1+z^{-1}}\right]\left[\dfrac{4-4z^{-1}+4+4z^{-1}}{1+z^{-1}}\right]}$$

$$= \frac{4(1+z^{-1})^2}{(7-z^{-1})\,8}$$

$$= \frac{1}{2}\,\frac{(1+z^{-1})^2}{(7-z^{-1})}$$

EXAMPLE 8.13 Obtain $H(z)$ from $H_a(s)$ when $T = 1$ s and

$$H_a(s) = \frac{3s}{s^2 + 0.5s + 2}$$

using the bilinear transformation.

Solution: Given $H_a(s) = \dfrac{3s}{s^2 + 0.5s + 2}$ and $T = 1$ s.

To get $H(z)$ using the bilinear transformation, put $s = \dfrac{2}{T}\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)$ in $H_a(s)$.

$$\therefore \qquad H(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = \left.\frac{3s}{s^2+0.5s+2}\right|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{3\times 2\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)}{\left[2\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)\right]^2 + 0.5\left[2\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)\right]+2}$$

$$= \frac{6\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)}{\dfrac{4(1-z^{-1})^2 + (1-z^{-1})(1+z^{-1}) + 2(1+z^{-1})^2}{(1+z^{-1})^2}}$$

$$= \frac{6(1+z^{-1})}{4(1-2z^{-1}+z^{-2}) + (1-z^{-2}) + 2(1+2z^{-1}+z^{-2})}$$

$$= \frac{6+6z^{-1}}{7-4z^{-1}+5z^{-2}}$$

EXAMPLE 8.14 Using the bilinear transformation, obtain $H(z)$ from $H_a(s)$ when $T = 1$s and $H_a(s) = \dfrac{s^3}{(s+1)(s^2+2s+2)}$

Solution: Given that $H_a(s) = \dfrac{s^3}{(s+1)(s^2+2s+2)}$ and $T = 1$ s.

To obtain $H(z)$ using the bilinear transformation, put $s = \dfrac{2}{T}\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)$ in $H_a(s)$.

Given $T = 1$ s,

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = \frac{s^3}{(s+1)(s^2+2s+2)}\Bigg|_{s=2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{\left[2\dfrac{(1-z^{-1})}{(1+z^{-1})}\right]^3}{\left[2\dfrac{(1-z^{-1})}{1+z^{-1}}+1\right]\left\{\left[2\dfrac{(1-z^{-1})}{1+z^{-1}}\right]^2 + 2\left[2\dfrac{(1-z^{-1})}{1+z^{-1}}\right]+2\right\}}$$

$$= \frac{8(1-z^{-1})^3}{\left[2(1-z^{-1})+(1+z^{-1})\right]\left[4(1-z^{-1})^2 + 4(1-z^{-1})(1+z^{-1}) + 2(1+z^{-1})^2\right]}$$

$$= \frac{8(1-z^{-1})^3}{(3-z^{-1})[10-4z^{-1}+2z^{-2}]}$$

$$= \frac{4(1-z^{-1})^3}{(3-z^{-1})(5-2z^{-1}+2z^{-2})}$$

$$= 4\,\frac{(1-3z^{-1}+3z^{-2}-z^{-3})}{15-11z^{-1}+8z^{-2}-2z^{-3}}$$

EXAMPLE 8.15 A digital filter with a 3 dB bandwidth of 0.4π is to be designed from the analog filter whose system response is:

$$H(s) = \frac{\Omega_c}{s+2\Omega_c}$$

Use the bilinear transformation and obtain $H(z)$.

Solution: We know that $\Omega_c = \frac{2}{T}\tan\frac{\omega_c}{2}$

Here the 3 dB bandwidth $\omega_c = 0.4\pi$

$$\therefore \qquad \Omega_c = \frac{2}{T}\tan\frac{0.4\pi}{2} = \frac{1.453}{T}$$

The system response of the digital filter is given by

$$H(z) = H_a(s)\Bigg|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{\Omega_c}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+2\Omega_c} = \frac{\frac{1.453}{T}}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+2\left(\frac{1.453}{T}\right)}$$

$$= \frac{1.453\,(1+z^{-1})}{2(1-z^{-1})+2(1+z^{-1})\,1.453}$$

$$= \frac{1+z^{-1}}{3.376\;-\;0.624\,z^{-1}}$$

EXAMPLE 8.16 The normalized transfer function of an analog filter is given by

$$H(s_n) = \frac{1}{s_n^2+1.6s_n+1}$$

Convert the analog filter to a digital filter with a cutoff frequency of 0.6 π, using the bilinear transformation.

Solution: The prewarping of analog filter has to be performed to preserve the magnitude response. For this the analog cutoff frequency is determined using the bilinear transformation, and the analog transfer function is unnormalized using this analog cutoff frequency. Then the analog transfer function is converted to digital transfer function using the bilinear transformation.

Given that, digital cutoff frequency, ω_c = 0.6π rad/s. Let T = 1s.

In the bilinear transformation,

$$\text{Analog cutoff frequency } \Omega_c = \frac{2}{T}\tan\frac{\omega_c}{2} = 2\tan\frac{0.6\,\pi}{2} = 2.753 \text{ rad/s.}$$

$$\text{Normalized analog transfer function } H_a(s_n) = \frac{1}{s_n^2 + 1.6\,s_n + 1}$$

The analog transfer function is unnormalized by replacing s_n by s/Ω_c. Therefore, unnormalized analog filter transfer function is given by

$$H_a(s) = \frac{1}{\left(\frac{s}{\Omega_c}\right)^2 + 1.6\left(\frac{s}{\Omega_c}\right) + 1} = \frac{1}{\left(\frac{s}{2.753}\right)^2 + 1.6\left(\frac{s}{2.753}\right) + 1}$$

$$= \frac{2.753^2}{s^2 + 1.6 \times 2.753\,s + 2.753^2} = \frac{7.579}{s^2 + 4.404\,s + 7.579}$$

The digital filter system function $H(z)$ is obtained by substituting $s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$ in $H_a(s)$. Here T = 1. Therefore, the digital filter transfer function is:

$$H(z) = \frac{7.579}{\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]^2 + 4.404\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right] + 7.579}$$

$$= \frac{7.579(1+z^{-1})^2}{4(1-2z^{-1}+z^{-2}) + 4.404(1+z^{-1})\,2(1-z^{-1}) + 7.579(1+z^{-1})^2}$$

$$= \frac{7.579[1+2z^{-1}+z^{-2}]}{20.387 + 7.158z^{-1} + 2.771z^{-2}}$$

$$= \frac{0.371 + 0.742z^{-1} + 0.371z^{-2}}{1 + 0.351z^{-1} + 0.136z^{-2}}$$

8.6 SPECIFICATIONS OF THE LOW-PASS FILTER

The magnitude response of low-pass filter in terms of gain and attenuation are shown in Figure 8.6.

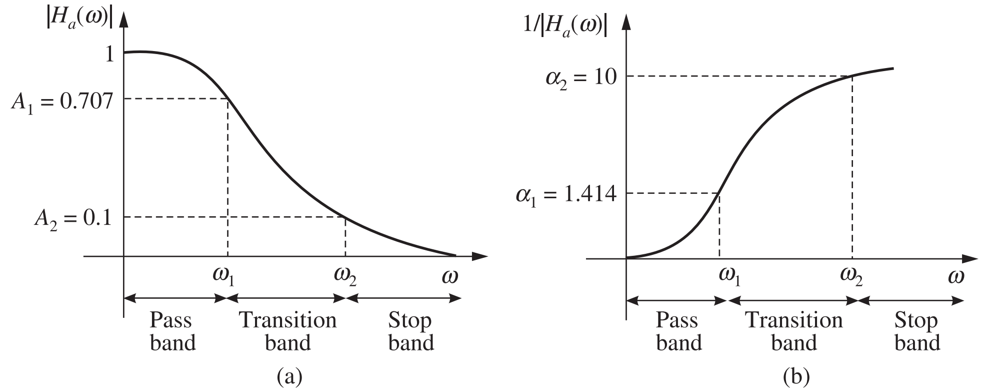


Figure 8.6 Magnitude response of low-pass filter (a) Gain vs ω and (b) Attenuation vs ω.

Let ω_1 = Passband frequency in rad/s.
ω_2 = Stopband frequency in rad/s.

Let the gain at the passband frequency ω_1 be A_1 and the gain at the stopband frequency ω_2 be A_2, i.e.

$$A_1 = |H(\omega)|\big|_{\omega=\omega_1} \quad \text{and} \quad A_2 = |H(\omega)|\big|_{\omega=\omega_2}$$

The filter may be expressed in terms of the gain or attenuation at the edge frequencies. Let α_1 be the attenuation at the passband edge frequency ω_1, and α_2 be the attenuation at the stopband edge frequency ω_2.

i.e.
$$\alpha_1 = \frac{1}{A_1} = \frac{1}{|H(\omega)|\big|_{\omega=\omega_1}} \quad \text{and} \quad \alpha_2 = \frac{1}{A_2} = \frac{1}{|H(\omega)|\big|_{\omega=\omega_2}}$$

The maximum value of normalized gain is unity, so A_1 and A_2 are less than 1 and α_1 and α_2 are greater than 1. In Figure 8.6, A_1 is assumed as $1/\sqrt{2}$ and A_2 is assumed as 0.1. Hence $\alpha_1 = \sqrt{2} = 1.414$ and $\alpha_2 = 1/0.1 = 10$.

Another popular unit that is used for filter specification is dB. When the gain is expressed in dB, it will be a negative dB. When the attenuation is expressed in dB, it will be a positive dB.

Let k_1 = Gain in dB at a passband frequency ω_1
k_2 = Gain in dB at a stopband frequency ω_2

The gain can be converted into normal values as follows:

$$20 \log A_1 = k_1 \qquad 20 \log A_2 = k_2$$
$$\log A_1 = k_1/20 \qquad \log A_2 = k_2/20$$
$$A_1 = 10^{k_1/20} \qquad A_2 = 10^{k_2/20}$$

When expressed in dB, the gain and attenuation will have only change in sign because log α = log(1/A) = –log A. (Hence when dB is positive it is attenuation and when dB is negative it is gain).

When $A_1 = 0.707$, $k_1 = 20 \log(0.707) = -3.0116 = -3$ dB
When $A_2 = 0.1$, $k_2 = 20 \log(0.1) = -20$ dB

The magnitude response of low-pass filter in terms of dB-attenuation is shown in Figure 8.7.

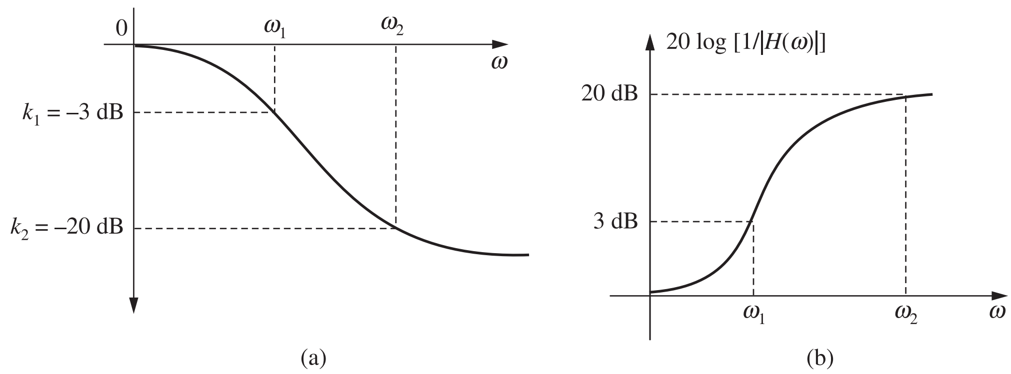


Figure 8.7 Magnitude response of low-pass filter (a) dB-Gain vs ω and (b) dB-attenuation vs ω.

Sometimes the specifications are given in terms of passband ripple δ_p and stopband ripple δ_s. In this case, the dB gain and attenuation can be estimated as follows:

$$k_1 = 20 \log (1 - \delta_p) \qquad \alpha_1 = -20 \log (1 - \delta_p)$$
$$k_2 = 20 \log \delta_s \qquad \alpha_2 = -20 \log \delta_s$$

If the ripples are specified in dB, then the minimum passband ripple is equal to k_1 and the negative of maximum passband attenuation is equal to k_2.

8.7 DESIGN OF LOW-PASS DIGITAL BUTTERWORTH FILTER

The popular methods of designing IIR digital filter involves the design of equivalent analog filter and then converting the analog filter to digital filter. Hence to design a Butterworth IIR digital filter, first an analog Butterworth filter transfer function is determined using the given specifications. Then the analog filter transfer function is converted to a digital filter transfer function using either impulse invariant transformation or bilinear transformation.

Analog Butterworth filter

The analog Butterworth filter is designed by approximating the ideal frequency response using an error function. The error function is selected such that the magnitude is maximally flat in the passband and monotonically decreasing in the stopband. (Strictly speaking the magnitude is maximally flat at the origin, i.e., at $\Omega = 0$, and monotonically decreasing with increasing Ω).

The magnitude response of low-pass filter obtained by this approximation is given by

$$|H_a(\omega)|^2 = \frac{1}{1 + \left(\dfrac{\Omega}{\Omega_c}\right)^{2N}}$$

where Ω_c is the 3 dB cutoff frequency and N is the order of the filter.

Frequency response of the Butterworth filter

The frequency response of Butterworth filter depends on the order N. The magnitude response for different values of N are shown in Figure 8.8. From Figure 8.8, it can be observed that the approximated magnitude response approaches the ideal response as the value of N increases. However, the phase response of the Butterworth filter becomes more nonlinear with increasing N.

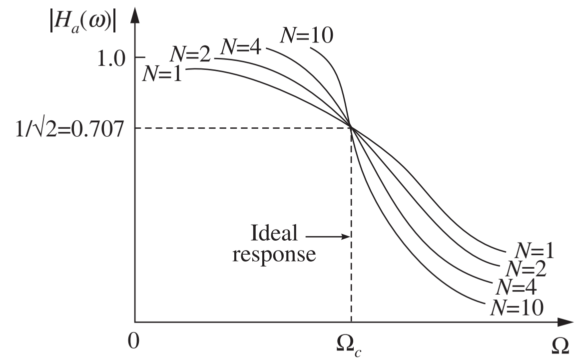


Figure 8.8 Magnitude response of Butterworth low-pass filter for various values of N.

Order of the filter

Since the frequency response of the filter depends on its order N, the order N has to be estimated to satisfy the given specifications.

Usually the specifications of the filter are given in terms of gain A or attenuation α at a passband or stopband frequency as given below:

$$A_1 \le |H(\omega)| \le 1, \qquad 0 \le \omega \le \omega_1$$

$$|H(\omega)| \le A_2, \qquad \omega_2 \le \omega \le \pi$$

The order of the filter is determined as given below.

Let Ω_1 and Ω_2 be the analog filter edge frequencies corresponding to digital frequencies ω_1 and ω_2. The values of Ω_1 and Ω_2 are obtained using the bilinear transformation or impulse invariant transformation.

$$\therefore \qquad A_1^2 \le \frac{1}{1+\left(\dfrac{\Omega_1}{\Omega_c}\right)^{2N}} \le 1$$

and

$$\frac{1}{1+\left(\dfrac{\Omega_2}{\Omega_c}\right)^{2N}} \le A_2^2$$

These two equations can be written in the form

$$\left(\frac{\Omega_1}{\Omega_c}\right)^{2N} \le \frac{1}{A_1^2} - 1$$

and

$$\left(\frac{\Omega_2}{\Omega_c}\right)^{2N} \ge \frac{1}{A_2^2} - 1$$

Assuming equality we can obtain the filter order N and the 3 dB cutoff frequency Ω_c. Dividing the first equation by the second, we have

$$\left(\frac{\Omega_1}{\Omega_2}\right)^{2N} = \frac{\dfrac{1}{A_1^2} - 1}{\dfrac{1}{A_2^2} - 1}$$

From this equation, the order of the filter N is obtained approximately as:

$$N = \frac{1}{2}\,\frac{\log\left\{\left(\dfrac{1}{A_2^2} - 1\right)\Big/\left(\dfrac{1}{A_1^2} - 1\right)\right\}}{\log\dfrac{\Omega_2}{\Omega_1}}$$

If N is not an integer, the value of N is chosen to be the next nearest integer. Also we can get

$$\Omega_c = \frac{\Omega_1}{\left[\dfrac{1}{A_1^2} - 1\right]^{1/2N}}$$

when parameters A_1 and A_2 are given in dB.

A_1 in dB is given by

$$A_1 \text{ dB} = -20 \log A_1$$

i.e.
$$\log A_1 = -\frac{A_1 \text{ dB}}{20}$$

or
$$A_1 = 10^{-\frac{A_1 \text{ dB}}{20}}$$

$\therefore$
$$\frac{1}{A_1^2} - 1 = \frac{1}{\left(10^{-\frac{A_1 \text{ dB}}{20}}\right)^2} - 1$$

i.e.
$$\frac{1}{A_1^2} - 1 = 10^{0.1 A_1 \text{ dB}} - 1$$

Similarly
$$\frac{1}{A_2^2} - 1 = 10^{0.1 A_2 \text{ dB}} - 1$$

$\therefore$
$$N = \frac{1}{2} \frac{\log\{[\frac{1}{A_2^2} - 1]/[\frac{1}{A_1^2} - 1]\}}{\log\left(\frac{\Omega_2}{\Omega_1}\right)} = \frac{1}{2} \frac{\log\left(\frac{10^{0.1 A_2 \text{dB}} - 1}{10^{0.1 A_1 \text{dB}} - 1}\right)}{\log\left(\frac{\Omega_2}{\Omega_1}\right)}$$

and Ω_c is given by

$$\Omega_c = \frac{\Omega_1}{(10^{0.1 A_1 \text{dB}} - 1)^{1/2N}} \quad \text{or} \quad \Omega_c = \frac{\Omega_2}{(10^{0.1 A_2 \text{dB}} - 1)^{1/2N}}$$

In fact,
$$\Omega_c = \frac{1}{2}\left[\frac{\Omega_1}{\left[10^{0.1 A_1 \text{dB}} - 1\right]^{1/2N}} + \frac{\Omega_2}{\left[10^{0.1 A_2 \text{dB}} - 1\right]^{1/2N}}\right]$$

Butterworth low-pass filter transfer function

The unnormalized transfer function of the Butterworth filter is usually written in factored form as:

$$H_a(s) = \prod_{k=1}^{N/2} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2} \quad \text{(when } N \text{ is even)}$$

or
$$H_a(s) = \frac{\Omega_c}{s+\Omega_c}\prod_{k=1}^{\frac{N-1}{2}} \frac{\Omega_c^2}{s^2 + b_k\Omega_c s + \Omega_c^2} \quad \text{(when } N \text{ is odd)}$$

where $b_k = 2\sin\left[\frac{(2k-1)\pi}{2N}\right]$

If s/Ω_c (where Ω_c is the 3 dB cutoff frequency of the low-pass filter) is replaced by s_n, then the normalized Butterworth filter transfer function is given by

$$H_a(s) = \prod_{k=1}^{N/2} \frac{1}{s_n^2 + b_k s_n + 1} \quad \text{(when } N \text{ is even)}$$

or
$$H_a(s) = \frac{1}{s_n+1}\prod_{k=1}^{\frac{N-1}{2}} \frac{1}{s_n^2 + b_k s_n + 1} \quad \text{(when } N \text{ is odd)}$$

where $b_k = 2\sin\left[\frac{(2k-1)\pi}{2N}\right]$

Design procedure for low-pass digital Butterworth IIR filter

The low-pass digital Butterworth filter is designed as per the following steps:

Let A_1 = Gain at a passband frequency ω_1
A_2 = Gain at a stopband frequency ω_2
Ω_1 = Analog frequency corresponding to ω_1
Ω_2 = Analog frequency corresponding to ω_2

Step 1 Choose the type of transformation, i.e., either bilinear or impulse invariant transformation.

Step 2 Calculate the ratio of analog edge frequencies Ω_2/Ω_1.
For bilinear transformation

$$\Omega_1 = \frac{2}{T}\tan\frac{\omega_1}{2},\ \Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} \quad \therefore\ \frac{\Omega_2}{\Omega_1} = \frac{\tan \omega_2/2}{\tan \omega_1/2}$$

For impulse invariant transformation,

$$\Omega_1 = \frac{\omega_1}{T},\quad \Omega_2 = \frac{\omega_2}{T} \quad \therefore\ \frac{\Omega_2}{\Omega_1} = \frac{\omega_2}{\omega_1}$$

Step 3 Decide the order N of the filter. The order N should be such that

$$N \geq \frac{1}{2}\,\frac{\log\left\{\left[\frac{1}{A_2^2}-1\right]\Big/\left[\frac{1}{A_1^2}-1\right]\right\}}{\log\frac{\Omega_2}{\Omega_1}}$$

Choose N such that it is an integer just greater than or equal to the value obtained above.

Step 4 Calculate the analog cutoff frequency $\Omega_c = \dfrac{\Omega_1}{\left[\dfrac{1}{A_1^2} - 1\right]^{1/2N}}$

For bilinear transformation $\Omega_c = \dfrac{\dfrac{2}{T}\tan \omega_1/2}{\left[\dfrac{1}{A_1^2} - 1\right]^{1/2N}}$

For impulse invariant transformation $\Omega_c = \dfrac{\omega_1/T}{\left[\dfrac{1}{A_1^2} - 1\right]^{1/2N}}$

Step 5 Determine the transfer function of the analog filter.
Let $H_a(s)$ be the transfer function of the analog filter. When the order N is even, for unity dc gain filter, $H_a(s)$ is given by

$$H_a(s) = \prod_{k=1}^{N/2} \frac{\Omega_c^2}{s^2 + b_k\Omega_c s + \Omega_c^2}$$

When the order N is odd, for unity dc gain filter, $H_a(s)$ is given by

$$H_a(s) = \frac{\Omega_c}{s + \Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{\Omega_c^2}{s^2 + b_k\Omega_c s + \Omega_c^2}$$

The coefficient b_k is given by

$$b_k = 2 \sin\left[\frac{(2k-1)\pi}{2N}\right]$$

For normalized case, $\Omega_c = 1$ rad/s

Step 6 Using the chosen transformation, transform the analog filter transfer function $H_a(s)$ to digital filter transfer function $H(z)$.

Step 7 Realize the digital filter transfer function $H(z)$ by a suitable structure.

Poles of the normalized Butterworth filter

The Butterworth low-pass filter has a magnitude squared response given by

$$|H_a(\omega)|^2 = \frac{1}{1 + \left(\dfrac{\Omega}{\Omega_c}\right)^{2N}}$$

We know that the frequency response $H_a(\Omega)$ of an analog filter is obtained by substituting $s = j\Omega$ in the analog transfer function $H_a(s)$. Hence the system transfer function is obtained by replacing Ω by (s/j) in the above equation.

$$\therefore \qquad H_a(s)\, H_a(-s) = \frac{1}{1 + \left(\dfrac{s}{j\Omega_c}\right)^{2N}} = \frac{1}{1 + \left(\dfrac{s^2}{j^2\Omega_c^2}\right)^{2N}}$$

In the above equation, when s/Ω_c is replaced by s_n (i.e. $\Omega_c = 1$ rad/s), the transfer function is called normalized transfer function.

$$\therefore \qquad H_a(s_n) H_a(-s_n) = \frac{1}{1 + (-s_n^2)N}$$

The transfer function of the above equation will have $2N$ poles which are given by the roots of the denominator polynomial. It can be shown that the poles of the transfer function symmetrically lie on a unit circle in s-plane with angular spacing of π/N.

For a stable and causal filter the poles should lie on the left half of the s-plane. Hence the desired filter transfer function is formed by choosing the N-number of left half poles. When N is even, all the poles are complex and exist in conjugate pairs. When N is odd, one of the pole is real and all other poles are complex and exist as conjugate pairs. Therefore, the transfer function of Butterworth filters will be a product of second order factors.

The poles of the Butterworth polynomial lie on a circle, whose radius is ω_c. To determine the number of poles of the Butterworth filter and the angle between them we use the following rules.

- Number of Butterworth poles = $2N$
- Angle between any two poles $\theta = 360°/(2N)$

If the order of the filter N is even, then the location of the first pole is at $\theta/2$ w.r.t. the positive real axis, with the angle measured in the counter-clockwise direction. The location of the subsequent poles are respectively, at

$$\left(\frac{\theta}{2} + \theta\right), \left(\frac{\theta}{2} + 2\theta\right), \left(\frac{\theta}{2} + 3\theta\right), \ldots, \left(360 - \frac{\theta}{2}\right)$$

If the order of the filter N is odd, then the location of the first pole is on the X-axis. The location of subsequent poles are at θ, 2θ, ..., $(360 - \theta)$ with the angle measured in the counter-clockwise direction.

If ϕ is the angle of a valid pole w.r.t. the X-axis, then the pole and its conjugate are located at $[\omega_c(\cos\phi \pm j\sin\phi)]$.

Properties of Butterworth filters

1. The Butterworth filters are all pole designs (i.e. the zeros of the filters exist at ∞).
2. The filter order N completely specifies the filter.

3. The magnitude response approaches the ideal response as the value of N increases.
4. The magnitude is maximally flat at the origin.
5. The magnitude is monotonically decreasing function of Ω.
6. At the cutoff frequency Ω_c, the magnitude of normalized Butterworth filter is $1/\sqrt{2}$. Hence the dB magnitude at the cutoff frequency will be 3 dB less than the maximum value.

EXAMPLE 8.17 Design a Butterworth digital filter using the bilinear transformation. The specifications of the desired low-pass filter are:

$$0.9 \le |H(\omega)| \le 1; \quad 0 \le \omega \le \frac{\pi}{2}$$

$$|H(\omega)| \le 0.2; \quad \frac{3\pi}{4} \le \omega \le \pi$$

with $T = 1$ s

Solution: The Butterworth digital filter is designed as per the following steps.

From the given specification, we have

$$A_1 = 0.9 \text{ and } \omega_1 = \frac{\pi}{2}$$

$$A_2 = 0.2 \text{ and } \omega_2 = \frac{3\pi}{4} \qquad \text{and } T = 1 \text{ s}$$

Step 1 Choice of the type of transformation

Here the bilinear transformation is already specified.

Step 2 Determination of the ratio of the analog filter's edge frequencies, Ω_2/Ω_1

$$\Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} = \frac{2}{1}\tan\left[\frac{(3\pi/4)}{2}\right] = 2\tan\frac{3\pi}{8} = 4.828$$

$$\Omega_1 = \frac{2}{T}\tan\frac{\omega_1}{2} = \frac{2}{1}\tan\left[\frac{(\pi/2)}{2}\right] = 2\tan\frac{\pi}{4} = 2$$

$$\therefore \qquad \frac{\Omega_2}{\Omega_1} = \frac{4.828}{2} = 2.414$$

Step 3 Determination of the order of the filter N

$$N \ge \frac{1}{2}\,\frac{\log\left\{\left[\dfrac{1}{A_2^2} - 1\right] \Big/ \left[\dfrac{1}{A_1^2} - 1\right]\right\}}{\log\dfrac{\Omega_2}{\Omega_1}}$$

$$\geq \frac{1}{2}\frac{\log\left\{\left[\frac{1}{(0.2)^2}-1\right]\Big/\left[\frac{1}{(0.9)^2}-1\right]\right\}}{\log 1.207}$$

$$\geq \frac{1}{2}\frac{\log\{24/0.2345\}}{\log 2.414} \geq 2.626$$

Since $N \geq 2.626$, choose $N = 3$.

Step 4 Determination of the analog cutoff frequency Ω_c (i.e., –3 dB frequency)

$$\Omega_c = \frac{\Omega_1}{\left[\frac{1}{A_1^2}-1\right]^{1/2N}} = \frac{2}{\left[\frac{1}{0.9^2}-1\right]^{1/2\times 3}} = 2.5467$$

Step 5 Determination of the transfer function of the analog Butterworth filter $H_a(s)$

For odd N, we have $H_a(s) = \frac{\Omega_c}{s+\Omega_c}\prod_{k=1}^{\frac{N-1}{2}}\frac{\Omega_c^2}{s^2+b_k\Omega_c s+\Omega_c^2}$

where $b_k = 2\sin\left[\frac{(2k-1)\pi}{2N}\right]$

For $N = 3$, we have

$$H_a(s) = \frac{\Omega_c}{s+\Omega_c}\,\frac{\Omega_c^2}{s^2+b_1\Omega_c s+\Omega_c^2}$$

where $b_1 = 2\sin\left[\frac{(2\times 1-1)\pi}{2\times 3}\right] = 2\sin\frac{\pi}{6} = 1$

Therefore, $H_a(s) = \left(\frac{2.5467}{s+2.5467}\right)\left(\frac{(2.5467)^2}{s^2+1(2.5467)\,s+(2.5467)^2}\right)$

Step 6 Conversion of $H_a(s)$ into $H(z)$

Since bilinear transformation is to be used, the digital filter transfer function is:

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = H_a(s)\Big|_{s=2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$H(z) = \left(\frac{2.5467}{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+2.5467}\right)\left[\frac{(2.5467)^2}{\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]^2+2.5467\left[2\,\frac{1-z^{-1}}{1+z^{-1}}\right]+(2.5467)^2}\right]$$

$$= \frac{0.2332(1+z^{-1})^3}{1+0.4394z^{-1}+0.3845z^{-2}+0.0416z^{-3}}$$

EXAMPLE 8.18 Design a digital Butterworth filter satisfying the following constraints:

$$0.8 \le |H(\omega)| \le 1; \qquad 0 \le \omega \le 0.2\pi$$

$$|H(\omega)| \le 0.2; \quad 0.32\pi \le \omega \le \pi$$

with $T = 1$ s. Apply impulse invariant transformation.

Solution: From the given specifications, we have

$$A_1 = 0.8 \qquad \omega_1 = 0.2\pi$$

$$A_2 = 0.2 \qquad \omega_2 = 0.32\pi \qquad \text{and } T = 1 \text{ s}$$

The Butterworth IIR digital filer is designed as per the following steps.

Step 1 Choice of the type of transformation
Here, the impulse invariant transformation is already specified.

Step 2 Determination of the ratio of analog filter's edge frequencies, Ω_2/Ω_1

$$\Omega_2 = \frac{\omega_2}{T} = \frac{0.32\pi}{1} = 0.32\pi$$

$$\Omega_1 = \frac{\omega_1}{T} = \frac{0.2\pi}{1} = 0.2\pi$$

$$\frac{\Omega_2}{\Omega_1} = \frac{0.32\pi}{0.2\pi} = 1.6$$

Step 3 Determination of the order of the filter N

$$N \ge \frac{1}{2}\frac{\log\left\{\left[\frac{1}{A_2^2}-1\right]\Big/\left[\frac{1}{A_1^2}-1\right]\right\}}{\log\frac{\Omega_2}{\Omega_1}}$$

$$\ge \frac{1}{2}\frac{\log\left\{\left[\frac{1}{0.2^2}-1\right]\Big/\left[\frac{1}{0.8^2}-1\right]\right\}}{\log 1.6}$$

$$\ge \frac{1}{2}\frac{\log\{24/0.5625\}}{\log 1.6} \ge 3.9931$$

So the order of the filter $N \ge 3.9931$. Choose $N = 4$.

Step 4 Determination of the analog cutoff frequency Ω_c (i.e., – 3 dB frequency)

$$\Omega_c = \frac{\Omega_1}{\left[\frac{1}{A_1^2} - 1\right]^{1/2N}} = \frac{0.2\pi}{\left[\frac{1}{0.8^2} - 1\right]^{1/2\times 4}} = 0.675 \text{ rad/s}$$

Step 5 Determination of the transfer function of analog low-pass Butterworth filter. The transfer function of the low-pass filter for even values of N is:

$$H_a(s) = \prod_{k=1}^{N/2} \frac{\Omega_c^2}{s^2 + b_k\Omega_c s + \Omega_c^2}$$

where $$b_k = 2\sin\left[\frac{(2k-1)\pi}{2N}\right]$$

Here $N = 4$; $\therefore$ $k = 1, 2$

When $k = 1$, $b_k = b_1 = 2\sin\left[\frac{(2-1)\pi}{2\times 4}\right] = 0.765$

When $k = 2$, $b_k = b_2 = 2\sin\left[\frac{(2\times 2-1)\pi}{2\times 4}\right] = 1.848$

$$\therefore\ H_a(s) = \frac{\Omega_c^2}{s^2 + b_1\Omega_c s + \Omega_c^2} \times \frac{\Omega_c^2}{s^2 + b_2\Omega_c s + \Omega_c^2}$$

$$= \frac{(0.675)^2}{s^2 + (0.765\times 0.675)s + 0.675^2} \times \frac{(0.675)^2}{s^2 + (1.848\times 0.675)s + (0.675)^2}$$

$$= \frac{0.2076}{(s^2 + 0.516s + 0.456)(s^2 + 1.247s + 0.456)}$$

Step 6 Determination of the digital filter transfer function $H(z)$
By partial fraction expansion, $H_a(s)$ can be expressed as:

$$H_a(s) = \frac{0.2076}{(s^2 + 0.516s + 0.456)\,(s^2 + 1.247s + 0.456)}$$

$$= \frac{As + B}{s^2 + 0.516s + 0.456} + \frac{Cs + D}{s^2 + 1.247s + 0.456}$$

On cross multiplying the above equation and simplifying, we get

$$0.2076 = (A + C)s^3 + (1.247A + B + 0.516C + D)s^2 + (0.456A + 1.247B + 0.456C + 0.516D)s + (0.456B + 0.456D)$$

On solving, we get

$$A = -0.622,\ B = -0.321,\ C = 0.622 \text{ and } D = 0.776$$

Therefore, $H_a(s)$ can be written as:

$$H_a(s) = \frac{-0.622s - 0.321}{s^2 + 0.516s + 0.456} + \frac{0.622s + 0.776}{s^2 + 1.247s + 0.456}$$

$$= \frac{-0.622(s + 0.516)}{(s^2 + 2 \times 0.258s + 0.258^2) + \left(\sqrt{0.456 - 0.258^2}\right)^2}$$

$$+ \frac{0.622(s + 1.248)}{(s^2 + 2 \times 0.624s + 0.624^2) + \left(\sqrt{0.456 - 0.624^2}\right)^2}$$

$$= \frac{-0.622\,(s + 0.258 + 0.258)}{(s + 0.258)^2 + (0.624)^2} + \frac{0.622(s + 0.624 + 0.624)}{(s + 0.624)^2 + (0.258)^2}$$

$$= -0.622\frac{s + 0.258}{(s + 0.258)^2 + (0.624)^2} - 0.257\frac{0.624}{(s + 0.258)^2 + (0.624)^2}$$

$$+ 0.622\frac{s + 0.624}{(s + 0.624)^2 + (0.258)^2} + 1.504\frac{0.258}{(s + 0.624)^2 + (0.258)^2}$$

The analog transfer function of the above equation can be transformed to digital transfer function using the following standard impulse invariant transformations.

$$\frac{s + a}{(s + a)^2 + b^2} \rightarrow \frac{1 - e^{-aT}(\cos bT)z^{-1}}{1 - 2e^{-aT}(\cos bT)z^{-1} + e^{-2aT}z^{-2}}$$

$$\frac{b}{(s + a)^2 + b^2} \rightarrow \frac{e^{-aT}(\sin bT)z^{-1}}{1 - 2e^{-aT}(\cos bT)z^{-1} + e^{-2aT}z^{-2}}$$

Taking $T = 1$ s, the above transformation can be applied to $H_a(s)$ to get $H(z)$.

$$H(z) = -0.622\frac{1 - e^{-0.258}(\cos 0.624)z^{-1}}{1 - 2e^{-0.258}(\cos 0.624)z^{-1} + e^{-2\times0.258}z^{-2}}$$

$$-0.257\frac{e^{-0.258}(\sin 0.624)z^{-1}}{1 - 2e^{-0.258}(\cos 0.624)z^{-1} + e^{-2\times0.258}z^{-2}}$$

$$+0.622\frac{1 - e^{-0.624}(\cos 0.258)z^{-1}}{1 - 2e^{-0.624}(\cos 0.258)z^{-1} + e^{-2\times0.624}z^{-2}}$$

$$+1.504\frac{e^{-0.624}(\sin 0.258)z^{-1}}{1 - 2e^{-0.624}(\cos 0.258)z^{-1} + e^{-2\times0.624}z^{-2}}$$

$$= \frac{-0.622 + 0.39 z^{-1}}{1 - 1.254 z^{-1} + 0.597 z^{-2}} + \frac{-0.116 z^{-1}}{1 - 1.254 z^{-1} + 0.597 z^{-2}}$$

$$+ \frac{0.622 - 0.322 z^{-1}}{1 - 1.036 z^{-1} + 0.287 z^{-2}} + \frac{0.206 z^{-1}}{1 - 1.036 z^{-1} + 0.287 z^{-2}}$$

$$= \frac{-0.622 + 0.274 z^{-1}}{1 - 1.254 z^{-1} + 0.597 z^{-2}} + \frac{0.622 - 0.116 z^{-1}}{1 - 1.036 z^{-1} + 0.287 z^{-2}}$$

$$= \frac{0.0224 z^{-1} + 0.0544 z^{-2} + 0.0094 z^{-3}}{1 - 2.29 z^{-1} + 2.1831 z^{-2} - 0.977 z^{-3} + 0.1713 z^{-4}}$$

EXAMPLE 8.19 Design a low-pass Butterworth digital filter to give response of 3 dB or less for frequencies upto 2 kHz and an attenuation of 20 dB or more beyond 4 kHz. Use the bilinear transformation technique and obtain $H(z)$ of the desired filter.

Solution: The specifications of the desired filter are given in terms of dB attenuation and frequency in Hz. First the gain is to be expressed as a numerical value and frequency in rad/s.

Here attenuation at passband frequency (ω_1) = 3 dB

Therefore, gain at passband edge frequency (ω_1) is $k_1 = -3$ dB

$$\therefore \qquad A_1 = 10^{k_1/20} = 10^{-3/20} = 0.707 = \frac{1}{\sqrt{2}}$$

Attenuation at stopband frequency (ω_2) = 20 dB

Therefore, gain at stopband edge frequency (ω_2) is $k_2 = -20$ dB

$$\therefore \qquad A_2 = 10^{k_2/20} = 10^{-20/20} = 0.1$$

Passband edge frequency = 2 kHz,

Stopband edge frequency = 4 kHz,

The design is performed as given below.

Let the sampling frequency be 10000 Hz.

$$\therefore \qquad \text{Normalized } \omega_1 = 2\pi \frac{f_1}{f_s} = 2\pi \frac{2000}{10000} = 0.4\pi$$

$$\text{Normalized } \omega_2 = 2\pi \frac{f_2}{f_s} = 2\pi \frac{4000}{10000} = 0.8\pi$$

Step 1 Bilinear transformation is chosen

Step 2 Ratio of analog filter edge frequencies Ω_2/Ω_1

$$\Omega_1 = \frac{2}{T} \tan \frac{\omega_1}{2} = \frac{2}{T} \tan \frac{0.4\pi}{2} = 14530.8 \text{ rad/s}$$

$$\Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} = \frac{2}{T}\tan\frac{0.8\pi}{2} = 61553.6 \text{ rad/s}$$

$$\therefore \qquad \frac{\Omega_2}{\Omega_1} = \frac{\tan\frac{\omega_2}{2}}{\tan\frac{\omega_1}{2}} = \frac{\tan 0.4\pi}{\tan 0.2\pi} = 4.236$$

Step 3 Order of the filter

$$N \ge \frac{1}{2}\,\frac{\log\left\{\left[\frac{1}{A_2^2}-1\right]\Big/\left[\frac{1}{A_1^2}-1\right]\right\}}{\log\frac{\Omega_2}{\Omega_1}}$$

$$\ge \frac{1}{2}\,\frac{\log\left\{\left[\frac{1}{(0.1)^2}-1\right]\Big/\left[\frac{1}{(1/\sqrt{2})^2}-1\right]\right\}}{\log 4.236}$$

$$\ge \frac{1}{2}\,\frac{\log[99/1]}{\log 4.236} \ge 1.59$$

$$\therefore \qquad N = 2$$

Step 4 Analog cutoff frequency Ω_c

$$\Omega_c = \frac{\Omega_1}{\left[\frac{1}{A_1^2}-1\right]^{1/2N}} = \frac{1.4530}{\left[\frac{1}{(1/\sqrt{2})^2}-1\right]^{1/2\times 2}} = 1.4530$$

Unnormalized $\Omega_c = f_s \times 1.4530 = 14530$ rad/s

Step 5 Transfer function $H_a(s)$

For $N = 2,\ H_a(s) = \dfrac{\Omega_c^2}{s^2 + b_1\,\Omega_c\, s + \Omega_c^2}$

where $b_1 = 2\sin\left[\dfrac{(2\times 1-1)\pi}{2\times 2}\right] = 2\sin\dfrac{\pi}{4} = 1.414$

$$\therefore \qquad H_a(s) = \frac{(14530)^2}{s^2 + 1.414\times 14530\,s + (14530)^2}$$

$$= \frac{2.1112\times 10^8}{s^2 + 20545.42\,s + 2.1112\times 10^8}$$

Step 6 Conversion of $H_a(s)$ into $H(z)$

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = H_a(s)\Big|_{s=20000\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$H(z) = \frac{2.112 \times 10^8}{\left[20 \times 10^3\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)\right]^2 + 2.0545 \times 10^4 \times 20 \times 10^3\left[\dfrac{1-z^{-1}}{1+z^{-1}}\right] + 2.112 \times 10^8}$$

$$= \frac{0.528}{2.5552 - 0.946z^{-1} + 0.5008z^{-2}}$$

EXAMPLE 8.20 Design a low-pass Butterworth filter using the bilinear transformation method for satisfying the following constraints:

Passband: 0–400 Hz
Stopband: 2.1–4 kHz
Passband ripple: 2 dB
Stopband attenuation: 20 dB
Sampling frequency: 10 kHz

Solution: Given

$\alpha_1 = 2$ dB, $\therefore\ k_1 = -2$ dB and $A_1 = 10^{k_1/20} = 10^{-2/20} = 0.794$

$\alpha_2 = 20$ dB, $\therefore\ k_2 = -20$ dB and $A_2 = 10^{k_2/20} = 10^{-20/20} = 0.1$

Step 1 Type of transformation
Bilinear transformation is already specified.

Step 2 Ratio of analog edge frequencies Ω_2/Ω_1.
Here $f_s = 10$ kHz
Passband edge frequency $f_1 = 400$ Hz
Stopband edge frequency $f_2 = 2.1$ kHz
Normalizing the frequencies, we have

$$\omega_1 = 2\pi\frac{f_1}{f_s} = 2\pi \times \frac{400}{10000} = 0.25 \text{ rad}$$

$$\omega_2 = 2\pi\frac{f_2}{f_s} = 2\pi \times \frac{2100}{10000} = 1.319 \text{ rad}$$

Therefore, the analog filter edge frequencies are:

$$\Omega_1 = \frac{2}{T}\tan\frac{\omega_1}{2} = 2 \times 10000 \tan\frac{0.25}{2} = 2513.102 \text{ rad/s}$$

and $$\Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} = 2 \times 10000 \tan\frac{1.319}{2} = 15{,}506.08 \text{ rad/s}$$

$\therefore$
$$\frac{\Omega_2}{\Omega_1} = \frac{15506.08}{2513.102} = 6.1703$$

Step 3 Order of the filter N

$$N \ge \frac{1}{2} \frac{\log\left\{\left[\frac{1}{A_2^2} - 1\right] \Big/ \left[\frac{1}{A_1^2} - 1\right]\right\}}{\log\left(\frac{\Omega_2}{\Omega_1}\right)} \quad \text{or} \quad N \ge \frac{1}{2} \frac{\log\left\{\frac{10^{0.1 A_2\,\text{dB}} - 1}{10^{0.1 A_1\,\text{dB}} - 1}\right\}}{\log(6.1703)}$$

i.e.
$$N \ge \frac{1}{2} \frac{\log\left\{\left\{\left[\frac{1}{(0.1)^2} - 1\right] \Big/ \left[\frac{1}{(0.794)^2} - 1\right]\right\}\right\}}{\log(6.1703)} \quad \text{or} \quad N \ge \frac{1}{2} \frac{\log\left\{\frac{10^{0.1 \times 20\,\text{dB}} - 1}{10^{0.1 \times 2\,\text{dB}} - 1}\right\}}{\log(6.1703)}$$

i.e. $N \ge 1.409 \approx 2$ or $N \ge 1.410 \approx 2$

Step 4 The cutoff frequency Ω_c

$$\Omega_c = \frac{\Omega_2}{\left[\frac{1}{A_2^2} - 1\right]^{1/2N}} \quad \text{or} \quad \Omega_c = \frac{\Omega_2}{[10^{0.1 A_2 \text{dB}} - 1]^{1/2N}}$$

i.e.

$$\Omega_c = \frac{15506.08}{\left[\frac{1}{(0.1)^2} - 1\right]^{1/2\times 2}} = 4915.7 \quad \text{or} \quad \Omega_c = \frac{15506.08}{[10^{0.1\times 20} - 1]^{1/2\times 2}} = 4915.788 \text{ rad/s}$$

Step 5 The system function $H_a(s)$

$$H_a(s) = \frac{\Omega_c^2}{s^2 + b_1 \Omega_c s + \Omega_c^2} \quad \text{where } b_1 = 2\sin\frac{(2\times 1 - 1)\pi}{2\times 2} = 1.414$$

$$= \frac{(4915.788)^2}{s^2 + 1.414 \times 4915.788 s + (4915.788)^2}$$

$$= \frac{2.416\times 10^7}{s^2 + 6950.92 s + 2.416\times 10^7}$$

Step 6 Digital transfer function $H(z)$

$$H(z) = H_a(s)\Big|_{s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)}$$

$$= \frac{2.416 \times 10^7}{\left[\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]^2 + 6950.92 \times \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] + 2.416 \times 10^7}$$

$$= \frac{2.416 \times 10^7}{\left[20000 \times \left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]^2 + 6950.92 \times 20000\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 2.416 \times 10^7}$$

$$= \frac{0.042 + 0.085 z^{-1} + 0.042 z^{-2}}{1 - 1.335 z^{-1} + 0.506 z^{-2}}$$

The poles are given by $P_k = \pm(\Omega_c)\left[e\,\frac{j(2k+N+1)\pi}{2N}\right],\ k = 0, 1 < N$

$$\therefore \qquad P_0 = \pm(\Omega_c) e^{j\left(\frac{3\pi}{4}\right)} = 4.915.788(-0.707 + j0.707) = -3475.6 + j3475.46$$

$$P_1 = \pm(\Omega_c) e^{j\left(\frac{5\pi}{4}\right)} = -3475.6 - j3475.6$$

EXAMPLE 8.21 A digital low-pass filter is required to meet the following specifications.

Passband attenuation $\le$ 1 dB	Passband edge = 4 kHz
Stopband attenuation $\ge$ 40 dB	Stopband edge = 8 kHz
Sampling rate = 24 kHz	

The filter is to be designed by performing the bilinear transformation on an analog system function. Design the Butterworth filter.

Solution: Given $\alpha_1 = 1$ dB, $\therefore\ k_1 = -1$ dB and $A_1 = 10^{k_1/20} = 10^{-1/20} = 0.8912$

$\alpha_2 = 40$ dB, $\therefore\ k_2 = -40$ dB and $A_2 = 10^{k_2/20} = 10^{-40/20} = 0.01$

Since $f_s = 24$ kHz, normalized angular frequencies are:

$$f_1 = 4 \text{ kHz}, \quad \therefore \quad \omega_1 = \frac{2\pi f_1}{f_s} = 2\pi \times \frac{4000}{24000} = 1.047 \text{ rad/s}$$

$$f_2 = 8 \text{ kHz}, \quad \therefore \quad \omega_2 = \frac{2\pi f_2}{f_s} = 2\pi \times \frac{8000}{24000} = 2.094 \text{ rad/s}$$

The Butterworth filter is designed as follows:

Step 1 Type of transformation

Bilinear transformation is already specified.

Step 2 Ratio of analog edge frequencies, Ω_2/Ω_1

$$\Omega_1 = \frac{2}{T}\tan\frac{\omega_1}{2} = 2\times 24000\tan\frac{1.047}{2} = 27706.49\,\text{rad/s}$$

$$\Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} = 2\times 24000\tan\frac{2.094}{2} = 83100.52\,\text{rad/s}$$

$$\therefore \quad \frac{\Omega_2}{\Omega_1} = \frac{83000.52}{27706.49} = 2.9957$$

Step 3 Order of the filter N

$$N \ge \frac{1}{2}\frac{\log\left\{\left[\frac{1}{A_2^2}-1\right]\Big/\left[\frac{1}{A_1^2}-1\right]\right\}}{\log(\Omega_2/\Omega_1)} \qquad \text{or} \qquad N \ge \frac{1}{2}\frac{\log\left(\frac{10^{0.1A_2\,\text{dB}}-1}{10^{0.1A_1\,\text{dB}}-1}\right)}{\log(\Omega_2/\Omega_1)}$$

$$\ge \frac{1}{2}\frac{\log\left\{\left[\frac{1}{(0.01)^2}-1\right]\Big/\left[\frac{1}{(0.8912)^2}-1\right]\right\}}{\log(2.9957)} \qquad \text{or} \qquad N \ge \frac{1}{2}\frac{\log\left(\frac{10^{0.1\times 40}-1}{10^{0.1\times 1}-1}\right)}{\log(2.9957)}$$

$$\ge \frac{1}{2}\frac{\log\{9999/0.2590\}}{\log(2.9957)} \qquad \text{or} \qquad N \ge \frac{1}{2}\frac{\log(9999/0.2589)}{\log(2.9957)}$$

$$\ge \frac{1}{2}\frac{4.586}{0.476} \qquad \text{or} \qquad N \ge \frac{1}{2}\frac{4.586}{0.476}$$

$$\ge 4.8 \approx 5 \qquad \text{or} \qquad N \ge 4.8 \approx 5$$

Step 4 The cutoff frequency Ω_c

$$\Omega_c = \frac{\Omega_1}{\left[\frac{1}{A_1^2}-1\right]^{1/2N}} \qquad \text{or} \qquad \Omega_c = \frac{\Omega_1}{[10^{0.1A_1\,\text{dB}}-1]^{1/2N}}$$

$$= \frac{27706.49}{\left[\frac{1}{(0.8912)^2}-1\right]^{1/2\times 5}} \qquad \text{or} \qquad \Omega_c = \frac{27706.49}{\left[10^{0.1\times 1}-1\right]^{1/2\times 5}}$$

$$= 31{,}715\ \text{rad/s} \qquad \text{or} \qquad \Omega_c = 31{,}715\ \text{rad/s}$$

Step 5 Analog filter transfer function $H_a(s)$

For $N = 5$, $H_a(s) = \left(\dfrac{\Omega_c}{s+\Omega_c}\right)\left(\dfrac{\Omega_c^2}{s^2+b_1\Omega_c s+\Omega_c^2}\right)\left(\dfrac{\Omega_c^2}{s^2+b_2\Omega_c s+\Omega_c^2}\right)$

where $$b_1 = 2\sin\left(\frac{(2\times 1-1)\pi}{2N}\right) = 2\sin\frac{\pi}{10} = 0.618$$

$$b_2 = 2\sin\left(\frac{(2\times 2-1)\pi}{10}\right) = 2\sin\frac{3\pi}{10} = 1.618$$

$$\therefore \quad H_a(s) = \left(\frac{31708}{s+31708}\right)\left(\frac{(31708)^2}{s^2+0.618\times 31708s+(31708)^2}\right)\left(\frac{(31708)^2}{s^2+1.618\times 31708s+(31708)^2}\right)$$

Step 6 Digital filter function $H(z)$

Using the bilinear transformation, we have

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = H_a(s)\Big|_{s=48000\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \left[\left(\frac{31708}{s+31708}\right)\left(\frac{(31708)^2}{s^2+0.618\times 31708s+(31708)^2}\right)\left(\frac{(31708)^2}{s^2+1.618\times 31708s+(31708)^2}\right)\right]\Bigg|_{s=48000\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

EXAMPLE 8.22 Design a digital IIR low-pass filter with passband edge at 1000 Hz and stopband edge at 1500 Hz for a sampling frequency of 5000 Hz. The filter is to have a passband ripple of 0.5 dB and a stopband ripple below 30 dB. Design a Butterworth filter using the bilinear transformation.

Solution: Given f_s = 5000 Hz, the normalized frequencies are given as:

Passband edge f_1 = 1000 Hz, $\therefore \ \omega_1 = 2\pi\frac{f_1}{f_s} = 2\pi\times\frac{1000}{5000} = 0.4\pi$ rad/s

Stopband edge f_2 = 1500 Hz, $\therefore \ \omega_2 = 2\pi\frac{f_2}{f_s} = 2\pi\times\frac{1500}{5000} = 0.6\pi$ rad/s

α_1 = 0.5 dB, $\therefore$ k_1 = –0.5 dB and $A_1 = 10^{k_1/20} = 10^{-0.5/20} = 0.9446$

α_2 = 30 dB, $\therefore$ k_2 = – 30 dB and $A_2 = 10^{k_2/20} = 10^{-30/20} = 0.0316$

The Butterworth filter is designed as follows:

Step 1 Type of transformation.
Bilinear transformation is to be used.

Step 2 Ratio of analog filter edge frequencies, Ω_2/Ω_1

$$\Omega_1 = \frac{2}{T}\tan\frac{\omega_1}{2} = 2\times 5000\tan\frac{0.4\pi}{2} = 7265.425 \text{ rad/s}$$

$$\Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} = 2\times 5000\tan\frac{0.6\pi}{2} = 13763.819 \text{ rad/s}$$

$$\frac{\Omega_2}{\Omega_1} = \frac{13763.819}{7265.425} = 1.8944$$

Step 3 Order of the filter N

$$N \ge \frac{1}{2}\frac{\log\left\{\left[\frac{1}{A_2^2}-1\right]\Big/\left[\frac{1}{A_1^2}-1\right]\right\}}{\log\,(\Omega_2/\Omega_1)}$$

$$\ge \frac{1}{2}\frac{\log\left\{\left[\frac{1}{(0.0316)^2}-1\right]\Big/\left[\frac{1}{(0.9446)^2}-1\right]\right\}}{\log\,(1.844)}$$

$$\ge \frac{1}{2}\frac{\log\{1000.44/0.1207\}}{\log(1.844)}$$

$$\ge 7.35 \approx 8$$

Step 4 The cutoff frequency Ω_c

$$\Omega_c = \frac{\Omega_1}{\left[\frac{1}{A_1^2}-1\right]^{1/2N}} = \frac{7265.425}{\left[\frac{1}{0.9446^2}-1\right]^{1/2\times 8}} = 8292 \text{ rad/s}$$

Step 5 The system function $H_a(s)$

$$H_a(s) = \prod_{k=1}^{N/2}\frac{\Omega_c^2}{s^2 + b_k\Omega_c s + \Omega_c^2}$$

where $b_k = 2\sin\left[\frac{(2k-1)\pi}{2N}\right]$

$$\therefore \qquad b_1 = 2\sin\left[\frac{\pi}{16}\right] = 0.390 \qquad b_2 = 2\sin\left[\frac{3\pi}{16}\right] = 1.111$$

$$b_3 = 2\sin\left[\frac{5\pi}{16}\right] = 1.662 \qquad b_4 = 2\sin\left[\frac{7\pi}{16}\right] = 1.961$$

$$H_a(s) = \left(\frac{(8292)^2}{s^2 + 0.39\times 8292s + (8292)^2}\right)\left(\frac{(8292)^2}{s^2 + 1.111 \times 8292s + (8292)^2}\right)$$
$$\left(\frac{(8292)^2}{s^2 + 1.662 \times 8292s + (8292)^2}\right)\left(\frac{(8292)^2}{s^2 + 1.961\times 8292s + (8292)^2}\right)$$

Step 6 Digital filter function $H(z)$

Using the bilinear transformation, we have

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = H_a(s)\Big|_{s=10000\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$H(z) = \left[\begin{matrix}\left(\dfrac{(8292)^2}{s^2 + 3233.8s + (8292)^2}\right)\left(\dfrac{(8292)^2}{s^2 + 9212.4s + (8292)^2}\right)\\ \left(\dfrac{(8292)^2}{s^2 + 13781.3s + (8292)^2}\right)\left(\dfrac{(8292)^2}{s^2 + 16260.6s + (8292)^2}\right)\end{matrix}\right]\Bigg|_{s=10000\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

EXAMPLE 8.23 Find the filter order for the following specifications:

$$\sqrt{0.5} \le |H(\omega)| \le 1 \qquad 0 \le \omega \le \pi/2$$
$$|H(\omega)| \le 0.2 \qquad 3\pi/4 \le \omega \le \pi$$

with $T = 1$ s. Use the impulse invariant method.

Solution: Given $A_1 = \sqrt{0.5}, \quad \omega_1 = \dfrac{\pi}{2}$

$$A_2 = 0.2, \qquad \omega_2 = \frac{3\pi}{4}$$

$T = 1$ s.

The impulse invariant transformation is to be used.

$$\therefore \qquad \frac{\Omega_2}{\Omega_1} = \frac{\omega_2/T}{\omega_1/T} = \frac{\omega_2}{\omega_1} = \frac{3\pi/4}{\pi/2} = 1.5$$

Order of the low-pass Butterworth filter N

$$N \geq \frac{1}{2}\,\frac{\log\left\{\left[\dfrac{1}{A_2^2} - 1\right]\Big/\left[\dfrac{1}{A_1^2} - 1\right]\right\}}{\log\,(\Omega_2/\Omega_1)}$$

$$\geq \frac{1}{2}\,\frac{\log\left\{\left[\dfrac{1}{(0.2)^2} - 1\right]\Big/\left[\dfrac{1}{(\sqrt{0.5})^2} - 1\right]\right\}}{\log(1.5)}$$

$$\geq \frac{1}{2}\,\frac{\log\,(24)}{\log\,(1.5)} \geq 3.919 \approx 4$$

EXAMPLE 8.24 Determine the order and the poles of a low-pass Butterworth filter that has a –3dB bandwidth of 500 Hz and an attenuation of 40 dB at 1000 Hz.

Solution: Given

Passband edge frequency f_1 = 500 Hz, $\quad\therefore\quad \omega_1 = 2\pi f_1 = 1000\,\pi$

Gain at passband edge k_1 = –3 dB, $\quad\therefore\quad A_1 = 10^{k_1/20} = 10^{-3/20} = 0.707$

Stopband edge frequency f_2 = 1000 Hz, $\quad\therefore\quad \omega_2 = 2\pi f_2 = 2000\,\pi$

Gain at stopband edge k_2 = – 40 dB, $\quad\therefore\quad A_2 = 10^{k_2/20} = 10^{-40/20} = 0.01$

Let the sampling frequency f_s = 2000 Hz.

The normalized frequencies are:

$$\omega_1 = 2\pi\,\frac{f_1}{f_s} = 2\pi\,\frac{500}{2000} = 0.5\pi$$

$$\omega_2 = 2\pi\,\frac{f_2}{f_s} = 2\pi\,\frac{1000}{2000} = \pi$$

For impulse invariant transformation,

$$\frac{\Omega_2}{\Omega_1} = \frac{\omega_2}{\omega_1} = 2$$

Therefore, order of the filter is:

$$N \geq \frac{1}{2}\,\frac{\log\left\{\left[\dfrac{1}{A_2^2} - 1\right]\Big/\left[\dfrac{1}{A_1^2} - 1\right]\right\}}{\log\left(\dfrac{\Omega_2}{\Omega_1}\right)}$$

$$\geq \frac{1}{2}\frac{\log\left\{\left[\dfrac{1}{(0.01)^2}-1\right]\Big/\left[\dfrac{1}{(0.707)^2}-1\right]\right\}}{\log 2}$$

$$\geq \frac{1}{2}\frac{\log\{999/1\}}{\log(2)} \geq 6.64 \approx 7$$

The pole positions are:

$$s_k = \Omega_c\, e^{j\left[\frac{\pi}{2}+(2k+1)\pi/2N\right]}$$

$$= 1000\pi\, e^{j\left[\frac{\pi}{2}+(2k+1)\pi/14\right]}, \quad k = 0, 1, 2, 3, 4, 5, 6$$

where Ω_c is 3 dB cutoff frequency.

EXAMPLE 8.25 Determine the order of a Butterworth low-pass filter satisfying the following specifications:

$$f_p = 0.10 \text{ Hz}, \qquad \alpha_p = 0.5 \text{ dB}$$
$$f_s = 0.15 \text{ Hz}, \qquad \alpha_s = 15 \text{ dB}; f = 1 \text{ Hz}$$

Solution: Given

$f_p = 0.10$ Hz, $\quad \therefore\ \omega_p = \omega_1 = 2\pi f_p = 2\pi(0.1) = 0.2\,\pi$

$f_s = 0.15$ Hz, $\quad \therefore\ \omega_s = \omega_2 = 2\pi f_s = 2\pi(0.15) = 0.30\,\pi$

$\alpha_p = \alpha_1 = 0.5$ dB, $\quad \therefore\ k_1 = -0.5$ dB, so $A_1 = 10^{k_1/20} = 10^{-0.5/20} = 0.944$

$\alpha_s = \alpha_2 = 15$ dB, $\quad \therefore\ k_2 = -15$ dB, so $A_2 = 10^{k_2/20} = 10^{-15/20} = 0.177$

$f = 1$ Hz, $\quad \therefore\ T = \frac{1}{f} = \frac{1}{1} = 1\text{s}.$

1. The type of transformation is not specified. Let us use bilinear transformation.

2. $$\frac{\Omega_2}{\Omega_1} = \frac{\frac{2}{T}\tan\frac{\omega_2}{2}}{\frac{2}{T}\tan\frac{\omega_1}{2}} = \frac{\frac{2}{1}\tan\frac{0.3\pi}{2}}{\frac{2}{1}\tan\frac{0.2\pi}{2}} = \frac{1.019}{0.649} = 1.57$$

3. $$N \geq \frac{1}{2}\frac{\log\left\{\left[\dfrac{1}{A_2^2}-1\right]\Big/\left[\dfrac{1}{A_1^2}-1\right]\right\}}{\log\left(\dfrac{\Omega_2}{\Omega_1}\right)}$$

$$\geq \frac{1}{2}\frac{\log\left\{\left[\frac{1}{0.177^2}-1\right]\Big/\left[\frac{1}{0.944^2}-1\right]\right\}}{\log(1.57)}$$

$$\geq 6.16 \approx 7$$

So the order of the low-pass Butterworth filter is $N = 7$.

8.8 DESIGN OF LOW-PASS CHEBYSHEV FILTER

For designing a Chebyshev IIR digital filter, first an analog filter is designed using the given specifications. Then the analog filter transfer function is transformed to digital filter transfer function by using either impulse invariant transformation or bilinear transformation.

The analog Chebyshev filter is designed by approximating the ideal frequency response using an error function. There are two types of Chebyshev approximations. In type-1 approximation, the error function is selected such that the magnitude response is equiripple in the passband and monotonic in the stopband. In type-2 approximation, the error function is selected such that the magnitude function is monotonic in the passband and equiripple in the stopband. The type-2 magnitude response is also called inverse Chebyshev response. The type-1 design is presented in this book.

The magnitude response of type-1 Chebyshev low-pass filter is given by

$$|H_a(\Omega)|^2 = \frac{1}{1+\varepsilon^2 c_N^2\left(\frac{\Omega}{\Omega_c}\right)}$$

where ε is attenuation constant given by $\varepsilon = \left[\frac{1}{A_1^2}-1\right]^{\frac{1}{2}}$

A_1 is the gain at the passband edge frequency ω_1 and $c_N\left(\frac{\Omega}{\Omega_c}\right)$ is the Chebyshev polynomial of the first kind of degree N given by

$$c_N(x) = \{\cos(N\cos^{-1}x), \quad \text{for } |x| \leq 1$$

$$= \{\cos(N\cosh^{-1}x), \quad \text{for } |x| \geq 1$$

and Ω_c is the 3 dB cutoff frequency.

The frequency response of Chebyshev filter depends on order N. The approximated response approaches the ideal response as the order N increases. The phase response of the Chebyshev filter is more nonlinear than that of the Butterworth filter for a given filter length N. The magnitude response of type-1 Chebyshev filter is shown in Figure 8.9.

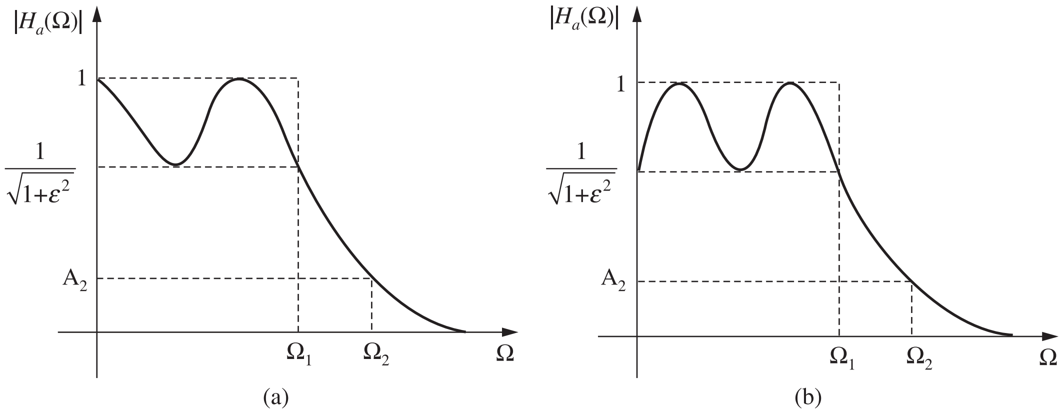


Figure 8.9 Magnitude response of type-1 Chebyshev filter.

The design parameters of the Chebyshev filter are obtained by considering the low-pass filter with the desired specifications as given below.

$$A_1 \le |H(\omega)| \le 1 \qquad 0 \le \omega \le \omega_1$$

$$|H(\omega)| \le A_2 \qquad \omega_2 \le \omega \le \pi$$

The corresponding analog magnitude response is to be obtained in the design process. We have

$$A_1^2 \le \frac{1}{1 + \varepsilon^2 c_N^2(\Omega_1/\Omega_2)} \le 1$$

$$\frac{1}{1 + \varepsilon^2 c_N^2(\Omega_1/\Omega_2)} \le A_2^2$$

Assuming $\Omega_c = \Omega_1$, we will have $c_N(\Omega_1/\Omega_c) = c_N(1) = 1$.
Therefore, from the above inequality involving A_1^2, we get

$$A_1^2 \le \frac{1}{1 + \varepsilon^2}$$

Assuming equality in the above equation, the expression for ε is

$$\varepsilon = \left[\frac{1}{A_1^2} - 1\right]^{\frac{1}{2}}$$

The order of the analog filter, N can be determined from the inequality for A_2^2.

Assuming $\Omega_c = \Omega_1$,

$$c_N(\Omega_2/\Omega_1) \geq \frac{1}{\varepsilon}\left[\frac{1}{A_2^2} - 1\right]^{\frac{1}{2}}$$

Since $\Omega_2 > \Omega_1$,

$$\cosh[N\cosh^{-1}(\Omega_2/\Omega_1)] \geq \frac{1}{\varepsilon}\left[\frac{1}{A_2^2} - 1\right]^{\frac{1}{2}}$$

or

$$N \geq \frac{\cosh^{-1}\left\{\frac{1}{\varepsilon}\left[\frac{1}{A_2^2} - 1\right]^{\frac{1}{2}}\right\}}{\cosh^{-1}(\Omega_2/\Omega_1)}$$

Choose N to be the next nearest integer to the value given above. The values of Ω_2 and Ω_1 are determined from ω_1 and ω_2 using either impulse invariant transformation or bilinear transformation.

The transfer function of Chebyshev filters are usually written in the factored form as given below.

When N is even, $H_a(s) = \prod_{k=1}^{\frac{N}{2}} \frac{B_k\Omega_c^2}{s^2 + b_k\Omega_c s + c_k\Omega_c^2}$

When N is odd, $H_a(s) = \frac{B_0\Omega_c}{s + \Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{B_k\Omega_c^2}{s^2 + b_k\Omega_c s + c_k\Omega_c^2}$

where

$$b_k = 2\,y_N \sin\left(\frac{(2k-1)\pi}{2N}\right)$$

$$c_k = y_N^2 + \cos^2\left(\frac{(2k-1)\pi}{2N}\right)$$

$$c_0 = y_N$$

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\varepsilon^2} + 1\right)^{\frac{1}{2}} + \frac{1}{\varepsilon}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\varepsilon^2} + 1\right)^{\frac{1}{2}} + \frac{1}{\varepsilon}\right]^{\frac{-1}{N}}\right\}$$

For even values of N and unity dc gain filter, the parameter B_k are evaluated using the equation:

$$H_a(s)\Big|_{s=0} = \frac{1}{[1+\varepsilon^2]^{1/2}}$$

For odd values of N and unity dc gain filter, the parameter B_k are evaluated using the equation:

$$H_a(s)\Big|_{s=0} = 1$$

Poles of a normalized Chebyshev filter

The transfer function of the analog system can be obtained from the equation for the magnitude squared response as:

$$H_a(s)H_a(-s) = \frac{1}{1+\varepsilon^2 c_N^2\left(\dfrac{s/j}{\Omega_c}\right)}$$

For the normalized transfer function, let us replace s/Ω_c by s_n.

$$\therefore \qquad H_a(s_n)H_a(-s_n) = \frac{1}{1+\varepsilon^2 c_N^2(-js_n)}$$

The normalized poles in the s-domain can be obtained by equating the denominator of the above equation to zero, i.e., $1+\varepsilon^2 c_N^2(-js_n)$ to zero.

The solution to the above expression gives us the $2N$ poles of the filter given by

$$s_n = -\sin x \sinh y + j\cos x \cosh y = \sigma_n + j\Omega_n$$

where $n = 1, 2, \ldots, (N+1)/2$ for N odd
$= 1, 2, \ldots, N/2$ for N even

and

$$x = \frac{(2n-1)\pi}{2N} \qquad n = 1,2, \ldots, N$$

$$y = \pm\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right) \qquad n = 1, 2, \ldots, N$$

The unnormalized poles, s'_n can be obtained from the normalized poles as shown below.

$$s'_n = s_n\Omega_c$$

The normalized poles lie on an ellipse in s-plane. Since for a stable filter all the poles should lie in the left half of s-plane, only the N poles on the ellipse which are in the left half of s-plane are considered.

For N even, all the poles are complex and exist in conjugate pairs. For N odd, one pole is real and all other poles are complex and occur in conjugate pairs.

Design procedure for low-pass digital Chebyshev IIR filter

The low-pass Chebyshev IIR digital filter is designed following the steps given below.

Step 1 Choose the type of transformation.
(Bilinear or impulse invariant transformation)

Step 2 Calculate the attenuation constant ε.

$$\varepsilon = \left[\frac{1}{A_1^2} - 1\right]^{\frac{1}{2}}$$

Step 3 Calculate the ratio of analog edge frequencies Ω_2/Ω_1.
For bilinear transformation,

$$\frac{\Omega_2}{\Omega_1} = \frac{\dfrac{2}{T}\tan\dfrac{\omega_2}{2}}{\dfrac{2}{T}\tan\dfrac{\omega_1}{2}} = \frac{\tan\dfrac{\omega_2}{2}}{\tan\dfrac{\omega_1}{2}}$$

For impulse invariant transformation,

$$\frac{\Omega_2}{\Omega_1} = \frac{\omega_2/T}{\omega_1/T} = \frac{\omega_2}{\omega_1}$$

Step 4 Decide the order of the filter N such that

$$N \geq \frac{\cosh^{-1}\left\{\dfrac{1}{\varepsilon}\left[\dfrac{1}{A_2^2} - 1\right]\right\}}{\cosh^{-1}\left\{\dfrac{\Omega_2}{\Omega_1}\right\}}$$

Step 5 Calculate the analog cutoff frequency Ω_c.
For bilinear transformation,

$$\Omega_c = \frac{\Omega_1}{\left[\dfrac{1}{A_1^2} - 1\right]^{1/2N}} = \frac{\dfrac{2}{T}\tan\dfrac{\omega_1}{2}}{\left[\dfrac{1}{A_1^2} - 1\right]^{1/2N}}$$

For impulse invariant transformation

$$\Omega_c = \frac{\Omega_1}{\left[\dfrac{1}{A_1^2} - 1\right]^{1/2N}} = \frac{\omega_1/T}{\left[\dfrac{1}{A_1^2} - 1\right]^{1/2N}}$$

Step 6 Determine the analog transfer function $H_a(s)$ of the filter.
When the order N is even, $H_a(s)$ is given by

$$H_a(s) = \prod_{k=1}^{N/2} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + c_k \Omega_c^2}$$

When the order N is odd, $H_a(s)$ is given by

$$H_a(s) = \frac{B_0 \Omega_c}{s + c_0 \Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + c_k \Omega_c^2}$$

where $b_k = 2y_N \sin\left(\dfrac{(2k-1)\pi}{2N}\right)$

$$c_k = y_N^2 + \cos^2 \frac{(2k-1)\pi}{2N}$$

$$c_0 = y_N$$

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\varepsilon^2} + 1\right)^{\frac{1}{2}} + \frac{1}{\varepsilon}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\varepsilon^2} + 1\right)^{\frac{1}{2}} + \frac{1}{\varepsilon}\right]^{\frac{-1}{N}}\right\}$$

For even values of N and unity dc gain filter, find B_k's such that

$$H_a(0) = \frac{1}{(1 + \varepsilon^2)^{1/2}}$$

For odd values of N and unity dc gain filter, find B_k's such that

$$\prod_{k=0}^{\frac{N-1}{2}} \frac{B_k}{c_k} = 1$$

(It is normal practice to take $B_0 = B_1 = B_2 = \ldots = B_k$)

Step 7 Using the chosen transformation, transform $H_a(s)$ to $H(z)$, where $H(z)$ is the transfer function of the digital filter.

[The high-pass, band pass and band stop filters are obtained from low-pass filter design by frequency transformation].

Properties of Chebyshev filters (Type 1)

1. The magnitude response is equiripple in the passband and monotonic in the stopband.

2. The chebyshev type-1 filters are all pole designs.
3. The normalized magnitude function has a value of $1/\sqrt{1+\varepsilon^2}$ at the cutoff frequency Ω_c.
4. The magnitude response approaches the ideal response as the value of N increases.

EXAMPLE 8.26 Design a Chebyshev IIR digital low-pass filter to satisfy the constraints.

$$0.707 \le |H(\omega)| \le 1, \qquad 0 \le \omega \le 0.2\pi$$

$$|H(\omega)| \le 0.1, \qquad 0.5\pi \le \omega \le \pi$$

using bilinear transformation and assuming $T = 1$ s.

Solution: Given

$$A_1 = 0.707, \qquad \omega_1 = 0.2\pi$$

$$A_2 = 0.1, \qquad \omega_2 = 0.5\pi$$

$T = 1$ s and bilinear transformation is to be used. The low-pass Chebyshev IIR digital filter is designed as follows:

Step 1 Type of transformation
Here bilinear transformation is to be used.

Step 2 Attenuation constant ε

$$\varepsilon = \left[\frac{1}{A_1^2} - 1\right]^{\frac{1}{2}} = \left[\frac{1}{0.707^2} - 1\right]^{\frac{1}{2}} = 1$$

Step 3 Ratio of analog edge frequencies, Ω_2/Ω_1.
Since bilinear transformation is to be used,

$$\frac{\Omega_2}{\Omega_1} = \frac{\frac{2}{T}\tan\frac{\omega_2}{2}}{\frac{2}{T}\tan\frac{\omega_1}{2}} = \frac{\tan\frac{0.5\pi}{2}}{\tan\frac{0.2\pi}{2}} = \frac{2}{0.6498} = 3.0779$$

Step 4 Order of the filter N

$$N \ge \frac{\cosh^{-1}\left\{\frac{1}{\varepsilon}\left[\frac{1}{A_2^2} - 1\right]^{\frac{1}{2}}\right\}}{\cosh^{-1}\left\{\frac{\Omega_2}{\Omega_1}\right\}} \ge \frac{\cosh^{-1}\left\{\frac{1}{1}\left[\frac{1}{0.1^2} - 1\right]^{0.5}\right\}}{\cosh^{-1}\{3.0779\}} \ge 1.669 \approx 2.$$

Step 5 Analog cutoff frequency Ω_c

$$\Omega_c = \frac{\Omega_1}{\left[\frac{1}{A_1^2} - 1\right]^{1/2N}} = \frac{\frac{2}{T}\tan\frac{\omega_1}{2}}{\left[\frac{1}{A_1^2} - 1\right]^{1/2N}} = \frac{0.6498}{\left[\frac{1}{0.7077} - 1\right]^{\frac{1}{4}}} = 0.6498$$

Step 6 Analog filter transfer function $H_a(s)$

$$H_a(s) = \prod_{k=1}^{N/2} \frac{B_k\Omega_c^2}{s^2 + b_k\Omega_c s + c_k\Omega_c^2} = \frac{B_1\Omega_c^2}{s^2 + b_1\Omega_c s + c_1\Omega_c^2}$$

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\varepsilon^2} + 1\right)^{\frac{1}{2}} + \frac{1}{\varepsilon}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\varepsilon^2} + 1\right)^{\frac{1}{2}} + \frac{1}{\varepsilon}\right]^{\frac{-1}{N}}\right\}$$

$$= \frac{1}{2}\left\{\left[\left(\frac{1}{1^2} + 1\right)^{\frac{1}{2}} + \frac{1}{1}\right]^{\frac{1}{2}} - \left[\left(\frac{1}{1^2} + 1\right)^{\frac{1}{2}} + \frac{1}{1}\right]^{\frac{-1}{2}}\right\}$$

$$= \frac{1}{2}\left\{[2.414]^{\frac{1}{2}} - [2.414]^{\frac{-1}{2}}\right\} = 0.455$$

$$b_1 = 2y_N \sin\left[\frac{(2k-1)\pi}{2N}\right] = 2 \times 0.455 \ \sin\left[\frac{(2\times 1 - 1)\pi}{2\times 2}\right] = 0.6435$$

$$c_1 = y_N^2 + \cos^2\left[\frac{(2k-1)\pi}{2N}\right] = (0.455)^2 + \cos^2\left[\frac{(2\times 1-1)\pi}{2\times 2}\right] = 0.707$$

For N even,

$$\prod_{k=1}^{\frac{N}{2}} \frac{B_k}{c_k} = \frac{A}{(1+\varepsilon^2)^{0.5}} = 0.707$$

That is $B_1 = c_1 \times 0.707 = 0.707 \times 0.707 = 0.5$.
Therefore, the system function is:

$$H_a(s) = \frac{0.5(0.6498)^2}{s^2 + (0.6435)(0.6498)s + (0.707)(0.6498)^2}$$

On simplifying, we get

$$H_a(s) = \frac{0.2111}{s^2 + 0.4181s + 0.2985}$$

Step 7 Digital filter transfer function $H(z)$

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{0.2111}{s^2 + 0.4181s + 0.2985}\Bigg|_{s=2\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{0.2111}{\left[2\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)\right]^2 + 0.4181\left[2\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)\right] + 0.2985}$$

$$= \frac{0.2111(1+z^{-1})^2}{5.1347 - 7.403z^{-1} + 3.463z^{-2}}$$

$$= \frac{0.0411(1+z^{-1})^2}{1 - 1.441z^{-1} + 0.6744z^{-2}}$$

EXAMPLE 8.27 Determine the system function $H(z)$ of the lowest order Chebyshev IIR digital filter with the following specifications:

3 dB ripple in passband $0 \le \omega \le 0.2\,\pi$

25 dB attenuation in stopband $0.45\pi \le \omega \le \pi$

Solution: Given

$\alpha_1 = 3$ dB, $\therefore$ $k_1 = -3\text{dB}$ and hence $A_1 = 10^{k_1/20} = 10^{-3/20} = 0.707$

$\alpha_2 = 25$ dB, $\therefore$ $k_2 = -25\text{dB}$ and hence $A_2 = 10^{k_2/20} = 10^{-25/20} = 0.0562$

$\omega_1 = 0.2\pi$ and $\omega_2 = 0.45\pi$

Let $T = 1$ and bilinear transformation is used

Attenuation constant $\varepsilon = \left[\dfrac{1}{A_1^2} - 1\right]^{\frac{1}{2}} = \left[\dfrac{1}{0.707^2} - 1\right] = 1$

Ratio of analog frequencies, $\dfrac{\Omega_2}{\Omega_1} = \dfrac{\dfrac{2}{T}\tan\dfrac{\omega_2}{2}}{\dfrac{2}{T}\tan\dfrac{\omega_1}{2}} = \dfrac{\tan\dfrac{0.45\pi}{2}}{\tan\dfrac{0.2\pi}{2}} = 2.628$

$$\text{Order of filter } N \geq \frac{\cosh^{-1}\left\{\dfrac{1}{\varepsilon}\left[\dfrac{1}{A_2^2}-1\right]^{\frac{1}{2}}\right\}}{\cosh^{-1}\left\{\dfrac{\Omega_2}{\Omega_1}\right\}}$$

$$\geq \frac{\cosh^{-1}\left\{\dfrac{1}{1}\left[\dfrac{1}{0.0562^2}-1\right]^{\frac{1}{2}}\right\}}{\cosh^{-1}\{2.628\}}$$

$$\geq \frac{3.569}{1.621} \geq 2.20 \approx 3$$

$$\text{Analog cutoff frequency } \Omega_c = \frac{\Omega_1}{\left[\dfrac{1}{A_1^2}-1\right]^{1/2N}} = \frac{\dfrac{2}{T}\tan\dfrac{\omega_1}{2}}{\left[\dfrac{1}{0.707^2}-1\right]^{1/6}} = 1.708$$

Analog filter transfer function for $N = 3$.

$$H_a(s) = \frac{B_0\Omega_c}{s + c_0\Omega_c}\;\frac{B_1\Omega_c^2}{s^2 + b_1\Omega_c s + c_1\Omega_c^2}$$

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\varepsilon^2}+1\right)^{\frac{1}{2}} + \frac{1}{\varepsilon}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\varepsilon^2}+1\right)^{\frac{1}{2}} + \frac{1}{\varepsilon}\right]^{\frac{-1}{N}}\right\}$$

$$\therefore \quad y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{1^2}+1\right)^{\frac{1}{2}} + \frac{1}{1}\right]^{\frac{1}{3}} - \left[\left(\frac{1}{1^2}+1\right)^{\frac{1}{2}} + \frac{1}{1}\right]^{\frac{-1}{3}}\right\} = 0.5959$$

$$c_0 = y_N = 0.5959$$

$$b_1 = 2y_N \sin\left[\frac{(2\times 1 - 1)\,\pi}{2N}\right] = 2\times 0.5959 \sin\frac{\pi}{6} = 0.5959$$

$$c_1 = y_N^2 + \cos^2\frac{(2\times 1 - 1)\pi}{2N} = 0.5959^2 + \cos^2\frac{\pi}{6} = 1.105$$

For N odd $\prod_{k=0}^{(N-1)/2} \frac{B_k}{c_k} = 1$

$\therefore \quad B_0 = c_0 = 0.5959, \quad B_1 = c_1 = 1.105$

$$H_a(s) = \left(\frac{0.5959 \times 1.708}{s + 0.5959 \times 1.708}\right)\left(\frac{1.105(1.708)^2}{s^2 + 0.5959 \times 1.708 s + 1.105(1.708)^2}\right)$$

$$= \left(\frac{1.01}{s + 1.01}\right)\left(\frac{3.223}{s^2 + 1.01 s + 3.223}\right)$$

Using bilinear transformation, $H(z)$ is given by

$$H(z) = H_a(s)\bigg|_{s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \left(\frac{1.01}{s+1.01}\right)\left(\frac{3.223}{s^2 + 0.1 s + 3.223}\right)\bigg|_{s = 2\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{3.25}{\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 1.01\right]\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.1 \times 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 3.223\right]}$$

$$= \frac{(3.25)\,(1 + z^{-1})^3}{7.423 - 1.554 z^{-1} + 7.023 z^{-2}}$$

EXAMPLE 8.28 The specification of the desired low-pass filter is:

$$0.9 \le |H(\omega)| \le 1.0; \qquad 0 \le \omega \le 0.3\pi$$

$$|H(\omega)| \le 0.15; \qquad 0.5\pi \le \omega \le \pi$$

Design a Chebyshev digital filter using the bilinear transformation.

Solution: Given

$$A_1 = 0.9, \quad \omega_1 = 0.3\pi$$
$$A_2 = 0.15, \quad \omega_2 = 0.5\pi$$

The Chebyshev filter is designed as per the following steps:

Step 1 The bilinear transformation is used.

Step 2 Attenuation constant ε

$$\varepsilon = \left[\frac{1}{A_1^2} - 1\right]^{1/2} = \left[\frac{1}{(0.9)^2} - 1\right]^{1/2} = 0.484$$

Step 3 Ratio of analog edge frequencies Ω_2/Ω_1

$$\frac{\Omega_2}{\Omega_1} = \frac{\frac{2}{T}\tan\frac{\omega_2}{2}}{\frac{2}{T}\tan\frac{\omega_1}{2}} = \frac{\tan 0.25\pi}{\tan 0.15\pi} = 1.962$$

Step 4 Order of the filter N

$$N \geq \frac{\cosh^{-1}\left[\frac{1}{\varepsilon}\left(\frac{1}{A_2^2}-1\right)^{\frac{1}{2}}\right]}{\cosh^{-1}\left(\frac{\Omega_2}{\Omega_1}\right)} \geq \frac{\cosh^{-1}\left[\frac{1}{0.484}\left(\frac{1}{0.15^2}-1\right)^{\frac{1}{2}}\right]}{\cosh^{-1}1.962}$$

$$\geq \frac{\cosh^{-1}13.618}{\cosh^{-1}1.962} \geq 2.55 = 3$$

So order of the filter is $N = 3$. Let $T = 1$ s.

Step 5 Analog cutoff frequency Ω_c

$$\Omega_c = \frac{\Omega_1}{\left[\frac{1}{A_1^2}-1\right]^{\frac{1}{2N}}} = \frac{\frac{2}{T}\tan\frac{\omega_1}{2}}{\left[\frac{1}{A_1^2}-1\right]^{\frac{1}{2N}}} = \frac{1.019}{\left[\frac{1}{0.92}-1\right]^{1/6}} = 1.13 \text{ rad/s}$$

Step 6 Analog transfer function $H_a(s)$

For $N = 3$, $$H_a(s) = \frac{B_0\Omega_c}{s+c_0\Omega_c}\;\frac{B_1\Omega_c^2}{s^2+b_1\Omega_c s+c_1\Omega_c^2}$$

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\varepsilon^2}+1\right)^{\frac{1}{2}}+\frac{1}{\varepsilon}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\varepsilon^2}+1\right)^{\frac{1}{2}}+\frac{1}{\varepsilon}\right]^{\frac{-1}{N}}\right\}$$

$$= \frac{1}{2}\left\{\left[\left(\frac{1}{(0.484)^2}+1\right)^{\frac{1}{2}}+\frac{1}{0.484}\right]^{\frac{1}{3}} - \left[\left(\frac{1}{(0.484)^2}+1\right)^{\frac{1}{2}}+\frac{1}{0.484}\right]^{\frac{-1}{3}}\right\}$$

$$= \frac{1}{2}\{1.634 - 0.612\} = 0.511$$

$$\therefore \qquad c_0 = y_N = 0.511$$

$$c_k = y_N^2 + \cos^2 \frac{(2k-1)\pi}{2N}$$

When $k = 1$, $c_1 = y_N^2 + \cos^2\left(\frac{\pi}{6}\right) = (0.511)^2 + 0.75 = 1.011$

$$b_k = 2y_N \sin\frac{(2k-1)\pi}{2N}$$

When $k = 1$, $b_1 = y_N + \sin\left(\frac{\pi}{6}\right) = 2 \times 0.511\left(\frac{1}{2}\right) = 0.511$

$$\therefore \quad H_a(s) = \left(\frac{B_0(1.13)}{s + 0.511 \times 1.13}\right)\left(\frac{B_1(1.13)^2}{s^2 + 0.511 \times 1.13s + 1.011(1.13)^2}\right)$$

When $s = 0$, $H_a(s) = H_a(0) = \dfrac{B_0B_1(1.442)}{(0.511)(1.13)(1.011)(1.13)^2} = 1.935\ B_0B_1$

$$\therefore \quad H_a(s) = \left(\frac{B_0(1.13)}{s + 0.511 \times 1.13}\right)\left(\frac{B_1(1.13)^2}{s^2 + 0.511 \times 1.13s + 1.011(1.13)^2}\right)$$

Let $H_a(0) = 1$, $\qquad \therefore\ 1.935\ B_0B_1 = 1$

Let $B_0 = B_1$, $\qquad B_0^2 = \dfrac{1}{1.935} = 0.516$ or $B_0 = 0.718$

$$\therefore \qquad B_0 = B_1 = 0.86$$

$$H_a(s) = \frac{0.516(1.442)}{(s + 0.577)(s^2 + 0.577s + 1.29)}$$

$$= \frac{0.744}{(s + 0.577)(s^2 + 0.577s + 1.29)}$$

Step 7 Digital transfer function

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = H_a(s)\Big|_{s=2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{0.744}{(s + 0.577)(s^2 + 0.577s + 1.29)}\Bigg|_{s=2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{0.744}{\left(2\frac{1-z^{-1}}{1+z^{-1}}+0.577\right)\left(\left(2\frac{1-z^{-1}}{1+z^{-1}}\right)^2+0.577\times 2\frac{1-z^{-1}}{1+z^{-1}}+1.29\right)}$$

$$= \frac{0.744(1+z^{-1})^3}{(2.577-1.423z^{-1})(6.83-5.42z^{-1}+3.75)}$$

EXAMPLE 8.29 Determine the system function of the lowest order Chebyshev digital filter that meets the following specifications.

2 dB ripple in the passband $0 \le \omega \le 0.25\pi$

Atleast 50 dB attenuation in stopband $0.4\pi \le \omega \le \pi$

Solution: Given

Ripple in passband = 2 dB, i.e. $k_1 = -2$ dB $\quad \therefore\ A_1 = 10^{k_1/20} = 10^{-2/20} = 0.794$

Attenuation in stopband = 50 dB, i.e. $k_2 = -50$ dB $\quad \therefore\ A_2 = 10^{k_2/20} = 10^{-50/20} = 0.0031$

$$\therefore \qquad A_1 = 0.794, \quad \omega_1 = 0.25\pi$$
$$A_2 = 0.003, \quad \omega_2 = 0.4\pi$$

The Chebyshev filter is designed as per the following steps:

Step 1 Type of transformation
Let us choose bilinear transformation.

Step 2 Attenuation constant ε

$$\varepsilon = \left[\frac{1}{A_1^2}-1\right]^{1/2} = \left[\frac{1}{0.794^2}-1\right]^{1/2} = 0.765$$

Step 3 Ratio of analog edge frequencies, Ω_2/Ω_1

$$\frac{\Omega_2}{\Omega_1} = \frac{\frac{2}{T}\tan\frac{\omega_2}{2}}{\frac{2}{T}\tan\frac{\omega_1}{2}} = \frac{\tan 0.4\pi/2}{\tan 0.25\pi/2} = \frac{1.453}{0.828} = 1.754$$

Step 4 Order of the filter N

$$N \ge \frac{\cosh^{-1}\left[\frac{1}{\varepsilon}\left(\frac{1}{A_2^2}-1\right)^{\frac{1}{2}}\right]}{\cosh^{-1}\left(\frac{\Omega_2}{\Omega_1}\right)}$$

$$\geq \frac{\cosh^{-1}\left[\frac{1}{0.765}\left(\frac{1}{(0.0031)^2}-1\right)^{\frac{1}{2}}\right]}{\cosh^{-1} 1.754}$$

$$\geq \frac{6.718}{1.161} \geq 5.786 \approx 6$$

$$\therefore \qquad N = 6$$

Step 5 Analog cutoff frequency Ω_c

$$\Omega_c = \frac{\Omega_1}{\left[\frac{1}{A_1^2}-1\right]^{\frac{1}{2N}}} = \frac{\frac{2}{T}\tan\frac{\omega_1}{2}}{\left[\frac{1}{A_1^2}-1\right]^{\frac{1}{2N}}} = \frac{0.828}{\left[\frac{1}{0.794^2}-1\right]^{1/12}} = 0.866 \text{ rad/s}$$

Step 6 Analog transfer function $H_a(s)$

For $N = 6$, $H_a(s) = \left(\frac{B_1\Omega_c^2}{s^2+b_1\Omega_c s+c_1\Omega_c^2}\right)\left(\frac{B_2\Omega_c^2}{s^2+b_2\Omega_c s+c_2\Omega_c^2}\right)\left(\frac{B_3\Omega_c^2}{s^2+b_3\Omega_c s+c_3\Omega_c^2}\right)$

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\varepsilon^2}+1\right)^{\frac{1}{2}}+\frac{1}{\varepsilon}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\varepsilon^2}+1\right)^{\frac{1}{2}}+\frac{1}{\varepsilon}\right]^{\frac{-1}{N}}\right\}$$

$$= \frac{1}{2}\left\{\left[\left(\frac{1}{(0.765)^2}+1\right)^{\frac{1}{2}}+\frac{1}{0.765}\right]^{\frac{1}{6}} - \left[\left(\frac{1}{(0.756)^2}+1\right)^{\frac{1}{2}}+\frac{1}{0.765}\right]^{\frac{-1}{6}}\right\}$$

$$= \frac{1}{2}\{1.197-0.83\} = 0.183$$

$$\therefore \qquad c_0 = y_N = 0.183$$

$$c_k = y_N^2 + \cos^2\frac{(2k-1)\pi}{2N}$$

$$c_1 = y_N^2 + \cos^2\frac{(2\times 1-1)\pi}{2\times 6} = (0.183)^2 + \cos^2\left(\frac{\pi}{12}\right) = 0.9664$$

$$b_1 = 2y_N \sin\frac{(2\times1-1)\pi}{2\times6} = 2\times0.183\sin\left(\frac{\pi}{12}\right) = 0.094$$

$$c_2 = y_N^2 + \cos^2\frac{(2\times2-1)\pi}{2\times6} = (0.183)^2 + \cos^2\left(\frac{3\pi}{12}\right) = 0.5334$$

$$b_2 = 2y_N \sin\frac{(2\times2-1)\pi}{2\times6} = 2\times0.183\sin\left(\frac{3\pi}{12}\right) = 0.258$$

$$c_3 = y_N^2 + \cos^2\frac{(2\times3-1)\pi}{2\times6} = 0.1$$

$$b_3 = 2y_N \sin\frac{(2\times3-1)\pi}{2\times6} = 0.353$$

Let $B_1 = B_2 = B_3$ and let $H_a(0) = 1$.

$$\therefore \quad \frac{B_1B_2B_3\Omega_c^6}{c_1c_2c_3\Omega_c^6} = 1$$

$$\therefore \quad B_1 = B_2 = B_3 = (c_1c_2c_3)^{\frac{1}{3}} = (0.964\times0.533\times0.1)^{\frac{1}{3}} = 0.371$$

$$\therefore \quad H_a(s) = \left(\frac{0.371\times(0.866)^2}{s^2+0.094\times0.866s+0.966\times(0.866)^2}\right)\left(\frac{0.371\times(0.866)^2}{s^2+0.258\times0.866s+0.533\times(0.866)^2}\right)\left(\frac{0.371\times(0.866)^2}{s^2+0.353\times0.866s+0.1\times(0.866)^2}\right)$$

$$= \left(\frac{0.278}{s^2+0.018s+0.724}\right)\left(\frac{0.278}{s^2+0.223s+0.399}\right)\left(\frac{0.278}{s^2+0.305s+0.074}\right)$$

Step 7 Digital filter transfer function $H(z)$ taking $T = 1$s.

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = H_a(s)\Big|_{s=2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$H(z) = \left[\frac{0.278}{\left\{\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.081 \times 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.724\right)\right\}}\right]$$

$$\left[\frac{0.278}{\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]^2 + 0.223\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right] + 0.399}\right]$$

$$\left[\frac{0.278}{\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]^2 + 0.305 \times 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.074}\right]$$

$$= \left[\frac{0.278(1+z^{-1})^2}{4.886 - 6.552z^{-1} + 4.562z^{-2}}\right]\left[\frac{0.278(1+z^{-1})^2}{4.845 - 7.202z^{-1} + 3.953z^{-2}}\right]$$

$$\left[\frac{0.278(1+z^{-1})^2}{4.684 - 7.852z^{-1} + 3.464z^{-2}}\right]$$

EXAMPLE 8.30 Find the Chebyshev filter order for the following specifications:

$$\sqrt{0.6} \le |H(\omega)| \le 1; \qquad 0 \le \omega \le \frac{\pi}{2}$$

$$|H(\omega)| \le 0.2; \qquad \frac{3\pi}{2} \le \omega \le \pi$$

with $T = 1$ s. Use the impulse invariant transformation.

Solution: Given

$$A_1 = \sqrt{0.6} = 0.774, \qquad \omega_1 = \frac{\pi}{2}$$

$$A_2 = 0.25, \qquad \omega_2 = \frac{3\pi}{2}$$

$T = 1$s and impulse invariant transformation is to be used.

The order of the filter is found as follows:

$$\text{Attenuation constant } \varepsilon = \left[\frac{1}{A_1^2} - 1\right]^{\frac{1}{2}} = \left[\frac{1}{(0.774)^2} - 1\right]^{\frac{1}{2}} = 0.818$$

$$\text{Analog passband edge frequency } \Omega_1 = \frac{\omega_1}{T} = \frac{\pi}{2}$$

$$\text{Analog stopband edge frequency } \Omega_2 = \frac{\omega_2}{T} = \frac{3\pi}{2}$$

$$\text{Ratio of edge frequencies } \frac{\Omega_2}{\Omega_1} = \frac{3\pi/2}{\pi/2} = 3$$

$$\text{Order of the filter } N \geq \frac{\cosh^{-1}\left[\frac{1}{\varepsilon}\left(\frac{1}{A_2^2} - 1\right)^{\frac{1}{2}}\right]}{\cosh^{-1}\left(\frac{\Omega_2}{\Omega_1}\right)}$$

$$\geq \frac{\cosh^{-1}\left[\frac{1}{0.818}\left(\frac{1}{0.25^2} - 1\right)^{\frac{1}{2}}\right]}{\cosh^{-1}(3)}$$

$$\geq 1.268 \approx 2$$

So the order of the filter is $N = 2$.

EXAMPLE 8.31 Find the filter order for the following specifications:

$$\sqrt{0.5} \leq |H(\omega)| \leq 1; \qquad 0 \leq \omega \leq \frac{\pi}{2}$$

$$|H(\omega)| \leq 0.2; \qquad \frac{3\pi}{4} \leq \omega \leq \pi$$

with $T = 1$ s. Use the impulse invariant method.

Solution: Given

$$A_1 = \sqrt{0.5} = 0.707, \qquad \omega_1 = \frac{\pi}{2}$$

$$A_2 = 0.2, \qquad \omega_2 = \frac{3\pi}{4}$$

$T = 1$s and impulse invariant transformation is to be used.

Since the type of filter is not specified, let us find the order of Chebyshev type-1 filter.

$$\text{Attenuation constant } \varepsilon = \left[\frac{1}{A_1^2} - 1\right]^{\frac{1}{2}} = \left[\frac{1}{(0.707)^2} - 1\right]^{\frac{1}{2}} = 1$$

$$\text{Ratio of analog edge frequencies } \frac{\Omega_2}{\Omega_1} = \frac{\omega_2/T}{\omega_1/T} = \frac{3\pi/4}{\pi/2} = 1.5$$

$$\text{Order of the filter } \quad N \geq \frac{\cosh^{-1}\left[\frac{1}{\varepsilon}\left(\frac{1}{A_2^2} - 1\right)^{\frac{1}{2}}\right]}{\cosh^{-1}\left(\frac{\Omega_2}{\Omega_1}\right)} \geq \frac{\cosh^{-1}\left[\frac{1}{1}\left(\frac{1}{0.2^2} - 1\right)^{\frac{1}{2}}\right]}{\cosh^{-1}(1.5)}$$

$$\geq \frac{2.271}{0.962} \geq 2.36 \approx 3$$

The order of the filter $N = 3$.

EXAMPLE 8.32 Determine the lowest order of Chebyshev filter that meets the following specifications:

(i) 1 dB ripple in the passband $0 \leq |\omega| \leq 0.3\pi$

(ii) Atleast 60 dB attenuation in the stopband $0.35\pi \leq |\omega| \leq \pi$

Use the bilinear transformation.

Solution: Given $\omega_1 = 0.3\pi$, $\omega_2 = 0.35\,\pi$

1 dB ripple, so $\alpha_1 = 1$ dB or $k_1 = -1$ dB $\quad \therefore \; A_1 = 10^{k_1/20} = 10^{-1/20} = 0.891$

60 dB attenuation, so $\alpha_2 = 60$ dB or $k_2 = -60$ dB $\quad \therefore \; A_2 = 10^{k_2/20} = 10^{-60/20} = 0.001$

Step 1 Bilinear transformation is to be used.

Step 2 Attenuation constant $\varepsilon = \left[\frac{1}{A_1^2} - 1\right]^{\frac{1}{2}} = \left[\frac{1}{(0.891)^2} - 1\right]^{\frac{1}{2}} = 0.509$

Step 3 Ratio of analog edge frequencies $\dfrac{\Omega_2}{\Omega_1} = \dfrac{\frac{2}{T}\tan\frac{\omega_2}{2}}{\frac{2}{T}\tan\frac{\omega_1}{2}} = \dfrac{\tan\frac{0.35\pi}{2}}{\tan\frac{0.3\pi}{2}} = 1.2$

Step 4 Order of the filter $N \geq \dfrac{\cosh^{-1}\left[\dfrac{1}{\varepsilon}\left(\dfrac{1}{A_2^2}-1\right)^{\frac{1}{2}}\right]}{\cosh^{-1}\left(\dfrac{\Omega_2}{\Omega_1}\right)} \geq \dfrac{\cosh^{-1}\left[\dfrac{1}{0.509}\left(\dfrac{1}{0.001^2}-1\right)^{\frac{1}{2}}\right]}{\cosh^{-1}(1.2)}$

$$\geq 13.338 \approx 14$$

So the lowest order of the filter is $N = 14$.

EXAMPLE 8.33 Determine the lowest order of Chebyshev filter for the following specifications.

(i) Maximum passband ripple is 1 dB for $\Omega \leq 4$ rad/s

(ii) Stopband attenuation is 40 dB for $\Omega \geq 4$ rad/s

Solution: Using the impulse invariant transformation,

$$\frac{\Omega_2}{\Omega_1} = \frac{\omega_2/T}{\omega_1/T} = \frac{\omega_2}{\omega_1} = \frac{4}{4} = 1$$

$\delta_p = 1$ dB, $\quad \therefore\ k_1 = -1\,\text{dB and } A_1 = 10^{k_1/20} = 10^{-1/20} = 0.891$

$\alpha_2 = 40$ dB, $\quad \therefore\ k_2 = -40\,\text{dB and } A_2 = 10^{k_2/20} = 10^{-40/20} = 0.01$

Attenuation constant $\varepsilon = \left[\dfrac{1}{A_1^2} - 1\right]^{\frac{1}{2}} = \left[\dfrac{1}{(0.891)^2} - 1\right]^{\frac{1}{2}} = 0.509$

Order of the filter $N \geq \dfrac{\cosh^{-1}\left[\dfrac{1}{\varepsilon}\left(\dfrac{1}{A_2^2}-1\right)^{\frac{1}{2}}\right]}{\cosh^{-1}\left(\dfrac{\Omega_2}{\Omega_1}\right)} \geq \dfrac{\cosh^{-1}\left[\dfrac{1}{0.509}\left(\dfrac{1}{0.01^2}-1\right)^{\frac{1}{2}}\right]}{\cosh^{-1}(1)}$

$$\geq \frac{5.97}{0} = \infty$$

So the order of the filter required is $N = \infty$.

8.9 INVERSE CHEBYSHEV FILTERS

Inverse Chebyshev filters are also called type-2 Chebyshev filters. A low-pass inverse Chebyshev filter has a magnitude response given by

$$|H(\Omega)| = \frac{\varepsilon c_N(\Omega_2/\Omega)}{[1+\varepsilon^2 c_N^2(\Omega_2/\Omega)]^{\frac{1}{2}}}$$

where ε is a constant and Ω_c is the 3 dB cutoff frequency. The Chebyshev polynomial $c_N(x)$ is given by

$$c_N(x) = \cos(N\cos^{-1} x), \qquad \text{for } |x| \le 1$$

$$= \cosh(N\cosh^{-1} x), \text{ for } |x| > 1$$

The magnitude response of the inverse Chebyshev filter is shown in Figure 8.10. The magnitude response has maximally flat passband and equiripple stopband, just the opposite of the Chebyshev filters response. That is why type-2 Chebyshev filters are called the inverse Chebyshev filters.

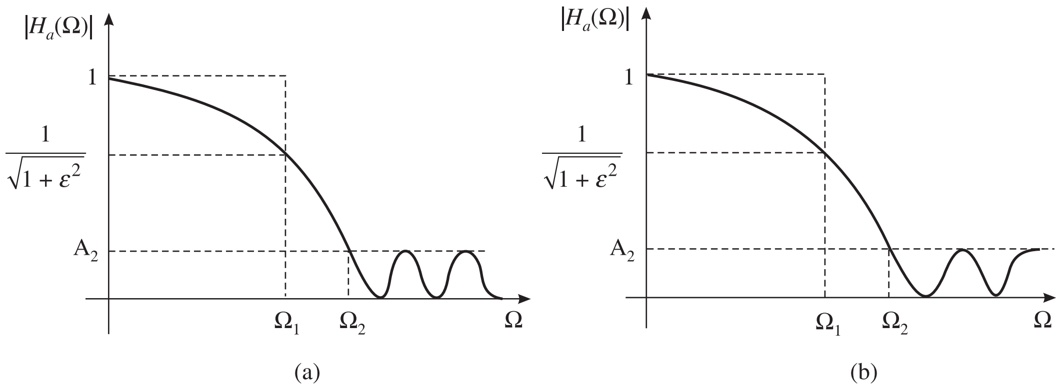


Figure 8.10 Magnitude response of the low-pass inverse Chebyshev filter.

The parameters of the inverse Chebyshev filter are obtained by considering the low-pass filter with the desired specifications:

$$0.707 \le |H(\Omega)| \le 1; \qquad 0 \le \Omega \le \Omega_c$$

$$|H(\Omega)| \le A_2; \qquad \Omega \le \Omega_2$$

The attenuation constant ε is given by

$$\varepsilon = \frac{A_2}{(1-A_2^2)^{\frac{1}{2}}}$$

The order of the filter N is given as:

$$N \ge \frac{\cosh^{-1}\left(\dfrac{1}{\varepsilon}\right)}{\cosh^{-1}(\Omega_2/\Omega_c)} = \frac{\cosh^{-1}\left(\dfrac{1}{A_2^2} - 1\right)^{\frac{1}{2}}}{\cosh^{-1}(\Omega_2/\Omega_c)}$$

The value of N is chosen to be the nearest integer greater than the value given above.

8.10 ELLIPTIC FILTERS

The elliptic filter is sometimes called the Cauer filter. This filter has equiripple passband and stopband. Among the filters discussed so far, for a given filter order, pass band and stop band deviations, elliptic filters have the minimum transition bandwidth. The magnitude response of an elliptic filter is given by

$$\left|H(\Omega)\right|^2 = \frac{1}{1+\varepsilon^2 U_N(\Omega/\Omega_c)}$$

where $U_N(x)$ is the Jacobian elliptic function of order N and ε is a constant related to the passband ripple.

8.11 FREQUENCY TRANSFORMATION

Basically there are four types of frequency selective filters, viz. low-pass, high-pass, band pass and band stop. In Figure 8.11, the frequency response of the ideal case is shown in solid lines and practical case in dotted lines.

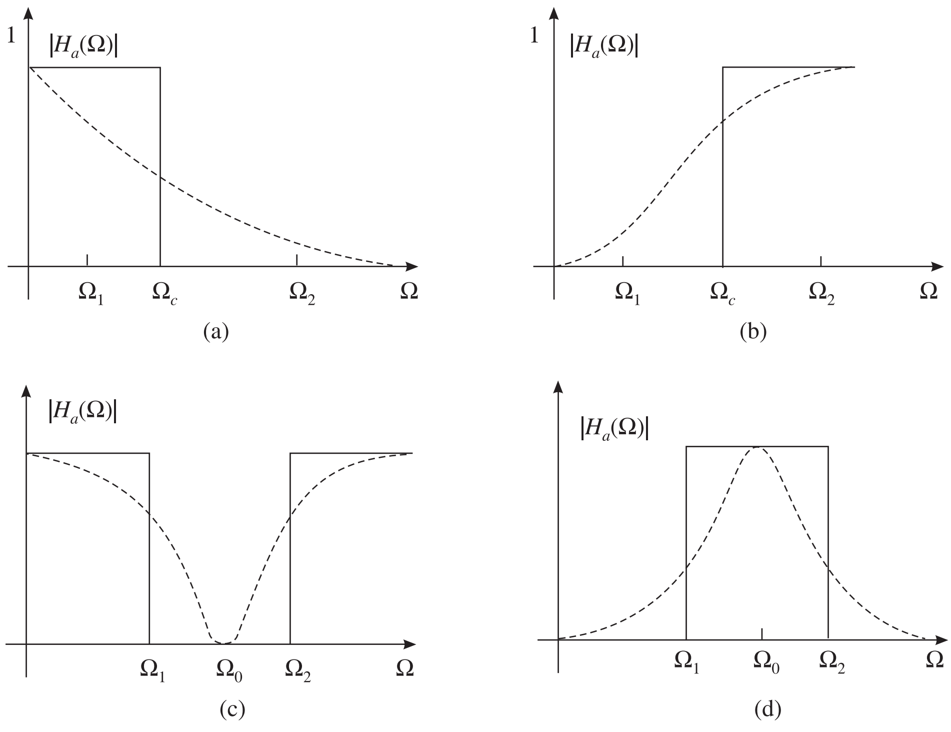


Figure 8.11 Frequency response of (a) Low-pass filter, (b) High-pass filter, (c) Band pass filter and (d) Band stop filter.

In the design techniques discussed so far, we have considered only low-pass filters. This low-pass filter can be considered as a prototype filter and its system function $H_p(s)$ can be determined. The high-pass or band pass or band stop filters are designed by designing a low-pass filter and then transforming that low-pass transfer function into the required filter function by frequency transformation. Frequency transformation can be accomplished in two ways.

(1) Analog frequency transformation
(2) Digital frequency transformation

8.11.1 Analog Frequency Transformation

In the analog frequency transformation, the analog system function $H_p(s)$ of the prototype filter is converted into another analog system function $H(s)$ of the desired filter (a low-pass filter with another cutoff frequency or a high-pass filter or a band pass filter or a band stop filter). Then using any of the mapping techniques (impulse invariant transformation or bilinear transformation) this analog filter is converted into the digital filter with a system function $H(z)$.

The frequency transformation formulae used to convert a prototype low-pass filter into a low-pass (with a different cutoff frequency), high-pass, band pass or band stop are given in Table 8.2. Here Ω_c is the cutoff frequency of the low-pass prototype filter. Ω_c^* cutoff frequency of new low-pass filter or high-pass filter and Ω_1 and Ω_2 are the cutoff frequencies of band pass or band stop filters.

TABLE 8.2 Analog Frequency Transformation

Type	*Transformation*
Low-pass	$s \to \Omega_c \dfrac{s}{\Omega_c^*}$
High-pass	$s \to \Omega_c \dfrac{\Omega_c^*}{s}$
Band pass	$s \to \Omega_c \dfrac{s^2 + \Omega_1 \Omega_2}{s(\Omega_2 - \Omega_1)}$
Band stop	$s \to \Omega_c \dfrac{s(\Omega_2 - \Omega_1)}{s^2 + \Omega_1 \Omega_2}$

Ω_0 is the centre frequency $\Omega_0 = \sqrt{\Omega_1 \Omega_2}$

$$\text{Quality factor } Q = \frac{\Omega_0}{\Omega_2 - \Omega_1}$$

EXAMPLE 8.34 A Prototype low-pass filter has the system function $H_p(s) = \dfrac{1}{s^2 + 3s + 2}$. Obtain a band pass filter with $\Omega_0 = 3$ rad/s and $Q = 12$.

Solution: We know that the centre frequency $\Omega_0 = \sqrt{\Omega_1 \Omega_2}$ and quality factor $Q = \dfrac{\Omega_0}{\Omega_2 - \Omega_1}$.

From Table 8.2, we have the low-pass to band pass transformation

$$s \to \Omega_c \frac{s^2 + \Omega_1\Omega_2}{s(\Omega_2 - \Omega_1)} = \Omega_c \frac{s^2 + \Omega_0^2}{s(\Omega_0/Q)}$$

$$s \to \Omega_c \frac{s^2 + 3^2}{s(3/12)} = 4\Omega_c \left(\frac{s^2 + 9}{s}\right)$$

Therefore, the transfer function of band pass filter is:

$$H(s) = H_p(s)\Big|_{s = 4\Omega_c\left(\frac{s^2+9}{s}\right)}$$

$$= \frac{1}{\left[4\Omega_c\left(\dfrac{s^2 + 9}{s}\right)\right]^2 + 3\left[4\Omega_c\left(\dfrac{s^2 + 9}{s}\right)\right] + 2}$$

$$= \frac{1}{16} \frac{s^2}{\Omega_c^2 s^4 + 0.75\Omega_c s^3 + (18\Omega_c^2 + 0.125)s^2 + 6.75\Omega_c s + 81\Omega_c^2}$$

EXAMPLE 8.35 Transform the prototype low-pass filter with system function $H(s) = \dfrac{\Omega_c}{s + 2\Omega_c}$ into a high-pass filter with a cutoff frequency Ω_c^*.

Solution: We know that the desired transformation from low-pass to high-pass is $s \to \Omega_c \dfrac{\Omega_c^*}{s}$.

Thus, we have
$$H_{hpf}(s) = \frac{\Omega_c}{\left(\dfrac{\Omega_c \Omega_c^*}{s}\right) + 2\Omega_c} = \frac{s}{2s + \Omega_c^*}$$

8.11.2 Digital Frequency Transformation

As in the analog domain, frequency transformation is possible in the digital domain also. The frequency transformation is done in the digital domain by replacing the variable z^{-1} by a function of z^{-1}, i.e., $f(z^{-1})$. This mapping must take into account the stability criterion. All the poles lying within the unit circle must map onto itself and the unit circle must also map onto itself. Table 8.3 gives the formulae for the transformation of the prototype low pass digital filter into a digital low-pass, high-pass, band pass or band stop filters.

TABLE 8.3 Digital Frequency Transformation

Type	*Transformation*	*Design parameter*
Low-pass	$z^{-1} \to \dfrac{z^{-1}-\alpha}{1-\alpha z^{-1}}$	$\alpha = \dfrac{\sin\left[(\omega_c-\omega_c^*)/2\right]}{\sin\left[(\omega_c+\omega_c^*)/2\right]}$
High-pass	$z^{-1} \to -\dfrac{z^{-1}+\alpha}{1+\alpha z^{-1}}$	$\alpha = -\dfrac{\cos\left[(\omega_c-\omega_c^*)/2\right]}{\cos\left[(\omega_c+\omega_c^*)/2\right]}$
Band pass	$z^{-1} \to -\dfrac{z^{-2}-\alpha_1 z^{-1}+\alpha_2}{\alpha_2 z^{-2}-\alpha_1 z^{-1}+1}$	$\alpha_1 = \dfrac{-2\alpha k}{(k+1)}$
		$\alpha_2 = \dfrac{(k-1)}{(k+1)}$
		$\alpha = \dfrac{\cos[(\omega_2+\omega_1)/2]}{\cos[(\omega_2-\omega_1)/2]}$
		$k = \cot\left(\dfrac{\omega_2-\omega_1}{2}\right)\tan\left(\dfrac{\omega_c}{2}\right)$
Band stop	$z^{-1} \to \dfrac{z^{-2}-\alpha_1 z^{-1}+\alpha_2}{\alpha_2 z^{-2}-\alpha_1 z^{-1}+1}$	$\alpha_1 = \dfrac{-2\alpha}{(k+1)}$
		$\alpha_2 = \dfrac{(1-k)}{(1+k)}$
		$\alpha = \dfrac{\cos[(\omega_2+\omega_1)/2]}{\cos[(\omega_2-\omega_1)/2]}$
		$k = \tan\left(\dfrac{\omega_2-\omega_1}{2}\right)\tan\left(\dfrac{\omega_c}{2}\right)$

The frequency transformation may be accomplished in any of the available two techniques, however, caution must be taken to which technique to use. For example, the impulse invariant transformation is not suitable for high-pass or bandpass filters whose resonant frequencies are higher. In such a case, suppose a low-pass prototype filter is converted into a high-pass filter using the analog frequency transformation and transformed later to a digital filter using the impulse invariant technique. This will result in aliasing problems. However, if the same prototype low-pass filter is first transformed into digital filter using the impulse invariant technique and later converted into a high-pass filter using the digital frequency transformation, then it will not have any aliasing problem. Whenever the bilinear transformation is used, it is of no significance whether analog frequency transformation is used or digital frequency transformation. In this case, both analog and digital frequency transformation techniques will give the same result.

SHORT QUESTIONS WITH ANSWERS

1. What are the basic types of digital filters and on what basis are they classified?

Ans. There are following two basic types of digital filters:

(i) Finite impulse response filters (FIR filters)

(ii) Infinite impulse response filters (IIR filters).

They are classified on the basis of the number of sample points used to determine the unit sample (i.e. impulse) response of a LTI discrete-time system.

2. Define an IIR filter.

Ans. An IIR (Infinite-duration Impulse Response) filter is a digital filter designed by considering all the infinite samples of impulse response.

3. Define an FIR filter.

Ans. An FIR (Finite-duration Impulse Response) filter is a digital filter designed by considering only a finite number of samples of impulse response.

4. Distinguish between IIR and FIR filters.

Ans. The filter design starts from ideal frequency response. The desired impulse response which consists of infinite number of samples is obtained by taking the inverse Fourier transform of the ideal frequency response of the system. The digital filters designed by using all the infinite samples of the impulse response are called IIR filters and the digital filters designed by using only a finite number of samples of the impulse response are called FIR filters.

5. Compare IIR and FIR filters.

Ans. The IIR and FIR filters are compared as follows:

IIR filter	*FIR filter*
(i) Design is based on all the infinite samples of the impulse response.	(i) Design is based on only a finite number of samples of impulse response.
(ii) The impulse response cannot be directly converted to digital filter transfer function.	(ii) The impulse response can be directly converted to digital filter transfer function.
(iii) The digital filter cannot be directly designed. First an analog filter is to be designed and then it has to be transformed to a digital filter.	(iii) The digital filter can be directly designed.
(iv) The specifications include the desired characteristics for magnitude response only.	(iv) The specifications include the desired characteristics for both magnitude and phase response.
(v) Linear phase characteristics cannot be achieved.	(v) Linear phase characteristics can be achieved.

6. Based on frequency response how are filters classified?

Ans. Based on frequency response, filters are classified as low-pass filters, high-pass filters, bandpass filters and bandstop filters.

7. What are the requirements for an analog filter to be causal and stable?

Ans. For an analog filter to be causal and stable, the requirements are as follows

(i) The analog filter transfer function $H_a(s)$ should be a rational function of s and the coefficients of s should be real.

(ii) The poles should lie on the left half of s-plane.

(iii) The number of zeros should be less than or equal to the number of poles.

8. What are the requirements for a digital filter to be causal and stable?

Ans. The requirements for a digital filter to be causal and stable are as follows:

(i) The digital filter transfer function $H(z)$ should be a rational function of z and the coefficients of z should be real.

(ii) The poles should lie inside the unit circle in z-plane.

(iii) The number of zeros should be less than or equal to the number of poles.

9. How a digital filter is designed?

Ans. For designing a digital IIR filter, first an equivalent analog filter is designed using any one of the approximation technique and the given specifications. The result of the analog filter design will be an analog transfer function $H_a(s)$. The analog filter transfer function is transformed to digital filter transfer function $H(z)$ using either bilinear or impulse invariant transformation.

10. Mention any two techniques for digitizing the transfer function of an analog filter.

Ans. Two techniques for digitizing the transfer function of an analog filter are: (i) Impulse invariant transformation, and (ii) Bilinear transformation.

11. Compare the analog and digital filters.

Ans. The analog and digital filters are compared as follows:

Analog filter	*Digital filter*
(i) It operates on analog signals (or actual signals)	(i) It operates on digital samples (or sampled version) of the signal.
(ii) It is governed by the linear differential equation.	(ii) It is governed by the linear difference equation.
(iii) It consists of electrical components like resistors, inductors and capacitors.	(iii) It consists of adders, multipliers and delay elements implemented in digital logic (either in hardware or software or both).
(iv) In analog filters, the approximation problem is solved to satisfy the desired frequency response.	(iv) In digital filters, the filter coefficients are designed to satisfy the desired frequency response.

12. Mention the important features of IIR filters.

Ans. The important features of IIR filters are as follows:

(i) The physically realizable IIR filters do not have linear phase.

(ii) The IIR filter specifications include the desired characteristics for the magnitude response only.

13. What is the impulse invariant transformation?

Ans. The transformation of analog filter to digital filter without modifying the impulse response of the filter is called impulse invariant transformation (i.e. in this transformation, the impulse response of the digital filter will be the sampled version of the impulse response of the analog filter).

14. What is the main objective of impulse invariant transformation?

Ans. The main objective of impulse invariant transformation is to develop an IIR filter transfer function whose impulse response is the sampled version of the impulse response of the analog filter. Therefore, the frequency response characteristics of the analog filter is preserved.

15. How analog poles are mapped to digital poles in the impulse invariant transformation (Bilinear transformation)?

Ans. In the impulse invariant transformation (or in bilinear transformation) the mapping of analog poles to digital poles is as follows:

(i) The analog poles on the imaginary axis of s-plane are mapped onto the unit circle in the z-plane.

(ii) The analog poles on the left half of s-plane are mapped into the interior of unit circle in z-plane.

(iii) The analog poles on the right half of s-plane are mapped into the exterior of unit circle in z-plane.

16. What is the importance of poles in filter design?

Ans. The importance of poles in filter design is the stability of a filter is related to the location of the poles. For a stable analog filter the poles should lie on the left half of *s*-plane. For a stable digital filter the poles should lie inside the unit circle in the *z*-plane.

17. What is aliasing?

Ans. The phenomena of high frequency sinusoidal components acquiring the identity of low frequency sinusoidal components after sampling is called aliasing (i.e., aliasing is higher frequencies impersonating lower frequencies). The aliasing problem will arise if the sampling rate does not satisfy the Nyquist sampling criteria.

18. What is bilinear transformation?

Ans. The bilinear transformation is a conformal mapping that transforms the *s*-plane to *z*-plane. In this mapping, the imaginary axis of *s*-plane is mapped into the unit circle in *z*-plane, the left half of *s*-plane is mapped into interior of unit circle in *z*-plane, and the right half of the *s*-plane is mapped into exterior of unit circle in *z*-plane. The bilinear mapping is a one-to-one mapping and it is accomplished when

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

19. What is frequency warping?

Ans. The distortion in frequency axis introduced when the *s*-plane is mapped into *z*-plane using the bilinear transformation, due to the nonlinear relation between analog and digital frequencies is called frequency warping.

20. What are the advantages and disadvantages of bilinear transformation?

Ans. The advantages and disadvantages of bilinear transformation are:

Advantages

(i) The bilinear transformation is one-to-one mapping.

(ii) There is no aliasing and so the analog filter need not have a band limited frequency response.

(iii) The effect of warping on amplitude response can be eliminated by prewarping the analog filter.

(iv) The bilinear transformation can be used to design digital filters with prescribed magnitude response with piecewise constant values.

Disadvantages

(i) The nonlinear relationship between analog and digital frequencies introduces frequency distortion which is called frequency warping.

(ii) Using the bilinear transformation, a linear phase analog filter cannot be transformed to a linear phase digital filter.

21. What is prewarping?

Ans. In IIR filter design using bilinear transformation, the conversion of the specified digital frequencies to analog equivalent frequencies is called prewarping. The prewarping is necessary to eliminate the effect of warping on amplitude response.

22. Explain the technique of warping.

Ans. In IIR filter design using bilinear transformation, the specified digital frequencies are converted to analog equivalent frequencies, which are called prewrap frequencies. Using the prewarp frequencies, the analog filter function is designed and then it is transformed to digital filter transfer function.

23. How bilinear transformation is preformed?

Ans. The bilinear transformation is performed by substituting $s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$ in the analog filter transfer function, i.e. $H(z) = [H_a(s)]\big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$.

24. Compare the impulse invariant and bilinear transformation.

Ans. The impulse invariant and bilinear transformations are compared as follows:

Impulse invariant transformation	*Bilinear transformation*
(i) It is many-to-one mapping.	(i) It is one-to-one mapping.
(ii) The relation between analog and digital frequency is linear.	(ii) The relation between analog and digital frequency is nonlinear.
(iii) To prevent the problem of aliasing, the analog filters should be band limited.	(iii) There is no problem of aliasing and so the analog filter need not be band limited.
(iv) The magnitude and phase responses of analog filter can be preserved by choosing low sampling time or high sampling frequency.	(iv) Due to the effect of warping, the phase response of analog filter cannot be preserved. But the magnitude response can be preserved by prewarping.

25. What is the relation between analog and digital frequencies in impulse invariant transformation?

Ans. The relation between analog and digital frequencies in impulse invariant transformation is given by

Digital frequency = Analog frequency × Sampling time period

i.e. $\omega = \Omega T$

26. What is the relation between digital and analog frequency in the bilinear transformation?

Ans. In bilinear transformation, the digital frequency is given by

$$\text{Digital frequency } \omega = 2\tan^{-1}\frac{\Omega T}{2}$$

where, Ω = Analog frequency, and T = Sampling time period.

27. Why impulse invariant transformation is not considered to be one-to-one?

Ans. In impulse invariant transformation any strip of width $2\pi/T$ in the s-plane for values of s in the range $(2k - 1)\pi/T \le \Omega \le (2k + 1)\pi/T$ (where k is an integer) is mapped into the entire z-plane. The left half portion of each strip in s-plane maps into the interior of the unit circle in z-plane, right half portion of each strip in s-plane maps into the exterior of the unit circle in z-plane and the imaginary axis of each strip in s-plane maps into the unit circle in z-plane. So the entire s-plane is mapped infinite number of times on to the entire z-plane. Hence the impulse invariant transformation is many-to-one and not one-to-one.

28. What is Butterworth approximation?

Ans. Butterworth approximation is one in which the error function is selected such that the magnitude response is maximally flat at the origin (i.e., at $\Omega = 0$) and monotonically decreasing with increasing Ω.

29. How the poles of Butterworth transfer function are located in s-plane?

Ans. The poles of the normalized Butterworth transfer function symmetrically lie on an unit circle in s-plane with angular spacing of π/N.

30. Write the magnitude function of low-pass Butterworth filter?

Ans. The magnitude function of low-pass Butterworth filter is given by

$$|H_a(\Omega)| = \frac{1}{\sqrt{1 + \left(\dfrac{\Omega}{\Omega_c}\right)^{2N}}}$$

where, Ω_c = Cutoff frequency, N = Order of the filter

31. How the order of the filter affects the frequency response of Butterworth filter?

Ans. The magnitude response of the Butterworth filter approaches the ideal response as the order of the filter is increased.

32. Write the transfer function of unnormalized Butterworth low-pass filter.

Ans. The transfer function of unnormalized Butterworth low-pass filter $H_a(s)$ is

When N is even, $$H_a(s) = \sum_{k=1}^{N/2} \frac{\Omega_c^2}{s^2 + b_k\Omega_c s + \Omega_c^2}$$

When N is odd, $$H_a(s) = \frac{\Omega_c}{s + \Omega_c} \sum_{k=1}^{(N-1)/2} \frac{\Omega_c^2}{s^2 + b_k\Omega_c s + \Omega_c^2}$$

where, $b_k = 2\sin\left[\dfrac{(2k-1)\pi}{2N}\right]$

N = Order of filter

Ω_c = Analog cutoff frequency

33. How will you choose the order N for a Butterworth filter?

Ans. The orders N for a Butterworth filter is chosen such that

$$N \geq \frac{1}{2} \frac{\log\left\{\left[\frac{1}{A_2^2} - 1\right] \Big/ \left[\frac{1}{A_1^2} - 1\right]\right\}}{\log\left(\frac{\Omega_2}{\Omega_1}\right)}$$

34. Write the properties of Butterworth filter.

Ans. The properties of Butterworth filter are as follows:

(i) The Butterworth filters are all pole designs.

(ii) At the cutoff frequency Ω_c, the magnitude of normalized Butterworth filter is $1/\sqrt{2}$.

(iii) The filter order N, completely specifies the filter and as the value of N increases the magnitude response approaches the ideal response.

(iv) The magnitude is maximally flat at the origin and monotonically decreasing with increasing Ω.

35. What is Chebyshev approximation?

Ans. Chebyshev approximation is one in which the approximation function is selected such that the error is minimized over a prescribed band of frequencies.

36. What is type-1 Chebyshev approximation?

Ans. Type-1 Chebyshev approximation is one in which the error function is selected such that the magnitude response is equiripple in the passband and monotonic in the stopband.

37. What is type-2 Chebyshev approximation?

Ans. Type-2 Chebyshev approximation is one in which the error function is selected such that the magnitude response is monotonic in the passband and equiripple in the stopband. The type-2 Chebyshev response is called inverse Chebyshev response.

38. Write the expression for the magnitude response of Chebyshev low-pass filter.

Ans. The magnitude response of type-1 Chebyshev low-pass filter is given by

$$|H_a(\Omega)| = \frac{1}{\sqrt{1 + \varepsilon^2 C_N^2\left(\frac{\Omega}{\Omega_c}\right)}}$$

where ε = attenuation constant

$C_N\left(\frac{\Omega}{\Omega_c}\right)$ = Chebyshev polynomial of the first kind of degree N.

39. How the order of the filter affects the frequency response of Chebyshev filter?

Ans. The magnitude response of type-1 Chebyshev filter approaches the ideal response as the order of the filter increases.

40. Write the transfer function of unnormalized Chebyshev low-pass filter?

Ans. The transfer function $H_a(s)$ of unnormalized type-1 Chebyshev low-pass filter is given as:

$$\text{When } N \text{ is even, } H_a(s) = \prod_{k=1}^{\frac{N}{2}} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + c_k \Omega_c^2}$$

$$\text{When } N \text{ is odd, } H_a(s) = \frac{B_0 \Omega_c}{s + c_0 \Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + c_k \Omega_c^2}$$

$$\text{where } b_k = 2 y_N \sin\left(\frac{(2k-1)\pi}{2N}\right); \; c_k = y_N^2 + \cos^2 \frac{(2k-1)\pi}{2N}; \; c_0 = y_N$$

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\varepsilon^2} + 1\right)^{\frac{1}{2}} + \frac{1}{\varepsilon}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\varepsilon^2} + 1\right)^{\frac{1}{2}} + \frac{1}{\varepsilon}\right]^{\frac{-1}{N}}\right\}$$

41. How will you determine the order N of Chebyshev filter?

Ans. The order N of a Chebyshev filter is such that

$$N \geq \frac{\cosh^{-1}\left\{\frac{1}{\varepsilon}\left[\frac{1}{A_2^2} - 1\right]^{\frac{1}{2}}\right\}}{\cosh^{-1}\left(\frac{\Omega_2}{\Omega_1}\right)}$$

where ε = Attenuation constant

A_2 = Gain at stopband edge frequency,

Ω_2 and Ω_1 = Analog stopband and passband edge frequencies.

42. How the poles of Chebyshev transfer function are located in s-plane?

Ans. The poles of the Chebyshev transfer function symmetrically lie on an ellipse in s-plane.

43. Write the properties of Chebyshev type-1 filters?

Ans. The properties of Chebyshev type-1 filter are as follows:

(i) The magnitude response is equiripple in the passband and monotonic in the stopband.

(ii) The type-1 Chebyshev filters are all pole designs.

(iii) The normalized magnitude function has a value of $1/\sqrt{1+\varepsilon^2}$ at the cutoff frequency Ω_c.
(iv) The magnitude response approaches the ideal response as the value of N increases.

44. Compare the Butterworth and Chebyshev type-1 filters.

Ans. The Butterworth and Chebyshev type-1 filters are compared as follows:

Butterworth	*Chebyshev type*-1
(i) All pole design.	(i) All pole design.
(ii) The poles lie on a circle in s-plane.	(ii) The poles lie on an ellipse in s-plane.
(iii) The magnitude response is maximally flat at the origin and monotonically decreasing function of Ω.	(iii) The magnitude response is equiripple in passband and monotonically decreasing in the stopband.
(iv) The normalized magnitude response has a value of $1/\sqrt{2}$ at the cutoff frequency Ω_c.	(iv) The normalized magnitude response has a value of $1/\sqrt{1+\varepsilon^2}$ at the cutoff frequency Ω_c.
(v) Only a few parameters have to be calculated to determine the transfer function.	(v) A large number of parameters have to be calculated to determine the transfer function.

REVIEW QUESTIONS

1. Compare analog and digital filters. State the advantages of digital filters over analog filters.
2. Define infinite impulse response and finite impulse response filters and compare.
3. Justify the statement IIR filter is less stable and give reason for it.
4. Describe digital IIR filter characterization in time domain.
5. Describe digital IIR filter characterization in z-domain.
6. Discuss the impulse invariant method.
7. What are the limitations of impulse invariant method?
8. Compare impulse invariant and bilinear transformation methods.
9. Discuss the magnitude and phase responses of digital filters.
10. Explain method of constructing Butterworth circle in the z-plane using the bilinear transformation method.
11. Compare Butterworth and Chebyshev approximations.
12. Discuss the magnitude characteristics of an analog Butterworth filter and give its pole locations. Discuss about the pole location for the digital Chebyshev filters.
13. What is frequency warping? How it will arise?

14. What is warping effect? Discuss influence of warping effect on amplitude response and phase response of a derived digital filter from a corresponding analog filter.

15. Discuss the concept of frequency transformation in analog domain.

16. Discuss the digital frequency transformation.

17. Obtain transformation for Butterworth filters between s and z using the bilinear transformation.

FILL IN THE BLANKS

1. Filters designed by considering _______ samples of the impulse response are called IIR filters.

2. The physically realizable IIR filters do not have _______ phase.

3. The IIR filter specifications includes the desired characteristics for the _______ response only.

4. Filters designed by considering _______ samples of the impulse response are called FIR filters.

5. The impulse response is obtained by taking the inverse Fourier transform of the _______.

6. The bandwidth of the discrete signal is limited by the _______.

7. The popular methods for design of IIR digital filters uses the technique of _______ an analog filter into an _______ digital filter.

8. The bandwidth of a real discrete signal is _______ the sampling frequency.

9. The three techniques used to transform an analog filter to digital filter are _______ _______, and _______.

10. The two properties which are to be preserved in analog to digital transformation are _______ and _______.

11. The tolerance in the passband and stopband are called _______.

12. In _______ transformation the impulse response of digital filter is the sampled version of the impulse response of analog filter.

13. In impulse invariant (bilinear) transformation, the _______ poles of s-plane are mapped into the interior of unit circle in z-plane.

14. In impulse invariant (bilinear) transformation, the right half poles of s-plane are mapped into the _______ of unit circle in z-plane.

15. In impulse invariant (bilinear) transformation, the poles on the imaginary axis of s-plane are mapped into the _______ in z-plane.

16. In impulse invariant transformation any strip of width _______ in s-plane is mapped into the entire z-plane.

17. The phenomenon of high frequency components acquiring the identity of low frequency components is called _______.

18. _______ is higher frequencies impersonating lower frequencies.
19. Aliasing occurs only in _______ transformation.
20. The impulse invariant mapping is _______ mapping, whereas bilinear mapping is a _______ mapping.
21. The _______ due to nonlinear relationship between analog and digital frequencies is called frequency warping.
22. In bilinear transformation, the effect of warping on _______ can be eliminated by _______ the analog filter.
23. A linear phase analog filter cannot be transformed into a linear phase digital filter using _______ transformation.
24. The two popular techniques used to approximate the ideal frequency response are _______ and _______ approximations.
25. In _______ approximation, the magnitude response is maximally flat at the origin and monotonically decreases with increasing frequency.
26. At the cutoff frequency, the magnitude of the Butterworth filter is _______ times the maximum value.
27. In _______ approximation, the magnitude response is equiripple in the passband and monotonic in the stopband.
28. In _______ approximation, the magnitude response is monotonic in the passband and equiripple in the stopband.
29. The type-2 Chebyshev response is also called _______ response.
30. In Chebyshev approximation, the normalized magnitude response has a value of _______ at the cutoff frequency.

OBJECTIVE TYPE QUESTIONS

1. The condition for a digital filter to be causal and stable is

(a) $h(n) = 0$ for $n < 0$ and $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$

(b) $h(n) = 0$ for $n > 0$ and $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$

(c) $h(n) = 0$ for $n < 0$ and $\sum_{n=-\infty}^{\infty} |h(n)| > \infty$

(d) $h(n) = 0$ for $n > 0$ and $\sum_{n=-\infty}^{\infty} |h(n)| > \infty$

2. IIR filters are
(a) recursive type (b) non-recursive type
(c) neither recursive nor non-recursive (d) none of the above

3. For same set of specifications
(a) IIR filter requires fewer filter coefficients than an FIR filter
(d) FIR filter requires fewer filter coefficients than IIR filter
(c) FIR and IIR filters require same number of filter coefficients
(d) none of the above

4. In the impulse invariant transformation, relationship between Ω and ω is
(a) $\Omega = \omega T$ (b) $\Omega = \omega/T$ (c) $\omega = \Omega/T$ (d) $\omega = T/\omega$

5. In the impulse invariant transformation

(a) $\dfrac{1}{s-p_i} \to \dfrac{1}{1-e^{p_iT}z^{-1}}$ (b) $\dfrac{1}{s-p_i} \to \dfrac{1}{1-e^{-p_iT}z^{-1}}$

(c) $\dfrac{1}{s-p_i} \to \dfrac{1}{1-e^{p_iT}z}$ (d) $\dfrac{1}{s-p_i} \to \dfrac{1}{1-e^{-p_iT}z}$

6. Non-linearity in the relationship between Ω and ω is known as
(a) aliasing (b) frequency warping
(c) unwarping (d) frequency mixing

7. In the bilinear transformation, the relationship between Ω and ω is

(a) $\Omega = 2\tan\dfrac{\omega}{2}$ (b) $\Omega = \dfrac{2}{T}\tan\dfrac{\omega}{2}$

(c) $\Omega = \dfrac{1}{T}\tan\dfrac{\omega}{2}$ (d) $\Omega = \tan\dfrac{\omega T}{2}$

8. In the bilinear transformation, the relation between s and z is

(a) $s = \dfrac{2}{T}\left(\dfrac{1+z^{-1}}{1-z^{-1}}\right)$ (b) $s = \dfrac{1}{T}\left(\dfrac{1+z^{-1}}{1-z^{-1}}\right)$

(c) $s = \dfrac{2}{T}\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)$ (d) $s = \dfrac{1}{T}\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)$

9. Butterworth filters have
(a) wideband transition region (b) sharp transition region
(c) oscillation in the transition region (d) none of the above

10. Chebyshev filters have
(a) wideband transition region (b) sharp transition region
(c) oscillation in the transition region (d) none of the above

11. Type-1 Chebyshev filter contains
(a) oscillations in the passband (b) oscillations in the stopband
(c) oscillations in stop and pass bands (d) oscillations in the transition band

12. Type-2 Chebyshev filter is also called
(a) inverse Chebyshev filter (b) elliptic filter
(c) reverse Chebyshev filter (d) none of the above

13. The attenuation constant ε in the design of Chebyshev filter is given by

(a) $\varepsilon = \left[\dfrac{1}{A_1^2} - 1\right]^{\frac{1}{2}}$ (b) $\varepsilon = \left[\dfrac{1}{A_2^2} - 1\right]^{\frac{1}{2}}$

(c) $\varepsilon = \left[\dfrac{1}{A_1^2} - 1\right]^{\frac{1}{2N}}$ (d) $\varepsilon = \left[\dfrac{1}{A_2^2} - 1\right]^{\frac{1}{2N}}$

14. The cutoff frequency Ω_c of a low-pass Butterworth filter is given by

(a) $\Omega_c = \dfrac{\Omega_1}{\left[\dfrac{1}{A_1^2} - 1\right]^{\frac{1}{2N}}}$ (b) $\Omega_c = \dfrac{\Omega_2}{\left[\dfrac{1}{A_1^2} - 1\right]^{\frac{1}{2N}}}$

(c) $\Omega_c = \dfrac{\Omega_1}{\left[\dfrac{1}{A_1^2} - 1\right]^{\frac{1}{2}}}$ (d) $\Omega_c = \dfrac{\Omega_1}{\left[\dfrac{1}{A_1^2} + 1\right]^{\frac{1}{2N}}}$

15. For Butterworth filter, when A_1 and A_2 are in dB, filter order N is given by

(a) $N \geq \dfrac{1}{2}\dfrac{\log\left[\dfrac{10^{0.1A_2\,\text{dB}} + 1}{10^{0.1A_1\,\text{dB}} + 1}\right]}{\log(\Omega_2/\Omega_1)}$ (b) $N \geq \dfrac{1}{2}\dfrac{\log\left[\dfrac{10^{0.1A_2\,\text{dB}} - 1}{10^{0.1A_1\,\text{dB}} - 1}\right]}{\log(\Omega_2/\Omega_1)}$

(c) $N \geq \dfrac{1}{2}\dfrac{\log\left[\dfrac{10^{0.1A_2\,\text{dB}} + 1}{10^{0.1A_1\,\text{dB}} + 1}\right]}{\log(\Omega_1/\Omega_2)}$ (d) $N \geq \dfrac{1}{2}\dfrac{\log\left[\dfrac{10^{0.1A_2\,\text{dB}} + 1}{10^{0.1A_1\,\text{dB}} + 1}\right]}{\log(\Omega_1/\Omega_2)}$

16. The magnitude response of a Butterworth filter is given by

(a) $|H_a(\Omega)|^2 = \dfrac{1}{1 + \left(\dfrac{\Omega_c}{\Omega_1}\right)^{2N}}$ (b) $|H_a(\Omega)|^2 = \dfrac{1}{1 + \left(\dfrac{\Omega}{\Omega_c}\right)^{2N}}$

(c) $|H_a(\Omega)|^2 = \dfrac{1}{1 + \left(\dfrac{\Omega_c}{\Omega_1}\right)^{\frac{1}{2N}}}$ (d) $|H_a(\Omega)|^2 = \dfrac{1}{1 + \left(\dfrac{\Omega}{\Omega_c}\right)^{\frac{1}{2N}}}$

17. The magnitude response of type-1 Chebyshev filter is given by

(a) $|H_a(\Omega)|^2 = \dfrac{1}{1 + \varepsilon^2 c_N^2\left(\dfrac{\Omega}{\Omega_c}\right)}$ (b) $|H_a(\Omega)|^2 = \dfrac{1}{1 + \varepsilon^2 c_N^2\left(\dfrac{\Omega}{\Omega_c}\right)^{\frac{1}{2N}}}$

(c) $|H_a(\Omega)|^2 = \dfrac{1}{1 + \varepsilon c_N\left(\dfrac{\Omega}{\Omega_c}\right)^2}$ (d) $|H_a(\Omega)|^2 = \dfrac{1}{1 + \varepsilon c_N\left(\dfrac{\Omega}{\Omega_c}\right)^{\frac{1}{2N}}}$

PROBLEMS

1. Use the backward difference for the derivative to convert the analog low-pass filter with system function given below to digital filter assuming $T = 1$ s.

(a) $H(s) = \dfrac{1}{s+4}$ (b) $H(s) = \dfrac{1}{s^2 + 25}$ (c) $H(s) = \dfrac{1}{(s+0.2)^2 + 16}$

2. Covert the analog filter with system function given below into a digital filter using impulse invariant transformation assuming $T = 1$s.

(a) $H(s) = \dfrac{1}{(s+3)(s+4)}$ (b) $H(s) = \dfrac{s+0.2}{(s+0.2)^2 + 9}$

(c) $H(s) = \dfrac{1}{(s+0.5)(s^2 + 0.5s + 2)}$

3. Convert the analog filter with system function $H(s) = \dfrac{s+0.3}{(s+0.3)^2 + 16}$ into a digital filter using the bilinear transformation. The digital filter should have resonant frequency of $\omega_r = \pi/2$.

4. Convert the analog filter with system function into a digital filter using bilinear transformation. Take $T = 1$s.

(a) $H(s) = \dfrac{4}{(s+1)(s+3)}$ (b) $H(s) = \dfrac{2s}{s^2 + 3s + 4}$

5. A digital filter with a 3 dB bandwidth of 0.4π is to be designed from the analog filter whose system response is $H(s) = \dfrac{\Omega_c}{s + 3\Omega_c}$.

Use the bilinear transformation and obtain $H(z)$.

6. The specification of the desired low-pass filter is

$$\frac{1}{\sqrt{2}} \le |H(\omega)| \le 1.0; \qquad 0 \le \omega < 0.2\pi$$

$$|H(\omega)| \le 0.08; \qquad 0.4\pi \le \omega \le \pi$$

Design a Butterworth digital filter using the bilinear transformation.

7. The specification of the desired low-pass digital filter is

$$0.9 \le |H(\omega)| \le 1.0; \qquad 0 \le \omega \le 0.25\pi$$

$$|H(\omega)| \le 0.24; \qquad 0.5\pi \le \omega \le \pi$$

Design a Chebyshev digital filter using the impulse invariant transformation.

8. Determine $H(z)$ for a Butterworth filter satisfying the following constraints:

$$\sqrt{0.5} \le |H(\omega)| \le 1; \qquad 0 \le \omega \le \pi/2$$

$$|H(\omega)| \le 0.2; \qquad 3\pi/4 \le \omega \le \pi$$

with $T = 1$ s. Apply the impulse invariant transformation.

9. Design (a) Butterworth low-pass digital filter, (b) Chebyshev low-pass digital filter satisfying the following specifications:

Sampling time = 1 s
Passband frequency = 0.06π rad/s
Stopband frequency = 0.75π rad/s
Passband attenuation = 6 dB
Stop band attenuation = 20 dB

10. Design a Butterworth low-pass digital filter satisfying the following specifications:

$$0.89 \le |H(\omega)| \le 1.0; \qquad 0 \le \omega \le 0.2\pi$$

$$|H(\omega)| \le 0.18; \qquad 0.3\pi \le \omega \le \pi$$

11. A prototype low-pass filter has the system response $H(s) = \dfrac{1}{s^2 + 2s + 4}$. Obtain a bandpass filter with $\Omega_0 = 4$ rad/s and $Q = 10$.

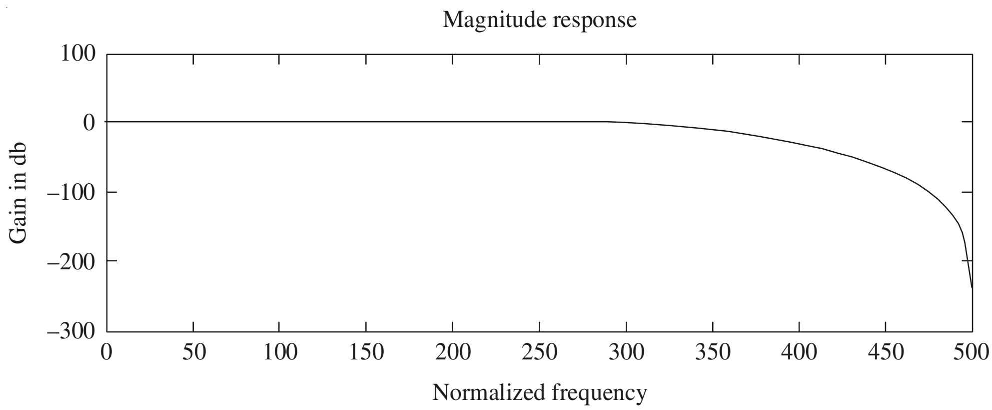
12. Transform the prototype low-pass filter with system function $H(s) = \dfrac{\Omega_c}{s + 3\,\Omega_c}$ into a high-pass filter with cutoff frequency Ω_c^*.

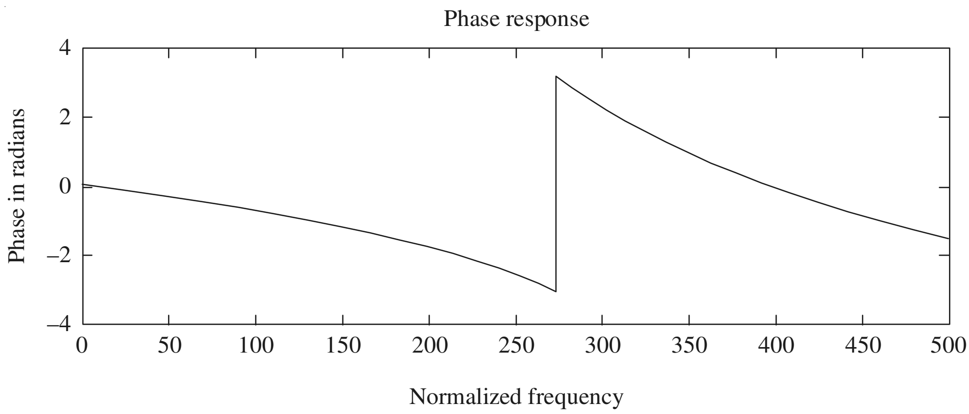
MATLAB PROGRAMS

Program 8.1

```
% Design of filter using Bilinear Transformation
clc; clear all; close all;
fs=1000; % sampling frequency
fn=fs/2;
fc=300; % cutoff frequency
n=5;
[z,p,k]=butter(n,fc/fn);
b=k*poly(z);% zeros
a=poly(p);% poles
[h,om]=freqz(b,a,512,fs);
subplot(2,1,1),plot(om,20*log10(abs(h)));
xlabel('Normalized Frequency')
ylabel('gain in db')
title('magnitude response')
subplot(2,1,2),plot(om,angle(h));
xlabel('Normalized Frequency')
ylabel('phase in radians')
title('phase response')
```

Output:



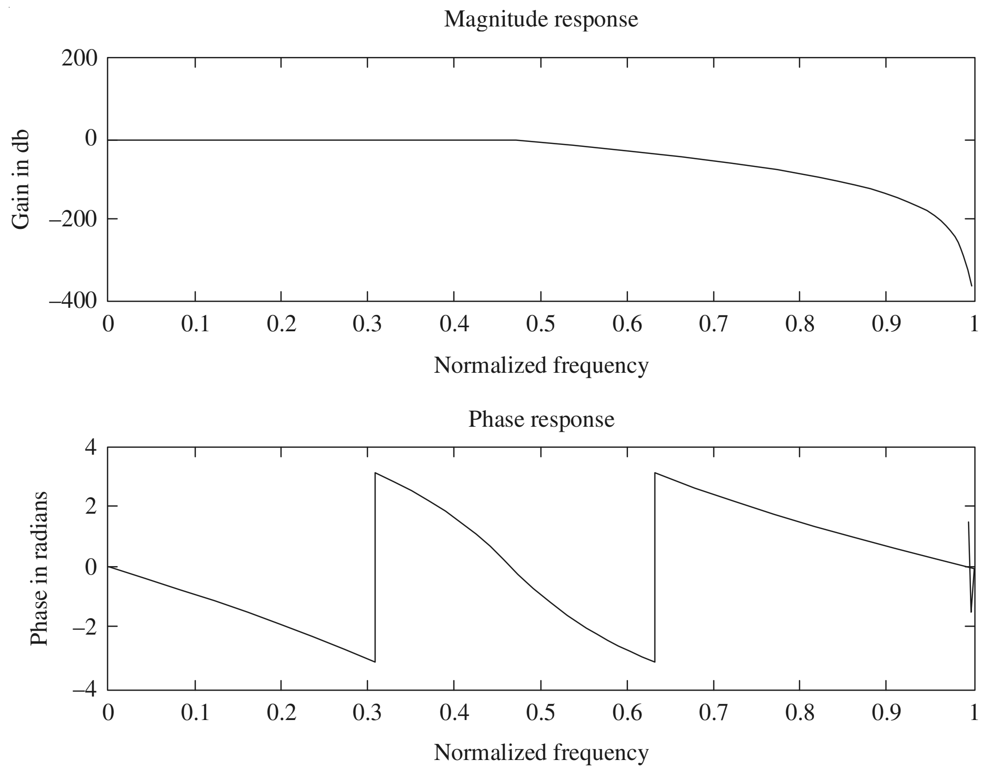


Program 8.2

% Butterworth low-pass filter

```
clc; clear all; close all;
alphas = 30; % pass band attenuation in dB
alphap = 0.5; % stop band attenuation in dB
fpass=1000; % pass band frequency in Hz
fstop=1500; % stop band frequency in Hz
fsam=5000; % sampling frequency in Hz
wp=2*fpass/fsam;
ws=2*fstop/fsam; % pass band and stop band frequencies
[n,wn] = buttord(wp,ws,alphap,alphas);  % minimal order, half-power frequency
[b,a] = butter(n,wn); % coefficients of designed filter
[h,w] = freqz(b,a);
subplot(2,1,1);plot(w/pi,20*log10(abs(h)));
xlabel('Normalized Frequency')
ylabel('gain in db')
title('magnitude response')
subplot(2,1,2);plot(w/pi,angle(h));
xlabel('Normalized Frequency')
ylabel('phase in radians')
title('phase response')
n =
    8
wn =
   0.4644
```

Output:

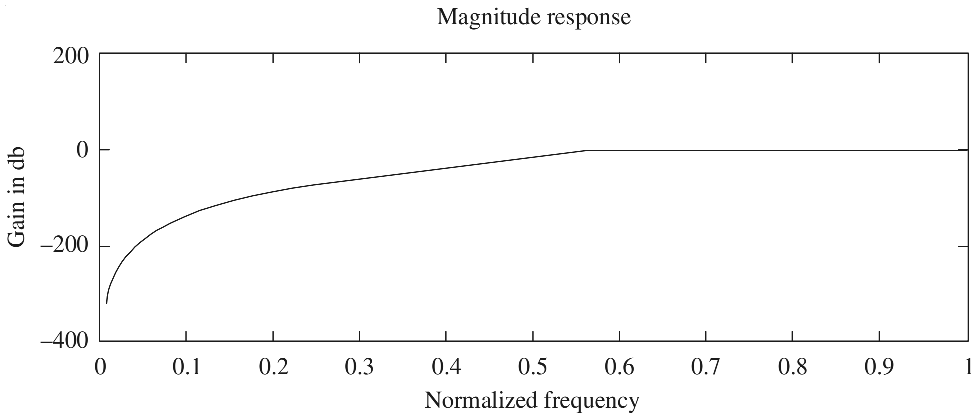


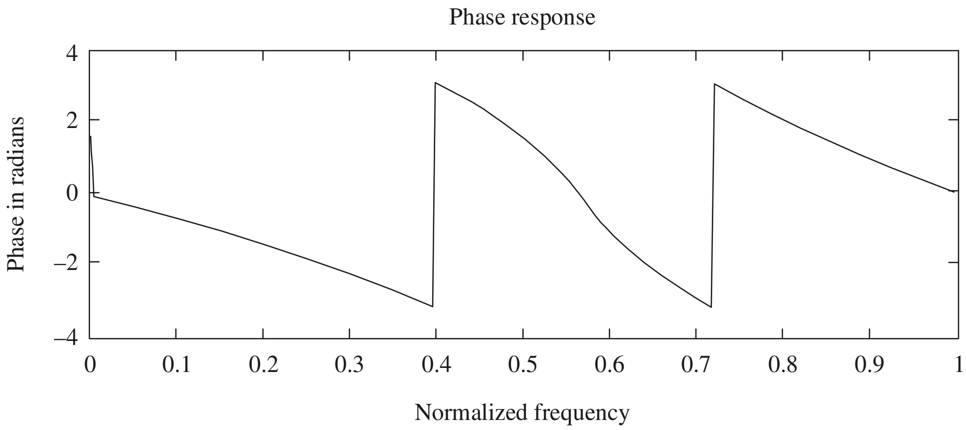
Program 8.3

```
% Butterworth high-pass filter
clc; clear all; close all;
alphas = 50; % pass band attenuation in dB
alphap= 1; % stop band attenuation in dB
fp=1050; % pass band frequency in Hz
fs=600; % stop band frequency in Hz
fsam=3500; % sampling frequency in Hz
wp=2*fp/fsam;
ws=2*fs/fsam;
[n,wn] = buttord(wp,ws,alphap,alphas); % minimal order, half-power frequency
[b,a] = butter(n,wn,'high'); % coefficients of the designed filter
[h,w] = freqz(b,a);
subplot(2,1,1),plot(w/pi,20*log10(abs(h)));
```

```
xlabel('Normalized Frequency')
ylabel('gain in db')
title('magnitude response')
subplot(2,1,2),plot(w/pi,angle(h));
xlabel('Normalized Frequency')
ylabel('phase in radians')
title('phase response')
n =
     8

wn =
    0.5646
```

Output:

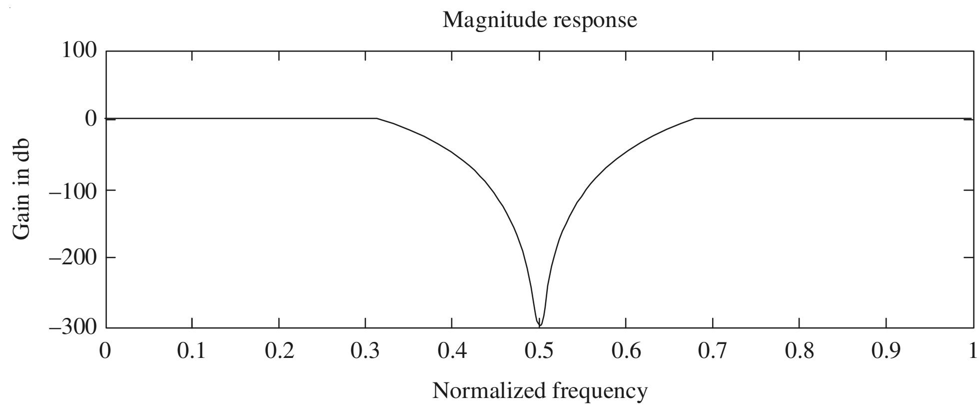


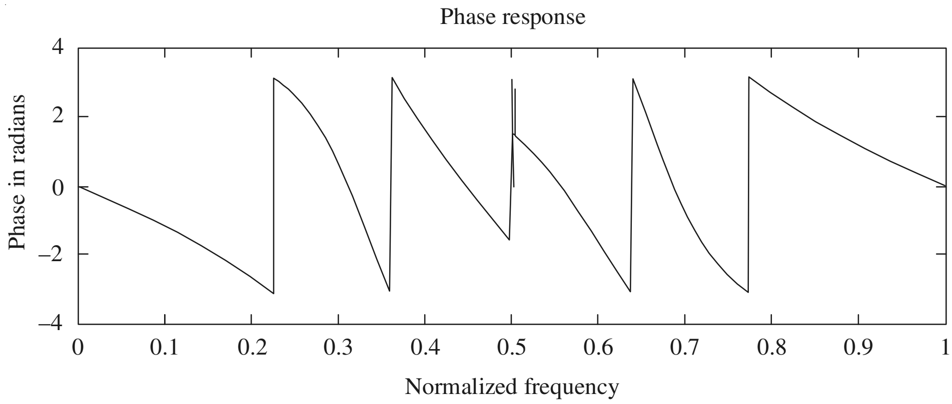


Program 8.4

```
% Butterworth band stop filter
clc; clear all; close all;
ws=[0.4 0.6]; % stop band frequency in radians
wp=[0.3 0.7]; % pass band frequency in radians
alphap=0.4; % pass band attenuation in dB
alphas=50; % stop band attenuation in dB
[n,wn] = buttord(wp,ws,alphap,alphas);
[b,a]=butter(n,wn,'stop');
[h,w] = freqz(b,a);
subplot(2,1,1);plot(w/pi,20*log10(abs(h)));
xlabel('Normalized Frequency')
ylabel('gain in db')
title('magnitude response')
subplot(2,1,2);plot(w/pi,angle(h));
xlabel('Normalized Frequency')
ylabel('phase in radians')
title('phase response')
n =
    9

wn =
   0.3243    0.6757
```

Output:

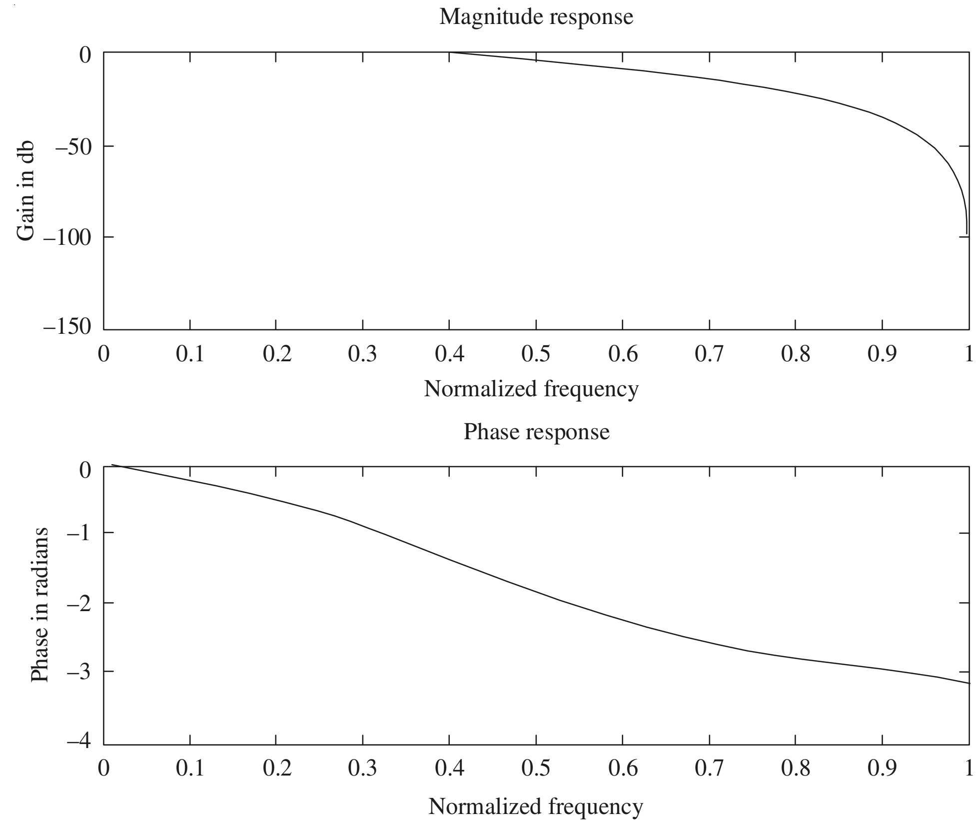




Program 8.5

```
% Chebyshev filter low-pass type-1
clc; clear all; close all;
alphap=0.15; % pass band attenuation in dB
alphas=0.9; % stop band attenuation in dB
wp=0.3*pi; % pass band frequency in radians
ws=0.5*pi; % stop band frequency in radians
[n,wn]=cheb1ord(wp/pi,ws/pi,alphap,alphas);
[b,a] = cheby1(n,alphap,wn);  % coefficients of designed filter
[h,w] = freqz(b,a);
subplot(2,1,1);plot(w/pi,20*log10(abs(h)));
xlabel('Normalized Frequency')
ylabel('gain in db')
title('magnitude response')
subplot(2,1,2);plot(w/pi,angle(h));
xlabel('Normalized Frequency')
ylabel('phase in radians')
title('phase response')
```

Output:

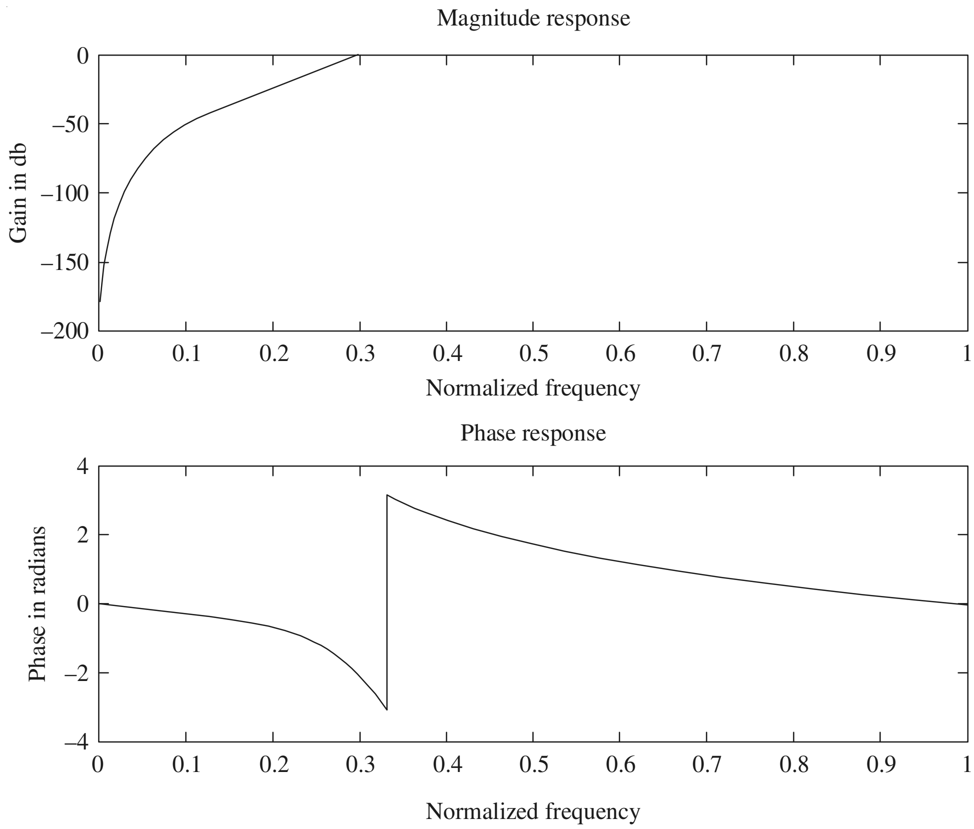


Program 8.6

```
% Chebyshev filter high-pass type-1
clc; clear all; close all;
alphap=1; % pass band attenuation in dB
alphas=15; % stop band attenuation in dB
wp=0.3*pi; % pass band frequency in radians
ws=0.2*pi; % stop band frequency in radians
[n,wn]=cheb1ord(wp/pi,ws/pi,alphap,alphas);
[b,a] = cheby1(n,alphap,wn,'high'); % coefficients of designed filter
[h,w] = freqz(b,a);
subplot(2,1,1),plot(w/pi,20*log10(abs(h)));
xlabel('Normalized Frequency')
ylabel('gain in db')
title('magnitude response')
```

```
subplot(2,1,2),plot(w/pi,angle(h));
xlabel('Normalized Frequency')
ylabel('phase in radians')
title('phase response')
```

Output:

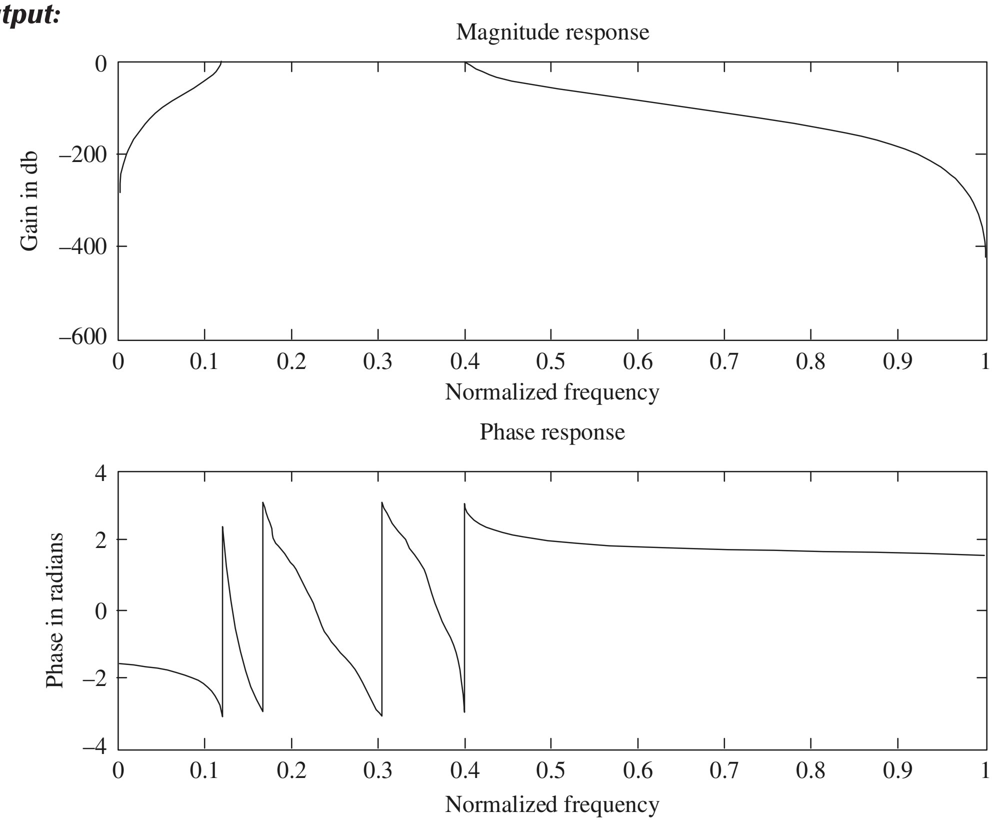


Program 8.7

```
% Chebyshev band pass filter type-1
clc; clear all; close all;
Wp = [60 200]/500;
Ws = [50 250]/500;
alphap = 3; % pass band attenuation in dB
alphas = 40; % stop band attenuation in dB
[n,Wp] = cheb1ord(Wp,Ws,alphap,alphas);
[b,a] = cheby1(n,alphap,Wp);
```

```
[h,w] = freqz(b,a);
subplot(2,1,1);plot(w/pi,20*log10(abs(h)));
xlabel('Normalized Frequency')
ylabel('gain in db')
title('magnitude response')
subplot(2,1,2);plot(w/pi,angle(h));
xlabel('Normalized Frequency')
ylabel('phase in radians')
title('phase response')
```

Output:

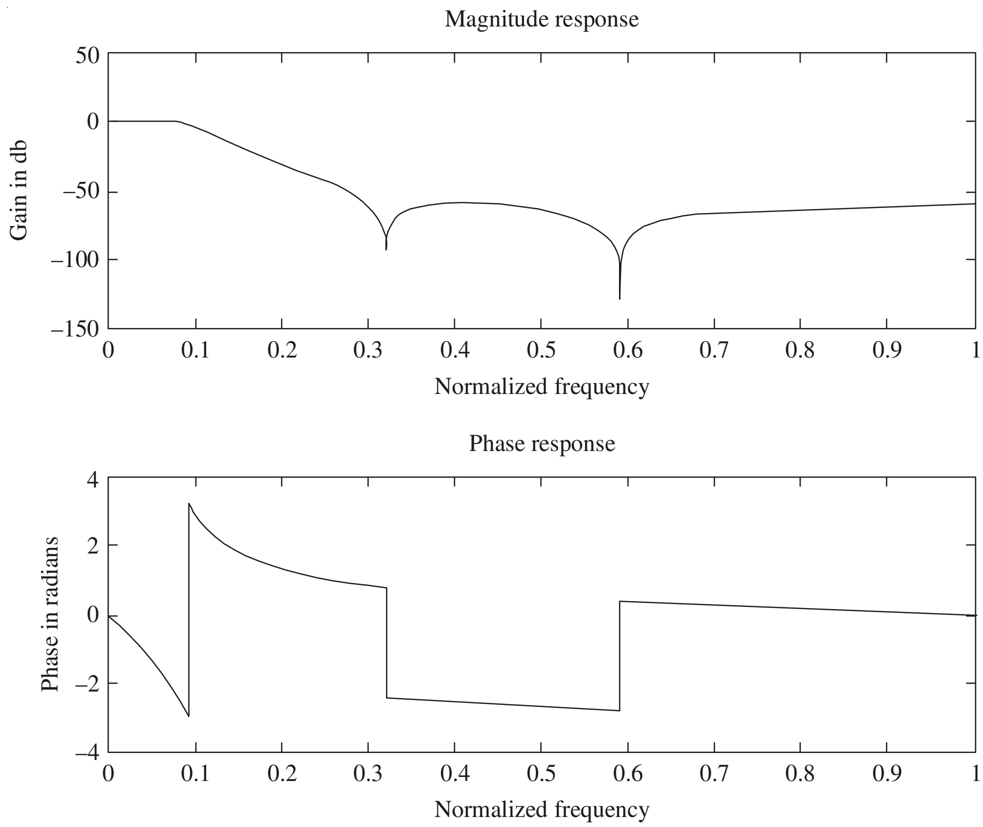


Program 8.8

```
% Chebyshev low-pass filter type-2
clc; clear all; close all;
Wp = 40/500; % pass band frequency in radians
Ws = 150/500;  % stop band frequency in radians
```

```
alphap = 3; % pass band attenuation in dB
alphas = 60; % stop band attenuation in dB
[n,Ws] = cheb2ord(Wp,Ws,alphap,alphas);
[b,a] = cheby2(n,alphas,Ws);
[h,w]=freqz(b,a);
subplot(2,1,1),plot(w/pi,20*log10(abs(h)));
xlabel('Normalized Frequency')
ylabel('gain in db')
title('magnitude response')
subplot(2,1,2),plot(w/pi,angle(h));
xlabel('Normalized Frequency')
ylabel('phase in radians')
title('phase response')
```

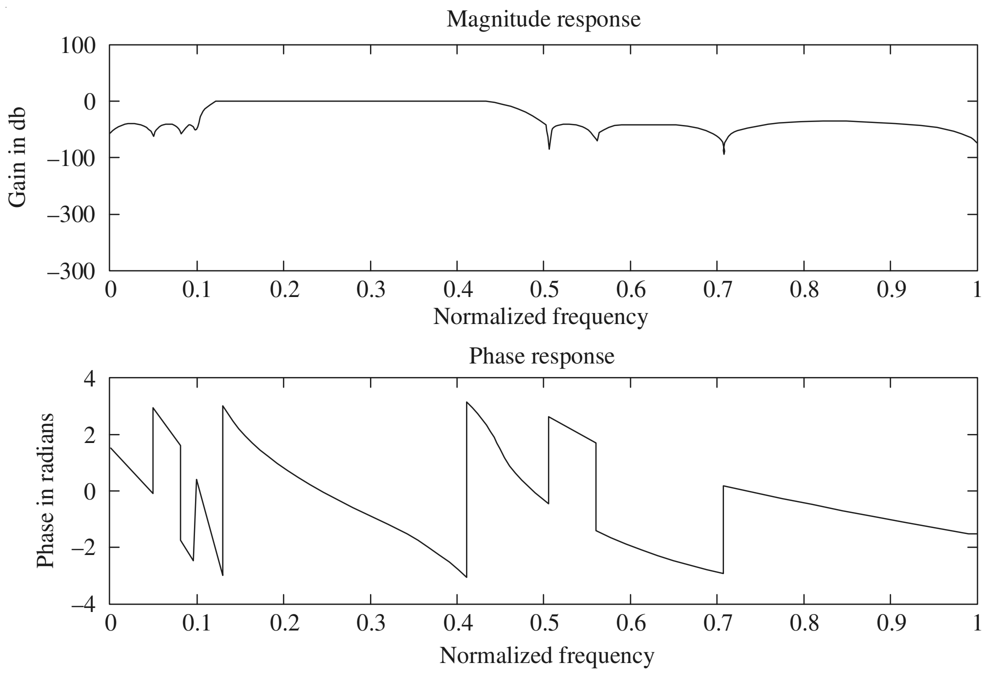
Output:

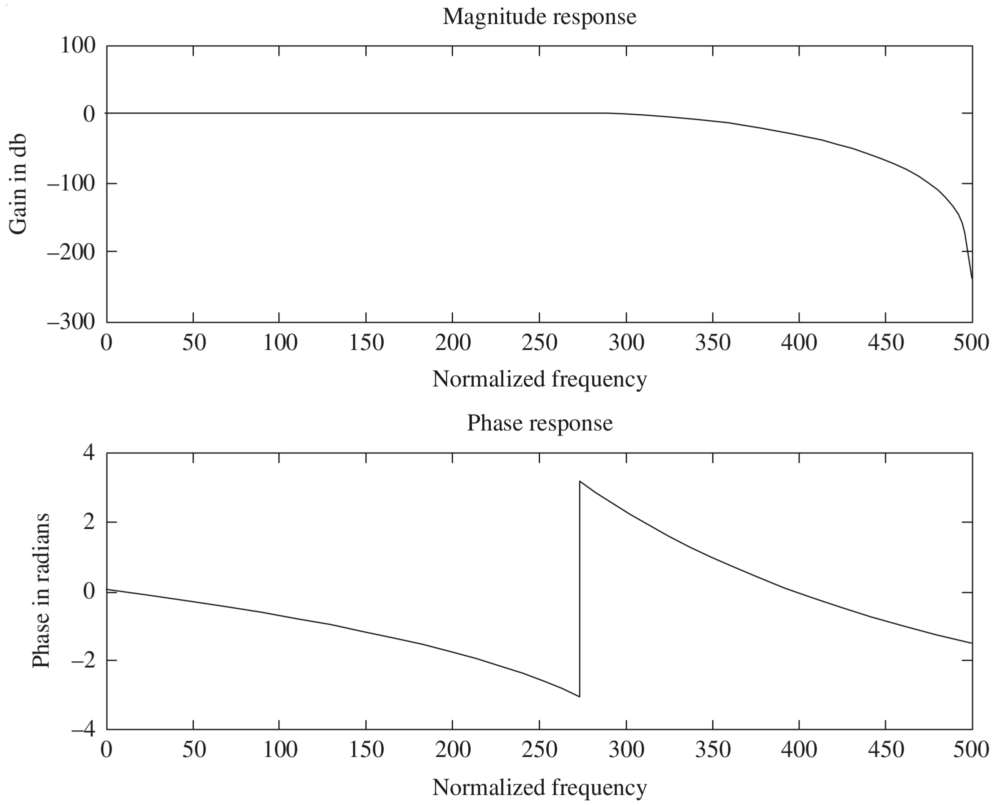


Program 8.9

```
% Chebyshev band pass filter type-2
clc; clear all; close all;
Wp = [60 200]/500; % pass band frequency in radians
Ws = [50 250]/500; % stop band frequency in radians
alphap = 3; % pass band attenuation in dB
alphas = 40; % stop band attenuation in dB
[n,Ws] = cheb2ord(Wp,Ws,alphap,alphas);
[b,a] = cheby2(n,alphas,Ws);
[h,w]=freqz(b,a);
subplot(2,1,1);plot(w/pi,20*log10(abs(h)));
xlabel('Normalized Frequency')
ylabel('gain in db')
title('magnitude response')
subplot(2,1,2);plot(w/pi,angle(h));
xlabel('Normalized Frequency')
ylabel('phase in radians')
title('phase response')
```

Output:




Magnitude response
Gain in db
100
0
–100
–200
–300
0 50 100 150 200 250 300 350 400 450 500
Normalized frequency
Phase response
Phase in radians
4
2
0
–2
–4
0 50 100 150 200 250 300 350 400 450 500
Normalized frequency

9

FIR Filters

9.1 INTRODUCTION

A filter is a frequency selective system. Digital filters are classified as finite duration unit impulse response (FIR) filters or infinite duration unit impulse response (IIR) filters, depending on the form of the unit impulse response of the system. In the FIR system, the impulse response sequence is of finite duration, i.e., it has a finite number of non-zero terms. The IIR system has an infinite number of non-zero terms, i.e., its impulse response sequence is of infinite duration. IIR filters are usually implemented using recursive structures (feedback-poles and zeros) and FIR filters are usually implemented using non-recursive structures (no feedback-only zeros). The response of the FIR filter depends only on the present and past input samples, whereas for the IIR filter, the present response is a function of the present and past values of the excitation as well as past values of the response.

The following are the main advantages of FIR filters over IIR filters:

1. FIR filters are always stable.
2. FIR filters with exactly linear phase can easily be designed.
3. FIR filters can be realized in both recursive and non-recursive structures.
4. FIR filters are free of limit cycle oscillations, when implemented on a finite word length digital system.
5. Excellent design methods are available for various kinds of FIR filters.

The disadvantages of FIR filters are as follows:

1. The implementation of narrow transition band FIR filters is very costly, as it requires considerably more arithmetic operations and hardware components such as multipliers, adders and delay elements.
2. Memory requirement and execution time are very high.

FIR filters are employed in filtering problems where linear phase characteristics within the pass band of the filter is required. If this is not required, either an FIR or an IIR filter may be employed. An IIR filter has lesser number of side lobes in the stop band than an FIR filter with the same number of parameters. For this reason if some phase distortion is tolerable, an IIR filter is preferable. Also, the implementation of an IIR filter involves fewer parameters, less memory requirements and lower computational complexity.

9.2 CHARACTERISTICS OF FIR FILTERS WITH LINEAR PHASE

The transfer function of a FIR causal filter is given by

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

where $h(n)$ is the impulse response of the filter. The frequency response [Fourier transform of $h(n)$] is given by

$$H(\omega) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$$

which is periodic in frequency with period 2π, i.e.,

$$H(\omega) = H(\omega + 2k\pi), \qquad k = 0, 1, 2, \ldots$$

Since $H(\omega)$ is complex it can be expressed as

$$H(\omega) = \pm \left| H(\omega) \right| e^{j\theta(\omega)}$$

where $\left| H(\omega) \right|$ is the magnitude response and $\theta(\omega)$ is the phase response.

We define the phase delay τ_p and group delay τ_g of a filter as:

$$\tau_p = -\frac{\theta(\omega)}{\omega} \text{ and } \tau_g = -\frac{d\theta(\omega)}{d\omega}$$

For FIR filters with linear phase, we can define

$$\theta(\omega) = -\alpha\omega \qquad -\pi \le \omega \le \pi$$

where α is constant phase delay in samples.

$$\tau_g = -\frac{d\theta(\omega)}{d\omega} = -\frac{d}{d\omega}(-\alpha\omega) = \alpha \text{ and } \tau_p = -\frac{\theta(\omega)}{\omega} = \frac{\alpha\omega}{\omega} = \alpha$$

i.e. $\tau_p = \tau_g = \alpha$ which means that α is independent of frequency.

We have

$$\sum_{n=0}^{N-1} h(n)\, e^{-j\omega n} = \pm \left| H(\omega) \right| e^{j\theta(\omega)}$$

i.e.
$$\sum_{n=0}^{N-1} h(n)[\cos \omega n - j \sin \omega n] = \pm |H(\omega)| [\cos \theta(\omega) + j \sin \theta(\omega)]$$

This gives us

$$\sum_{n=0}^{N-1} h(n) \cos \omega n = \pm |H(\omega)| \cos \theta(\omega)$$

and
$$-\sum_{n=0}^{N-1} h(n) \sin \omega n = \pm |H(\omega)| \sin \theta(\omega)$$

Therefore,
$$\frac{\sum_{n=0}^{N-1} h(n) \sin \omega n}{\sum_{n=0}^{N-1} h(n) \cos \omega n} = \frac{\sin \theta(\omega)}{\cos \theta(\omega)} = \frac{\sin \alpha\omega}{\cos \alpha\omega}$$

i.e.
$$\sum_{n=0}^{N-1} h(n) [\sin \omega n \cos \alpha\omega - \cos \omega n \sin \alpha\omega] = 0$$

i.e.
$$\sum_{n=0}^{N-1} h(n) \sin (\alpha - n)\omega = 0$$

This will be zero when

$$h(n) = h(N-1-n) \text{ and } \alpha = \frac{N-1}{2}, \qquad \text{for } 0 \le n \le N-1$$

This shows that FIR filters will have constant phase and group delays when the impulse response is symmetrical about $\alpha = (N-1)/2$.

The impulse response satisfying the symmetry condition $h(n) = h(N-1-n)$ for odd and even values of N is shown in Figure 9.1. When $N = 9$, the centre of symmetry of the sequence occurs at the fourth sample and when $N = 8$, the filter delay is $3\frac{1}{2}$ samples.

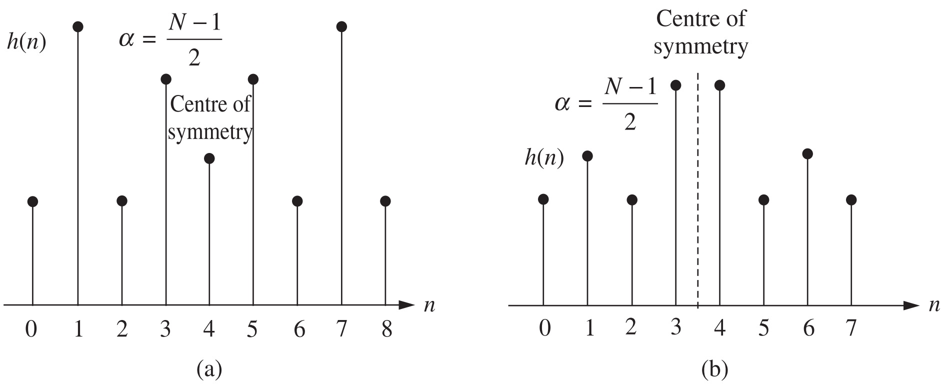


Figure 9.1 Impulse response sequence of symmetrical sequences for (a) N odd (b) N even.

If only constant group delay is required and not the phase delay, we can write

$$\theta(\omega) = \beta - \alpha\omega$$

Now, we have

$$H(\omega) = \pm|H(\omega)|e^{j(\beta-\alpha\omega)}$$

i.e. $$\sum_{n=0}^{N-1} h(n)e^{-j\omega n} = \pm|H(\omega)|e^{j(\beta-\alpha\omega)}$$

i.e. $$\sum_{n=0}^{N-1} h(n)[\cos\omega n - j\sin\omega n] = \pm|H(\omega)|[\cos(\beta-\alpha\omega) + j\sin(\beta-\alpha\omega)]$$

This gives

$$\sum_{n=0}^{N-1} h(n)\cos\omega n = \pm|H(\omega)|\cos(\beta-\alpha\omega)$$

and $$-\sum_{n=0}^{N-1} h(n)\sin\omega n = \pm|H(\omega)|\sin(\beta-\alpha\omega)$$

$\therefore$ $$-\frac{\sum_{n=0}^{N-1} h(n)\sin\omega n}{\sum_{n=0}^{N-1} h(n)\cos\omega n} = \frac{\sin(\beta-\alpha\omega)}{\cos(\beta-\alpha\omega)}$$

Cross multiplying and rearranging, we get

$$\sum_{n=0}^{N-1} h(n)[\cos\omega n\sin(\beta-\alpha\omega) + \sin\omega n\cos(\beta-\alpha\omega)] = 0$$

i.e. $$\sum_{n=0}^{N-1} h(n)\sin[\beta-(\alpha-n)\omega] = 0$$

If $\beta = \dfrac{\pi}{2}$, the above equation can be written as:

$$\sum_{n=0}^{N-1} h(n)\cos(\alpha-n)\,\omega = 0$$

This equation will be satisfied when

$$h(n) = -h(N-1-n) \text{ and } \alpha = \frac{N-1}{2}$$

This shows that FIR filters have constant group delay τ_g and not constant phase delay when the impulse response is antisymmetrical about $\alpha = (N-1)/2$.

The impulse response satisfying the antisymmetry condition is shown in Figure 9.2. When $N = 9$, the centre of antisymmetry occurs at fourth sample and when $N = 8$, the centre of antisymmetry occurs at $3\frac{1}{2}$ samples. From Figure 9.2, we find that $h[(N-1)/2] = 0$ for antisymmetric odd sequence.

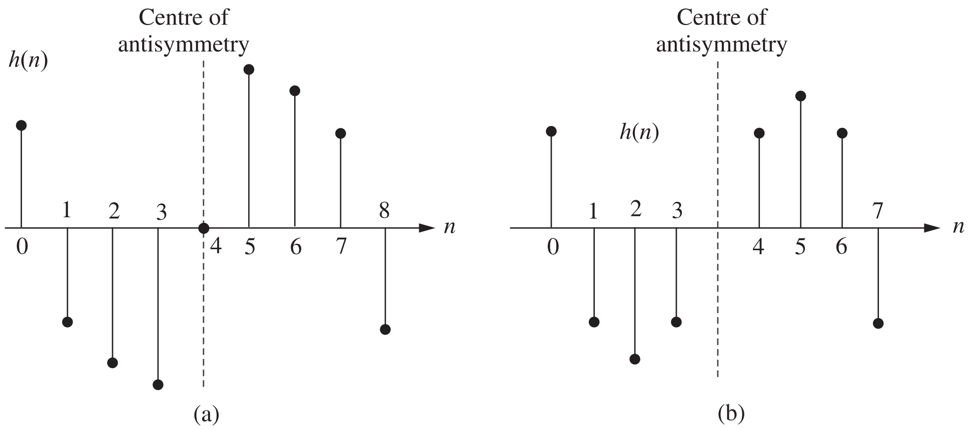


Figure 9.2 Impulse response sequence of antisymmetric sequences for (a) N odd (b) N even.

EXAMPLE 9.1 The length of an FIR filter is 7. If this filter has a linear phase, show that the equation $\sum_{n=0}^{N-1} h(n)\sin(\alpha - n)\omega = 0$ is satisfied.

Solution: The length of the filter is 7. Therefore, for linear phase,

$$\alpha = \frac{N-1}{2} = \frac{7-1}{2} = 3$$

The condition for symmetry when N is odd, is $h(n) = h(N-1-n)$.

Therefore, the filter coefficients are $h(0) = h(6)$, $h(1) = h(5)$, $h(2) = h(4)$ and $h(3)$. Therefore,

$$\begin{aligned}\sum_{n=0}^{N-1} h(n)\sin(\alpha - n)\omega &= \sum_{n=0}^{6} h(n)\sin(3-n)\omega \\ &= h(0)\sin 3\omega + h(1)\sin 2\omega + h(2)\sin\omega + h(3)\sin 0 + h(4)\sin(-\omega) \\ &\quad + h(5)\sin(-2\omega) + h(6)\sin(-3\omega) \\ &= 0\end{aligned}$$

Hence, the equation $\sum_{n=0}^{N-1} h(n)\sin(\alpha - n)\omega = 0$ is satisfied.

EXAMPLE 9.2 The following transfer function characterizes an FIR filter (N = 9). Determine the magnitude response and show that the phase and group delays are constant.

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

Solution: The transfer function of the filter is given by

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7} + h(8)z^{-8}$$

The phase delay $\alpha = \dfrac{N-1}{2} = \dfrac{9-1}{2} = 4$. Since α = 4, the transfer function can be expressed as:

$$H(z) = z^{-4}\,[h(0)\,z^4 + h(1)\,z^3 + h(2)\,z^2 + h(3)\,z^1 + h(4)\,z^0 + h(5)\,z^{-1} + h(6)\,z^{-2} + h(7)\,z^{-3} + h(8)\,z^{-4}]$$

Since $h(n) = h(N - 1 - n)$

$$H(z) = z^{-4}[h(0)\,(z^4 + z^{-4}) + h(1)\,(z^3 + z^{-3}) + h(2)\,(z^2 + z^{-2}) + h(3)\,(z + z^{-1}) + h(4)]$$

The frequency response is obtained by replacing z with $e^{j\omega}$.

$$H(\omega) = e^{-j4\omega}[h(0)\,[e^{j4\omega} + e^{-j4\omega}] + h(1)\,[e^{j3\omega} + e^{-j3\omega}] + h(2)\,[e^{j2\omega} + e^{-j2\omega}] + h(3)\,[e^{j\omega} + e^{-j\omega}] + h(4)]$$

$$= e^{-j4\omega}\left[h(4) + 2\sum_{n=0}^{3} h(n)\cos(4-n)\,\omega\right]$$

$$= e^{-j4\omega}\left|H(\omega)\right|$$

where $\left|H(\omega)\right|$ is the magnitude response and $\theta(\omega) = -5\omega$ is the phase response. The phase delay τ_p and group delay τ_g are given by

$$\tau_p = -\frac{\theta(\omega)}{\omega} = 5 \text{ and } \tau_g = \frac{d(\theta(\omega))}{d\omega} = -\frac{d(-5\omega)}{d\omega} = 5$$

Thus, the phase delay and the group delay are the same and are constants.

9.3 FREQUENCY RESPONSE OF LINEAR PHASE FIR FILTERS

The frequency response of the filter is the Fourier transform of its impulse response. If $h(n)$ is the impulse response of the system, then the frequency response of the system is denoted by $H(e^{j\omega})$ or $H(\omega)$. $H(\omega)$ is a complex function of frequency ω and so it can be expressed as magnitude function $|H(\omega)|$ and phase function $\angle H(\omega)$.

Depending on the value of N (odd or even) and the type of symmetry of the filter impulse response sequence (symmetric or antisymmetric), there are following four possible types of impulse response for linear phase FIR filters.

1. Symmetrical impulse response when N is odd.
2. Symmetrical impulse response when N is even.
3. Antisymmetric impulse response when N is odd.
4. Antisymmetric impulse response when N is even.

9.3.1 Frequency Response of Linear Phase FIR Filter when Impulse Response is Symmetrical and *N* is Odd

Let $h(n)$ be the impulse response of the system. The frequency response of the system $H(\omega)$ is given as:

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n}$$

Since the impulse response of the FIR filter has only N samples, the limits of summation can be changed to $n = 0$ to $N-1$.

$$\therefore \qquad H(\omega) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$$

When N is odd number, the symmetrical impulse response will have the centre of symmetry at $n = (N-1)/2$. Hence $H(\omega)$ is expressed as:

$$H(\omega) = \sum_{n=0}^{(N-3)/2} h(n)\, e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=(N+1)/2}^{N-1} h(n)\, e^{-j\omega n}$$

Let $m = N - 1 - n$, $\qquad \therefore\ n = N - 1 - m$

When $n = \dfrac{N+1}{2}$, $\qquad m = (N-1) - \left(\dfrac{N+1}{2}\right) = \dfrac{N-3}{2}$

When $n = N - 1$, $\qquad m = (N-1) - (N-1) = 0$

Therefore,

$$H(\omega) = \sum_{n=0}^{(N-3)/2} h(n)\, e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{m=0}^{(N-3)/2} h(N-1-m)\, e^{-j\omega(N-1-m)}$$

Replacing m by n, we get

$$H(\omega) = \sum_{n=0}^{(N-3)/2} h(n)\, e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{(N-3)/2} h(N-1-n)\, e^{-j\omega(N-1-n)}$$

For symmetrical impulse response, $h(n) = h(N-1-n)$.

Hence

$$H(\omega) = \sum_{n=0}^{(N-3)/2} h(n)\, e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{(N-3)/2} h(n)\, e^{-j\omega(N-1)-j\omega(-n)}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left\{ h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{(N-3)/2} h(n) \left[e^{-j\omega n + j\omega\left(\frac{N-1}{2}\right)} + e^{-j\omega(N-1) + j\omega\frac{N-1}{2} - j\omega(-n)} \right] \right\}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left\{ h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{(N-3)/2} h(n) \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left[(N-1) - \frac{N-1}{2} - n\right]} \right] \right\}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left\{ h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{(N-3)/2} h(n) \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)} \right] \right\}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left\{ h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{(N-3)/2} h(n)\, 2\cos\left[\left(\frac{N-1}{2} - n\right)\omega\right] \right\}$$

Let $k = \dfrac{N-1}{2} - n$, $\qquad \therefore\ n = \dfrac{N-1}{2} - k$

When $n = 0$, $\qquad k = \dfrac{N-1}{2}$

When $n = \dfrac{N-3}{2}$, $\qquad k = \dfrac{N-1}{2} - \dfrac{N-3}{2} = 1$

$$\therefore \qquad H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)} \left\{ h\left(\frac{N-1}{2}\right) + \sum_{k=1}^{(N-1)/2} 2h\left(\frac{N-1}{2} - k\right) \cos \omega k \right\}$$

Replacing k by n, we get

$$H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{(N-1)/2} 2h\left(\frac{N-1}{2} - n\right)\cos\omega n\right\}$$

The above equation for $H(\omega)$ is the frequency response of linear phase FIR filter when impulse response is symmetrical and N is odd.

The magnitude function of $H(\omega)$ is given by

$$|H(\omega)| = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{(N-1)/2} 2h\left(\frac{N-1}{2} - n\right)\cos\omega n$$

The phase function of $H(\omega)$ is given by

$$\angle H(\omega) = -\omega\left(\frac{N-1}{2}\right) = -\omega\alpha \qquad \text{where } \alpha = \frac{N-1}{2}$$

Figure 9.3(a) shows a symmetrical impulse response when $N = 9$ and Figure 9.3(b) shows the corresponding magnitude function of frequency response. From these figures it can be observed that the magnitude function of $H(\omega)$ is symmetric with $\omega = \pi$, when the impulse response is symmetric and N is odd number.

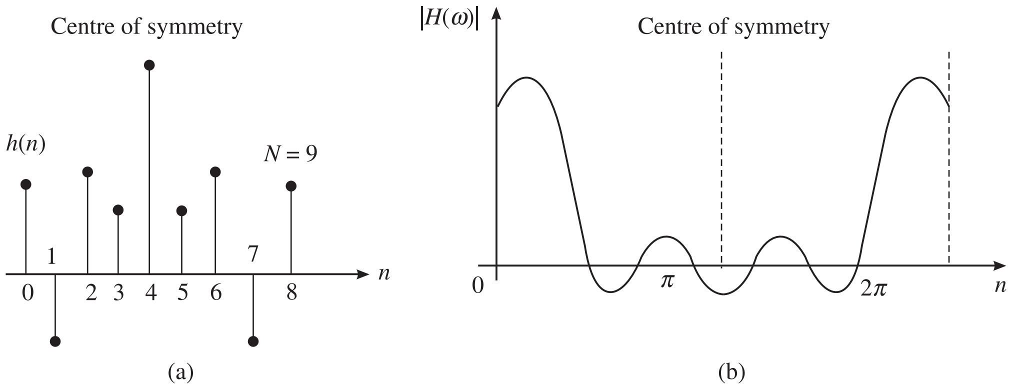


Figure 9.3 (a) Symmetrical impulse response, $N = 9$ (b) Magnitude function of $H(\omega)$.

9.3.2 Frequency Response of Linear Phase FIR Filter when Impulse Response is Symmetrical and *N* is Even

The Frequency response of FIR filter, with impulse response $h(n)$ of length N is:

$$H(\omega) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$$

For symmetrical impulse response with even number of samples (i.e. when N is even), the centre of symmetry lies between $n = (N/2) - 1$ and $n = N/2$. Hence $H(\omega)$ is expressed as:

$$H(\omega) = \sum_{n=0}^{(N/2)-1} h(n)\, e^{-j\omega n} + \sum_{n=N/2}^{N-1} h(n)\, e^{-j\omega n}$$

Let $m = N - 1 - n$, $\quad\therefore\quad n = N - 1 - m$

When $n = \dfrac{N}{2}$, $\quad m = N - 1 - \dfrac{N}{2} = \dfrac{N}{2} - 1$

When $n = N - 1$, $\quad m = N - 1 - (N - 1) = 0$

Therefore, the above equation for $H(\omega)$ can be written as:

$$H(\omega) = \sum_{n=0}^{(N/2)-1} h(n)\, e^{-j\omega n} + \sum_{m=0}^{(N/2)-1} h(N-1-m)\, e^{-j\omega(N-1-m)}$$

Replacing m by n, we get

$$H(\omega) = \sum_{n=0}^{(N/2)-1} h(n)\, e^{-j\omega n} + \sum_{n=0}^{(N/2)-1} h(N-1-n)\, e^{-j\omega(N-1-n)}$$

By the symmetry condition, $h(N - 1 - n) = h(n)$

Hence $H(\omega)$ can be written as:

$$H(\omega) = \sum_{n=0}^{(N/2)-1} h(n)\, e^{-j\omega n} + \sum_{n=0}^{(N/2)-1} h(n)\, e^{-j\omega(N-1) - j\omega(-n)}$$

$$= e^{-j\omega \frac{N-1}{2}} \left\{ \sum_{n=0}^{(N/2)-1} h(n) \left[e^{-j\omega n + j\omega \frac{N-1}{2}} + e^{-j\omega(-n) - j\omega(N-1) + j\omega \frac{N-1}{2}} \right] \right\}$$

$$= e^{-j\omega \frac{N-1}{2}} \left\{ \sum_{n=0}^{(N/2)-1} h(n) \left[e^{j\omega\left(\frac{N-1}{2} - n\right)} + e^{-j\omega\left(\frac{N-1}{2} - n\right)} \right] \right\}$$

$$= e^{-j\omega \frac{N-1}{2}} \left\{ \sum_{n=0}^{(N/2)-1} h(n)\, 2\cos\left(\omega\left(\frac{N-1}{2} - n \right) \right) \right\}$$

Let $k = \dfrac{N}{2} - n$, $\quad\therefore\quad n = \dfrac{N}{2} - k$

When $n = 0$, $\qquad k = \dfrac{N}{2}$

When $n = \dfrac{N}{2} - 1$, $\qquad k = \dfrac{N}{2} - \left(\dfrac{N}{2} - 1\right) = 1$

Therefore, the above expression for $H(\omega)$ becomes

$$H(\omega) = e^{-j\omega\frac{N-1}{2}}\left\{\sum_{k=1}^{N/2} 2h\left(\frac{N}{2}-k\right)\cos\omega\left(k-\frac{1}{2}\right)\right\}$$

On replacing k by n, we get

$$H(\omega) = e^{-j\omega\frac{N-1}{2}}\left\{\sum_{n=1}^{N/2} 2h\left(\frac{N}{2}-n\right)\cos\omega\left(n-\frac{1}{2}\right)\right\}$$

This is the expression for frequency response of linear phase FIR filter when impulse response is symmetrical and N is even. The magnitude function of $H(\omega)$ is given by

$$|H(\omega)| = \left[\sum_{n=1}^{N/2} 2h\left(\frac{N}{2}-n\right)\cos\omega\left(n-\frac{1}{2}\right)\right]$$

The phase function of $H(\omega)$ is given by

$$\angle H(\omega) = -\omega\left(\frac{N-1}{2}\right) = -\omega\alpha \qquad \text{where } \alpha = \frac{N-1}{2}$$

Figure 9.4(a) shows a symmetrical impulse response when $N = 8$, and Figure 9.4(b) shows the corresponding magnitude function of frequency response. From these figures it can be observed that the magnitude function of $H(\omega)$ is antisymmetric with $\omega = \pi$, when impulse response is symmetric and N is even number.

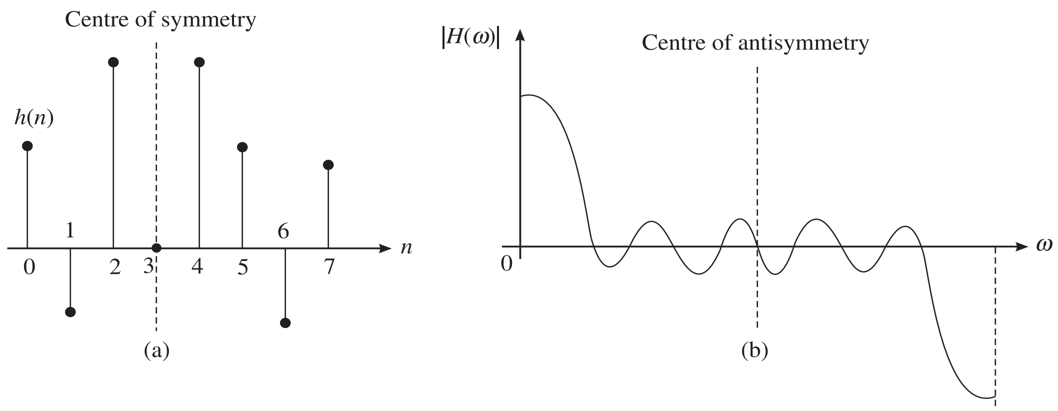


Figure 9.4 (a) Symmetrical impulse response, $N=8$, (b) Magnitude function of $H(\omega)$.

9.3.3 Frequency Response of Linear Phase FIR Filter when Impulse Response is Antisymmetric and *N* is Odd

The frequency response of linear phase FIR filter with impulse response $h(n)$ of length N is:

$$H(\omega) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$$

The impulse response is antisymmetric with centre of antisymmetry at $n = (N - 1)/2$. Also $h[(N - 1)/2] = 0$. Hence $H(\omega)$ can be expressed as:

$$H(\omega) = \sum_{n=0}^{(N-3)/2} h(n)\, e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=(N+1)/2}^{N-1} h(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{(N-3)/2} h(n)\, e^{-j\omega n} + \sum_{n=(N+1)/2}^{N-1} h(n)\, e^{-j\omega n}$$

Let $m = N - 1 - n$, $\quad\therefore\ n = N - 1 - m$

When $n = \dfrac{N+1}{2}$, $\quad m = N - 1 - \left(\dfrac{N+1}{2}\right) = \dfrac{N-3}{2}$

When $n = N - 1$, $\quad m = N - 1 - (N - 1) = 0$

$$\therefore \qquad H(\omega) = \sum_{n=0}^{(N-3)/2} h(n)\, e^{-j\omega n} + \sum_{m=0}^{(N-3)/2} h(N-1-m)\, e^{-j\omega(N-1-m)}$$

On replacing m by n, we get

$$H(\omega) = \sum_{n=0}^{(N-3)/2} h(n)\, e^{-j\omega n} + \sum_{n=0}^{(N-3)/2} h(N-1-n)\, e^{-j\omega(N-1-n)}$$

For antisymmetric impulse response, $h(N - 1 - n) = -h(n)$. Hence, the above equation for $H(\omega)$ can be written as:

$$H(\omega) = \sum_{n=0}^{(N-3)/2} h(n)\, e^{-j\omega n} + \sum_{n=0}^{(N-3)/2} - h(n)\, e^{-j\omega(-n)-j\omega(N-1)}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left\{ \sum_{n=0}^{(N-3)/2} h(n) \left[e^{-j\omega n + j\omega\left(\frac{N-1}{2}\right)} - e^{-j\omega(-n) - j\omega(N-1) + j\omega\left(\frac{N-1}{2}\right)} \right] \right\}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left\{ \sum_{n=0}^{(N-3)/2} h(n) \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} - e^{-j\omega\left(\frac{N-1}{2}-n\right)} \right] \right\}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left\{ \sum_{n=0}^{(N-3)/2} h(n)\, 2j \sin\left(\omega\left(\frac{N-1}{2} - n \right)\right) \right\}$$

Since $j = e^{j\pi/2}$

$$H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)} \left[\sum_{n=0}^{(N-3)/2} 2\, h(n)\, e^{j\frac{\pi}{2}} \sin\left(\omega\left(\frac{N-1}{2} - n \right)\right) \right]$$

$$= e^{j\left(\frac{\pi}{2} - \omega\frac{N-1}{2}\right)} \left[\sum_{n=0}^{(N-3)/2} 2h(n) \sin\left(\omega\left(\frac{N-1}{2} - n \right)\right) \right]$$

Let $k = \dfrac{N-1}{2} - n,$ $\qquad \therefore \; n = \dfrac{N-1}{2} - k$

When $n = 0,$ $\qquad k = \dfrac{N-1}{2}$

When $n = \dfrac{N-3}{2},$ $\qquad k = \dfrac{N-1}{2} - \dfrac{N-3}{2} = 1$

$$\therefore \qquad H(\omega) = e^{j\left(\frac{\pi}{2} - \omega\left(\frac{N-1}{2}\right)\right)} \left[\sum_{k=1}^{(N-1)/2} 2h\left(\frac{N-1}{2} - k \right) \sin \omega k \right]$$

Replacing k by n, we get

$$H(\omega) = e^{j\left(\frac{\pi}{2} - \omega\left(\frac{N-1}{2}\right)\right)} \left[\sum_{n=1}^{(N-1)/2} 2h\left(\frac{N-1}{2} - n \right) \sin \omega n \right]$$

This is the equation for frequency response of linear phase FIR filter when impulse response is antisymmetric and n odd. The magnitude function is given by

$$|H(\omega)| = \sum_{n=1}^{(N-1)/2} 2h\left(\frac{N-1}{2} - n \right) \sin \omega n$$

The phase function is given by

$$\angle H(\omega) = \frac{\pi}{2} - \omega\left(\frac{N-1}{2}\right) = \beta - \alpha\omega$$

where $$\beta = \frac{\pi}{2} \text{ and } \alpha = \frac{N-1}{2}$$

Figure 9.5(a) shows an antisymmetric impulse response when $N = 9$, and Figure 9.5(b) shows the corresponding magnitude function of frequency response. From these figures, it can be observed that the magnitude function is antisymmetic with $\omega = \pi$, when the impulse response is antisymmetric and N is odd.

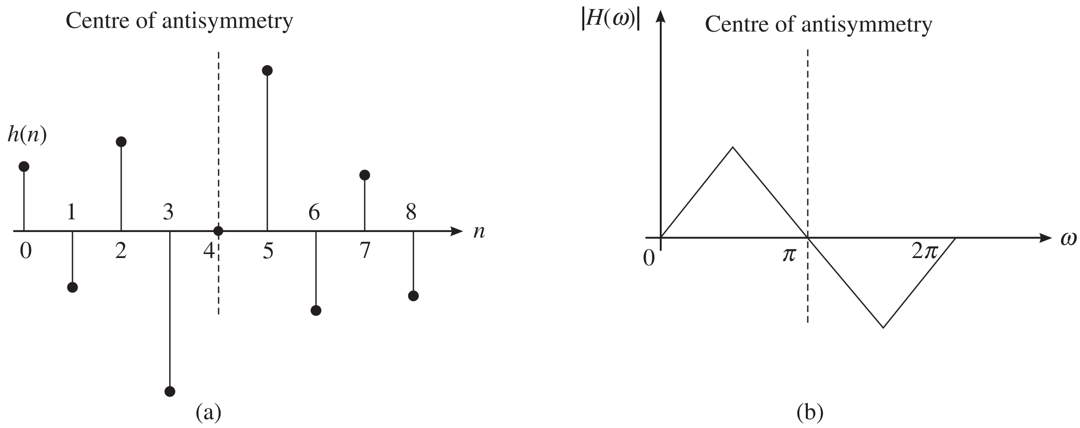


Figure 9.5 (a) Antisymmetric impulse response for $N = 9$, (b) Magnitude function of $H(\omega)$.

9.3.4 Frequency Response of Linear Phase FIR Filter when Impulse Response is Antisymmetric and *N* is Even

The frequency response of linear phase FIR filter with impulse response $h(n)$ of length N is:

$$H(\omega) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$$

The impulse response $h(n)$ is antisymmetric with centre of antisymmetry in between $n = (N/2) - 1$ and $n = (N/2)$. Hence $H(\omega)$ can be expressed as:

$$H(\omega) = \sum_{n=0}^{(N/2)-1} h(n)\, e^{-j\omega n} + \sum_{n=N/2}^{N-1} h(n)\, e^{-j\omega n}$$

Let $m = N - 1 - n$, $\quad \therefore\ n = N - 1 - m$

When $n = \dfrac{N}{2}$, $\quad m = N - 1 - \dfrac{N}{2} = \dfrac{N}{2} - 1$

When $n = N - 1$, $\quad m = N - 1 - (N - 1) = 0$

$$\therefore \qquad H(\omega) = \sum_{n=0}^{(N/2)-1} h(n)\, e^{-j\omega n} + \sum_{m=0}^{(N/2)-1} h(N-1-m)\, e^{-j\omega(N-1-m)}$$

Replacing m by n, we have

$$H(\omega) = \sum_{n=0}^{(N/2)-1} h(n)\, e^{-j\omega n} + \sum_{n=0}^{(N/2)-1} h(N-1-n)\, e^{-j\omega(N-1-n)}$$

For antisymmetric impulse response, $h(N-1-n) = -h(n)$. Hence the above equation for $H(\omega)$ can be written as:

$$H(\omega) = \sum_{n=0}^{(N/2)-1} h(n)\, e^{-j\omega n} + \sum_{n=0}^{(N/2)-1} -h(n)\, e^{-j\omega(-n)-j\omega(N-1)}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left[\sum_{n=0}^{(N/2)-1} h(n)\left[e^{-j\omega n + j\omega\left(\frac{N-1}{2}\right)} - e^{-j\omega(-n)-j\omega(N-1)+j\omega\left(\frac{N-1}{2}\right)}\right]\right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left[\sum_{n=0}^{(N/2)-1} h(n)\left[e^{j\omega\left(\frac{N-1}{2}-n\right)} - e^{-j\omega\left(\frac{N-1}{2}-n\right)}\right]\right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left[\sum_{n=0}^{(N/2)-1} h(n)\, 2j \sin\left(\omega\left(\frac{N-1}{2}-n\right)\right)\right]$$

Replacing j by $e^{j(\pi/2)}$, we have

$$H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left[\sum_{n=0}^{(N/2)-1} 2h(n)\, e^{j(\pi/2)} \sin\left(\omega\left(\frac{N-1}{2}-n\right)\right)\right]$$

$$= e^{j\left(\frac{\pi}{2}-\omega\frac{N-1}{2}\right)}\left[\sum_{n=0}^{(N/2)-1} 2h(n) \sin\left(\omega\left(\frac{N-1}{2}-n\right)\right)\right]$$

Let $k = \dfrac{N}{2} - n$, $\qquad \therefore \; n = \dfrac{N}{2} - k$

When $n = 0$, $\qquad k = \dfrac{N}{2}$

When $n = \dfrac{N}{2} - 1$, $\qquad k = \dfrac{N}{2} - \left(\dfrac{N}{2} - 1\right) = 1$

$$\therefore \qquad H(\omega) = e^{j\left(\frac{\pi}{2} - \omega\frac{N-1}{2}\right)}\left[\sum_{k=1}^{N/2} 2h\left(\frac{N}{2}-k\right)\sin\left(\omega\left(k-\frac{1}{2}\right)\right)\right]$$

Replacing k by n, we get

$$H(\omega) = e^{j\left(\frac{\pi}{2} - \omega\frac{N-1}{2}\right)}\left[\sum_{n=1}^{N/2} 2h\left(\frac{N}{2}-n\right)\sin\left(\omega\left(n-\frac{1}{2}\right)\right)\right]$$

This is the equation for the frequency response of linear phase FIR filter when impulse response is antisymmetric and N is even.

The magnitude function is given by

$$\left|H(\omega)\right| = \sum_{n=1}^{N/2} 2h\left(\frac{N}{2}-n\right)\sin\left(\omega\left(n-\frac{1}{2}\right)\right)$$

The phase function is given by

$$\angle H(\omega) = \frac{\pi}{2} - \omega\frac{N-1}{2} = \beta - \alpha\omega$$

where $\beta = \dfrac{\pi}{2}$ and $\alpha = \dfrac{N-1}{2}$.

Figure 9.6(a) shows an antisymmetric impulse response when $N = 8$, and Figure 9.6(b) shows its corresponding magnitude function of frequency response. From Figure 9.6, it can be observed that the magnitude function of $H(\omega)$ is symmetric with $\omega = \pi$ when the impulse response is antisymmetric and N is even number.

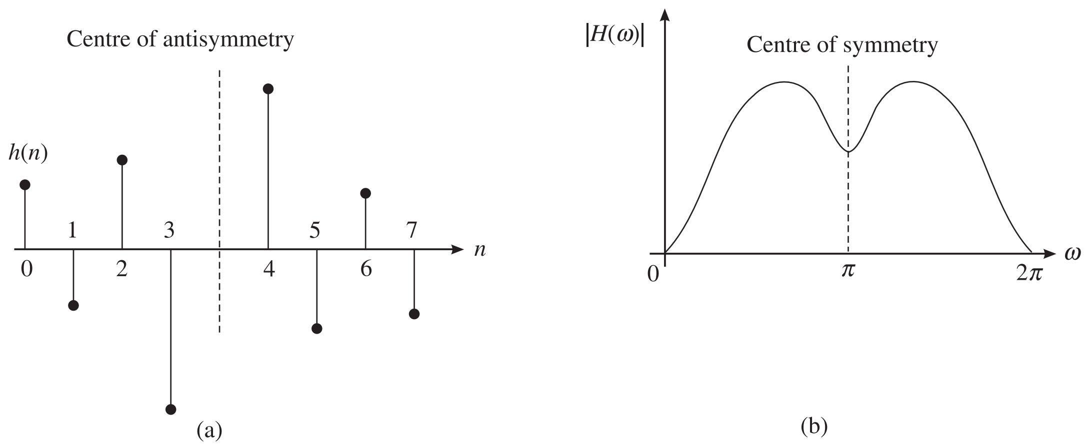


Figure 9.6 (a) Antisymmetrical impulse response for $N = 8$, (b) Magnitude function of $H(\omega)$.

9.4 DESIGN TECHNIQUES FOR FIR FILTERS

The well known methods of designing FIR filters are as follows:

1. Fourier series method
2. Window method
3. Frequency sampling method
4. Optimum filter design

In Fourier series method, the desired frequency response $H_d(\omega)$ is converted to a Fourier series representation by replacing ω by $2\pi fT$, where T is the sampling time. Then using this expression, the Fourier coefficients are evaluated by taking inverse Fourier transform of $H_d(\omega)$, which is the desired impulse response of the filter $h_d(n)$. The Z-transform of $h_d(n)$ gives $H_d(z)$ which is the transfer function of the desired filter. The $H_d(z)$ obtained from $H_d(n)$ will be a transfer function of unrealizable non causal digital filter of infinite duration. A finite duration impulse response $h(n)$ can be obtained by truncating the infinite duration impulse response $h_d(n)$ to N-samples. Now, take Z-transform of $h(n)$ to get $H(z)$. This $H(z)$ corresponds to a non-causal filter. So multiply this $H(z)$ by $z^{-(N-1)/2}$ to get the transfer function of realizable causal filter of finite duration.

In window method, we begin with the desired frequency response specification $H_d(\omega)$ and determine the corresponding unit sample response $h_d(n)$. The $h_d(n)$ is given by the inverse Fourier transform of $H_d(\omega)$. The unit sample response $h_d(n)$ will be an infinite sequence and must be truncated at some point, say, at $n = N - 1$ to yield an FIR filter of length N. The truncation is achieved by multiplying $h_d(n)$ by a window sequence $w(n)$. The resultant sequence will be of length N and can be denoted by $h(n)$. The Z-transform of $h(n)$ will give the filter transfer function $H(z)$. There have been many windows proposed like Rectangular window, Triangular window, Hanning window, Hamming window, Blackman wndow and Kaiser window that approximate the desired characteristics.

In frequency sampling method of filter design, we begin with the desired frequency response specification $H_d(\omega)$, and it is sampled at N-points to generate a sequence $\tilde{H}(k)$ which corresponds to the DFT coefficients. The N-point IDFT of the sequence $\tilde{H}(k)$ gives the impulse response of the filter $h(n)$. The Z-transform of $h(n)$ gives the transfer function $H(z)$ of the filter.

In optimum filter design method, the weighted approximation error between the desired frequency response and the actual frequency response is spread evenly across the pass band and evenly across the stop band of the filter. This results in the reduction of maximum error. The resulting filter have ripples in both the pass band and the stop band. This concept of design is called optimum equiripple design criterion.

The various steps in designing FIR filters are as follows:

1. Choose an ideal(desired) frequency response, $H_d(\omega)$.
2. Take inverse Fourier transform of $H_d(\omega)$ to get $h_d(n)$ or sample $H_d(\omega)$ at finite number of points (N-points) to get $\tilde{H}(k)$.
3. If $h_d(n)$ is determined, then convert the infinite duration $h_d(n)$ to a finite duration $h(n)$ (usually $h(n)$ is an N-point sequence) or if $\tilde{H}(k)$ is determined, then take N-point inverse DFT to get $h(n)$.

4. Take Z-transform of $h(n)$ to get $H(z)$, where $H(z)$ is the transfer function of the digital filter.
5. Choose a suitable structure and realize the filter.

9.5 FOURIER SERIES METHOD OF DESIGN OF FIR FILTERS

The frequency response of a digital filter is periodic with period equal to the sampling frequency. From Fourier series analysis, we know that any periodic function can be expressed as a linear combination of complex exponentials. Therefore, the desired frequency response of an FIR digital filter can be represented by the Fourier series as:

$$H_d(\omega)\big|_{\omega=\omega T} = H_d(\omega T) = \sum_{n=-\infty}^{\infty} h_d(n)\, e^{-j\omega nT}$$

where the Fourier coefficients $h_d(n)$ are the desired impulse response sequence of the filter. The samples of $h_d(n)$ can be determined using the equation:

$$h_d(n) = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} H_d(\omega T)\, e^{j\omega nT}\, d\omega$$

where ω_s is sampling frequency in rad/sec, F_s is sampling frequency in Hz. $T = 1/F_s$ is sampling period in sec.

The impulse response $h_d(n)$ from the above equation is an infinite duration sequence. For FIR filters, we truncate this infinite impulse response to a finite duration sequence of length N, where N is odd. Therefore,

$$h(n) = \begin{cases} h_d(n), & \text{for } n = -\left(\dfrac{N-1}{2}\right) \text{ to } \left(\dfrac{N-1}{2}\right) \\ 0, & \text{otherwise} \end{cases}$$

Taking Z-transform of the above equation for $h(n)$, we get

$$H(z) = \sum_{n=-(N-1)/2}^{(N-1)/2} h(n)\, z^{-n}$$

This transfer function of the filter $H(z)$ represents a non-causal filter (due to the presence of positive powers of z). Hence the transfer function represented by the above equation for $H(z)$ is multiplied by $z^{-(N-1)/2}$. Therefore

$$H(z) = z^{-\left(\frac{N-1}{2}\right)} \left[\sum_{n=-(N-1)/2}^{(N-1)/2} h(n) z^{-n} \right]$$

$$= z^{-\left(\frac{N-1}{2}\right)}\left[\sum_{n=-(N-1)/2}^{-1} h(n)\, z^{-n} + h(0) + \sum_{n=1}^{(N-1)/2} h(n)\, z^{-n}\right]$$

$$= z^{-\left(\frac{N-1}{2}\right)}\left[\sum_{n=1}^{(N-1)/2} h(-n)\, z^{n} + h(0) + \sum_{n=1}^{(N-1)/2} h(n)\, z^{-n}\right]$$

Since $h(n) = h(-n)$, we express $H(z)$ as:

$$H(z) = z^{-(N-1)/2}\left[h(0) + \sum_{n=1}^{(N-1)/2} h(n)\,[z^{n} + z^{-n}]\right]$$

Hence we see that casualty is brought about by multiplying the transfer function by the delay factor $\alpha = (N-1)/2$. This modification does not affect the amplitude response of the filter, however the abrupt truncation of the Fourier series results in oscillations in the pass band and stop band. These oscillations are due to the slow convergence of the Fourier series, particularly near the points of discontinuity. This effect is known as Gibbs phenomenon. The undesirable oscillations can be reduced by multiplying the desired frequency response coefficients by an appropriate window function.

Summarizing the above, the procedure for designing FIR filters by Fourier series method is as follows:

Step 1: Choose the desired frequency response $H_d(\omega)$ of the filter.
Step 2: Evaluate the Fourier series coefficients of $H_d(\omega T)$ which gives the desired impulse response $h_d(n)$.
Step 3: Truncate the infinite sequence $h_d(n)$ to a finite sequence $h(n)$.
Step 4: Take Z-transform of $h(n)$ to get a non-causal filter transfer function $H(z)$.
Step 5: Multiply $H(z)$ by $z^{-(N-1)/2}$ to convert the non-causal transfer function to a realizable causal FIR filter transfer function.

EXAMPLE 9.3 Design a low-pass FIR filter with five stage. [Given: Sampling time 1 ms; f_c = 200 Hz]. Also find the frequency response of the filter.

Solution: Given that f_c = 200 Hz and $f_s = \dfrac{1}{1\text{ ms}} = 1$ kHz

The normalized cutoff frequency $\omega_c = 2\pi f_c/f_s = 2\pi \times 200/1000 = 0.4\pi$ rad/sec. The given filter can be expressed by the following specifications:

$$H_d(\omega) = \begin{cases} 1, & \text{for } -\omega_c \le \omega \le \omega_c \\ 0, & \text{for } -\pi \le \omega \le -\omega_c \\ 0, & \text{for } \omega_c \le \omega \le \pi \end{cases}$$

The desired impulse response of the filter is given by

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-0.4\pi}^{0.4\pi} e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-0.4\pi}^{0.4\pi} = \frac{1}{2\pi}\left[\frac{e^{j0.4\pi n} - e^{-j0.4\pi n}}{jn}\right] = \frac{1}{n\pi}\sin 0.4\pi n$$

When $n \neq 0$,

$$h_d(n) = \frac{1}{n\pi}\sin 0.4\pi n$$

When $n = 0$, the factor $\frac{1}{n\pi}$ sin 0.4 πn becomes 0/0, which is indeterminate.

Hence using L'hospital rule, when $n = 0$, then

$$h_d(n) = h_d(0) = \underset{n\to 0}{\text{Lt}}\ \frac{1}{n\pi}\sin 0.4\pi n = 0.4$$

The impulse response of FIR filter is obtained by truncating $h_d(n)$ to 5 samples.

So $\qquad N = 5,\ N - 1 = 5 - 1 = 4$, and $(N - 1)/2 = 2$

Therefore, $h(n) = h_d(n)$ for $-(N-1)/2 \le n \le (N-1)/2$, i.e., for $-2 \le n \le 2$.

$\therefore \qquad h(n) = h_d(n) = 0.4$; for $n = 0$

and $\qquad h(n) = \dfrac{\sin 0.4\pi n}{n\pi}$; for $n \neq 0$, for $-2 \le n \le 2$

When $n = 0$, $\qquad h(n) = h(0) = 0.4\pi$

When $n = 1$, $\qquad h(n) = h(1) = \dfrac{\sin 0.4\pi}{\pi} = 0.3027 = h(-1)$

When $n = 2$, $\qquad h(n) = h(2) = \dfrac{\sin 2(0.4\pi)}{2\pi} = 0.0935 = h(-2)$

The coefficients of the designed filter are:

$$h(0) = 0.4\pi,\ h(1) = h(-1) = 0.3027,\ h(2) = h(-2) = 0.0935$$

The above coefficients correspond to a non-causal filter which is not realizable. The transfer function of the realizable digital filter is given by

$$H(z) = z^{-(N-1)/2}\left[h(0) + \sum_{n=1}^{(N-1)/2} h(n)\,[z^{-n} + z^{n}]\right] = z^{-2}\left[h(0) + \sum_{n=1}^{2} h(n)\,[z^{-n} + z^{n}]\right]$$

$$= z^{-2}\left[0.4\pi + 0.3027[z + z^{-1}] + 0.0935[z^2 + z^{-2}]\right]$$

$$= 0.0935 + 0.3027z^{-1} + 0.4\pi z^{-2} + 0.3027z^{-3} + 0.0935z^{-4}$$

Therefore, the coefficients of the realizable digital filter are:

$$h(0) = 0.0935,\ h(1) = 0.3027,\ h(2) = 0.4\pi,\ h(3) = 0.3027,\ h(4) = 0.0935$$

The frequency response of the filter is:

$$H(\omega) = e^{-j\omega(N-1)/2}\left[h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{(N-1)/2} 2h\left(\frac{N-1}{2} - n\right)\cos \omega n\right]$$

$$= e^{-j2\omega}[h(2) + 2h(1)\cos\omega + 2h(0)\cos 2\omega]$$

$$= e^{-j2\omega}[0.4\pi + 2(0.3027)\cos\omega + 2h(0.0935)\cos 2\omega]$$

$$= e^{-j2\omega}[0.4\pi + 0.6054\cos\omega + 0.1870\cos 2\omega]$$

The magnitude function of the filter is:

$$|H(\omega)| = 0.4\pi + 0.6054\cos\omega + 0.1870\cos 2\omega$$

EXAMPLE 9.4 Design an FIR digital filter to approximate an ideal low-pass filter with pass band gain of unity, cutoff frequency of 1 kHz and working at a sampling frequency of f_s = 4 kHz. The length of the impulse response should be 11. Use Fourier series method.

Solution: The desired frequency response of the ideal low-pass filter is given by

$$H_d(\omega) = \begin{cases} 1, & -1000\text{ Hz} \le f \le 1000\text{ Hz} \\ 0, & |f| > 1000\text{ Hz} \end{cases}$$

The above response can be equivalently specified in terms of the normalized ω_c. The normalized

$$\omega_c = \frac{2\pi f_c}{f_s} = 2\pi\left(\frac{1000}{4000}\right) = 1.570\text{ rad/sec}$$

Hence, the desired response is:

$$H_d(\omega) = \begin{cases} 1, & 0 \le |\omega| \le 1.570 \\ 0, & 1.570 < |\omega| \le \pi \end{cases}$$

The filter coefficients are given by

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-1.57}^{1.57}(1)e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-1.57}^{1.57} = \frac{1}{n\pi}\left[\frac{e^{j1.57n} - e^{-j1.57n}}{2j}\right]$$

$$= \frac{1}{n\pi}\sin 1.57n,\ n \neq 0$$

and when $n = 0$, $h_d(n) = 0/0$ is indeterminate. So when $n = 0$, using L'Hospital rule, we have

$$h_d(0) = \underset{n\to 0}{\text{Lt}}\ h_d(n) = \underset{n\to 0}{\text{Lt}}\ \frac{\sin 1.57n}{n\pi} = \frac{1.57}{\pi} = 0.5$$

The impulse response of FIR filter is obtained by truncating $h_d(n)$ to 11 samples. Since $N = 11$, the impulse response of the filter

$$h(n) = h_d(n) \text{ for } -\frac{(N-1)}{2} \le n \le \frac{(N-1)}{2}, \text{ i.e., for } -5 \le n \le 5$$

Therefore,

$$h(0) = 0.5$$

$$h(1) = \frac{1}{\pi}\sin(1.57)1 = 0.318 = h(-1)$$

$$h(2) = \frac{1}{2\pi}\sin(1.57)2 = 0.000253 \approx 0 = h(-2)$$

$$h(3) = \frac{1}{3\pi}\sin(1.57)3 = -0.1061 = h(-3)$$

$$h(4) = \frac{1}{4\pi}\sin(1.57)4 = -0.000253 \approx 0 = h(-4)$$

$$h(5) = \frac{1}{5\pi}\sin(1.57)5 = 0.0636 = h(-5)$$

Therefore, the designed filter coefficients are:

$$h(0) = 0.5,\ h(1) = 0.318 = h(-1),\ h(2) = 0 = h(-2),\ h(3) = -0.1061 = h(-3),$$

$$h(4) = 0 = h(-4),\ h(5) = 0.0636 = h(-5)$$

The above coefficients correspond to a non-causal filter which is unrealizable. The realizable digital filter transfer function is given by

$$H(z) = z^{-(N-1)/2}\left[h(0) + \sum_{n=1}^{(N-1)/2} h(n)\,[z^n + z^{-n}]\right] = z^{-5}\left[h(0) + \sum_{n=1}^{5} h(n)(z^n + z^{-n})\right]$$

$$= z^{-5}\left[\begin{array}{l} h(0) + h(1)(z + z^{-1}) + h(2)(z^2 + z^{-2}) + h(3)(z^3 + z^{-3}) \\ + h(4)(z^4 + z^{-4}) + h(5)\,(z^5 + z^{-5}) \end{array}\right]$$

$$= z^{-5}\left[0.5 + 0.318(z + z^{-1}) - 0.1061\,(z^3 + z^{-3}) + 0.0636(z^5 + z^{-5})\right]$$

$$= 0.0636 - 0.1061z^{-2} + 0.3183z^{-4} + 0.5z^{-5} + 0.3183z^{-6} - 0.1061z^{-8} + 0.0636z^{-10}$$

The coefficients of the realizable causal filter are:

$h(0) = h(10) = 0.0636$, $h(1) = h(9) = 0$, $h(2) = h(8) = -0.1061$, $h(3) = h(7) = 0$,

$h(4) = h(8) = 0.318$, $h(5) = 0.5$

The frequency response of the causal filter is:

$$H(\omega) = e^{-j\omega(N-1)/2}\left[h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{(N-1)/2} 2h\left(\frac{N-1}{2} - n\right)\cos\omega n\right]$$

$$= e^{-j5\omega}\left[h(5) + 2h(4)\cos\omega + 2h(3)\cos 2\omega + 2h(2)\cos 3\omega + 2h(1)\cos 4\omega + 2h(0)\cos 5\omega\right]$$

$$= e^{-j5\omega}\left[0.5 + 2(0.318)\cos\omega + 2(-0.1061)\cos 3\omega + 2(0.0636)\cos 5\omega\right]$$

$$= e^{-j5\omega}\left[0.5 + 0.636\cos\omega - 0.2122\cos 3\omega + 0.1272\cos 5\omega\right]$$

The magnitude function of the filter is:

$$|H(\omega)| = 0.5 + 0.636\cos\omega - 0.2122\cos 3\omega + 0.1272\cos 5\omega$$

EXAMPLE 9.5 A low-pass filter should have the frequency response given below. Find the filter coefficients $h_d(n)$. Also determine τ so that $h_d(n) = h_d(-n)$.

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\tau}, & -\omega_c \le \omega \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

Solution: For the given frequency response,

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau}, & -\omega_c \le \omega \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

The filter coefficients are given by

$$h_d(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} H_d(\omega)e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\tau}\,e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-\tau)}\,d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{\pi(n-\tau)}\left[\frac{e^{j(n-\tau)\omega_c} - e^{-j(n-\tau)\omega_c}}{2j}\right]$$

$$= \frac{\sin(n-\tau)\,\omega_c}{\pi(n-\tau)}, \quad \text{for } n \neq \tau$$

and $$h_d(\tau) = \frac{\omega_c}{\pi}, \quad \text{for } n = \tau \quad \text{(using L' Hospital rule)}$$

When $$h_d(n) = h_d(-n)$$

$$\frac{\sin(n-\tau)\,\omega_c}{\pi(n-\tau)} = \frac{\sin(-n-\tau)\,\omega_c}{\pi(-n-\tau)} = \frac{\sin(n+\tau)\omega_c}{\pi(n+\tau)}$$

This is possible only when $(n - \tau) = (n + \tau)$ or $\tau = 0$.

Table 9.1 shows the idealized frequency response and idealized impulse response of various filters.

9.6 DESIGN OF FIR FILTERS USING WINDOWS

The procedure for designing FIR filter using windows is:

1. Choose the desired frequency response of the filter $H_d(\omega)$.
2. Take inverse Fourier transform of $H_d(\omega)$ to obtain the desired impulse response $h_d(n)$.
3. Choose a window sequence $w(n)$ and multiply $h_d(n)$ by $w(n)$ to convert the infinite duration impulse response to a finite duration impulse response $h(n)$.
4. The transfer function $H(z)$ of the filter is obtained by taking Z-transform of $h(n)$.

9.6.1 Rectangular Window

The weighting function (window function) for an N-point rectangular window is given by

$$w_R(n) = \begin{cases} 1, & -\frac{(N-1)}{2} \le n \le \left(\frac{N-1}{2}\right) \\ 0, & \text{elsewhere} \end{cases} \quad \text{or} \quad w_R(n) = \begin{cases} 1, & 0 \le n \le (N-1) \\ 0, & \text{elsewhere} \end{cases}$$

The spectrum (frequency response) of rectangular window $W_R(\omega)$ is given by the Fourier transform of $w_R(n)$.

TABLE 9.1 The normalized ideal (desired) frequency response and impulse response for FIR filter design using windows.

Type of filter	*Ideal (desired) frequency response*	*Ideal (desired) impulse response*
Low-pass filter	$H_d(\omega) = \begin{cases} e^{-j\alpha\omega}; & -\omega_c \le \omega \le \omega_c \\ 0 ; & -\pi \le \omega < -\omega_c \\ 0 ; & \omega_c < \omega \le \pi \end{cases}$	$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n} d_\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\alpha} e^{j\omega n} d\omega \\ &= \frac{\sin\omega_c(n-\alpha)}{\pi(n-\alpha)} \end{aligned}$
High-pass filter	$H_d(\omega) = \begin{cases} e^{-j\alpha\omega}; & -\pi \le \omega \le -\omega_c \\ e^{-j\alpha\omega}; & \omega_c \le \omega \le \pi \\ 0 ; & -\omega_c < \omega < \omega_c \end{cases}$	$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{-j\omega\alpha} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{-j\omega\alpha} e^{j\omega n} d\omega \\ &= \frac{\sin(n-\alpha)\pi - \sin\omega_c(n-\alpha)}{\pi(n-\alpha)} \end{aligned}$
Band-pass filter	$H_d(\omega) = \begin{cases} e^{-j\omega\alpha}; -\omega_{c2} \le \omega \le -\omega_{c1} \\ e^{-j\omega\alpha}; \omega_{c1} \le \omega < \omega_{c2} \\ 0 \quad ; -\pi \le \omega < -\omega_{c2} \\ 0 \quad ; -\omega_{c1} < \omega < \omega_{c1} \\ 0 \quad ; \omega_{c2} < \omega \le \pi \end{cases}$	$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_{c2}}^{-\omega_{c1}} e^{-j\omega\alpha} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\omega_{c1}}^{\omega_{c2}} e^{-j\omega\alpha} e^{j\omega n} d\omega \\ &= \frac{\sin\omega_{c2}(n-\alpha) - \sin\omega_{c1}(n-\alpha)}{\pi(n-\alpha)} \end{aligned}$
Band-stop filter	$H_d(\omega) = \begin{cases} e^{-j\omega\alpha}; -\pi \le \omega \le -\omega_{c2} \\ e^{-j\omega\alpha}; -\omega_{c1} \le \omega \le \omega_{c1} \\ e^{-j\omega\alpha}; \omega_{c2} < \omega < \pi \\ 0 \quad ; -\omega_{c2} \le \omega \le -\omega_{c1} \\ 0 \quad ; \omega_{c1} < \omega < \omega_{c2} \end{cases}$	$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{-\omega_{c2}} e^{-j\omega\alpha} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{-\omega_{c1}}^{\omega_{c1}} e^{-j\omega\alpha} e^{j\omega n} d\omega \\ &\quad + \frac{1}{2\pi}\int_{\omega_{c2}}^{\pi} e^{-j\omega\alpha} e^{j\omega n} d\omega \\ &= \frac{\sin\omega_{c1}(n-\alpha) + \sin\pi(n-\alpha) - \sin\omega_{c2}(n-\alpha)}{\pi(n-\alpha)} \end{aligned}$

$$W_R(\omega) = \sum_{n=-(N-1)/2}^{(N-1)/2} e^{-j\omega n} = \sum_{n=0}^{N-1} e^{-j\omega\left(n - \frac{N-1}{2}\right)}$$

$$= \sum_{n=0}^{N-1} e^{-j\omega n}\, e^{j\omega\frac{N-1}{2}} = e^{j\omega\frac{N-1}{2}} \sum_{n=0}^{N-1} e^{-j\omega n}$$

$$= e^{j\omega\left(\frac{N-1}{2}\right)} \left[\frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}\right]$$

$$= e^{j\frac{\omega N}{2}}\, e^{-j\frac{\omega}{2}}\, \frac{e^{-j\frac{\omega N}{2}}\left[e^{j\frac{\omega N}{2}} - e^{-j\frac{\omega N}{2}}\right]}{e^{-j\frac{\omega}{2}}\left[e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right]}$$

$$= \frac{e^{j\frac{\omega N}{2}} - e^{-j\frac{\omega N}{2}}}{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}} = \frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}}$$

The frequency spectrum for $N = 31$ is shown in Figure 9.7. The spectrum $W_R(\omega)$ has two features that are important. They are the width of the main lobe and the side lobe amplitude. The frequency response is real and its zero occurs when $\omega = 2k\pi/N$ where k is an integer. The response for ω between $-2\pi/N$ and $2\pi/N$ is called the main lobe and the other lobes are called side lobes. For rectangular window the width of main lobe is $4\pi/N$. The first side lobe will be 13 dB down the peak of the main lobe and the roll off will be at 20 dB/decade. As the window is made longer, the main lobe becomes narrower and higher, and the side lobes become more concentrated around $\omega = 0$, but the amplitude of side lobes is unaffected. So increase in length does not reduce the amplitude of ripples, but increases the frequency when rectangular window is used.

If we design a low-pass filter using rectangular window, we find that the frequency response differs from the desired frequency response in many ways. It does not follow quick transitions in the desired response. The desired response of a low-pass filter changes abruptly from pass band to stop band, but the actual frequency response changes slowly. This region of gradual change is called filter's transition region, which is due to the convolution of the desired response with the window response's main lobe. The width of the transition region depends on the width of the main lobe. As the filter length N increases, the main lobe becomes narrower decreasing the width of the transition region.

The convolution of the desired response and the window response's side lobes gives rise to the ripples in both the pass band and stop band. The amplitude of the ripples is

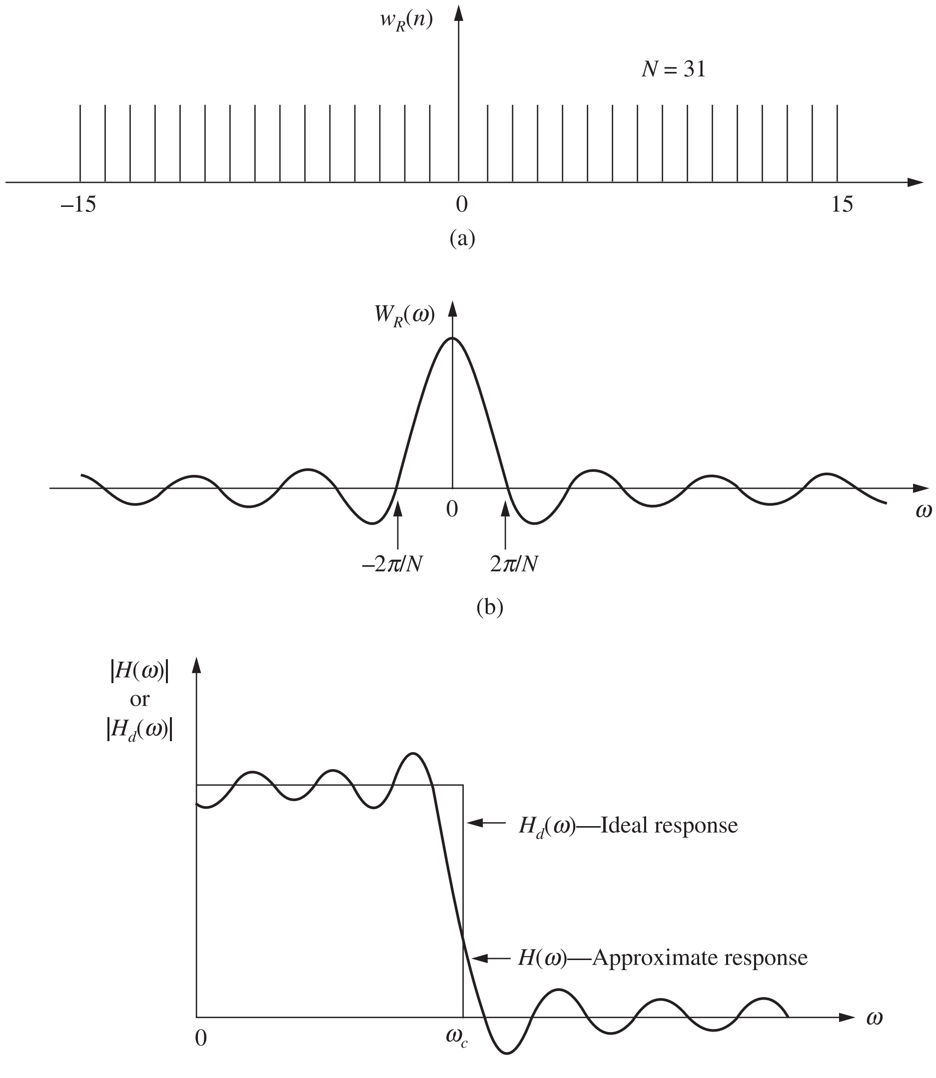


Figure 9.7 (a) Rectangular window sequence, (b) Magnitude response of rectangular window, (c) Magnitude response of low-pass filter approximated using rectangular window.

dictated by the amplitude of the side lobes. This effect, where maximum ripple occurs just before and just after the transition band, is known as Gibb's phenomenon.

The Gibbs phenomenon can be reduced by using a less abrupt truncation of filter coefficients. This can be achieved by using a window function that tapers smoothly towards zero at both ends.

9.6.2 Triangular or Bartlett Window

The triangular window has been chosen such that it has tapered sequences from the middle on either side. The window function $w_T(n)$ is defined as

$$w_T(n) = \begin{cases} 1 - \dfrac{2|n|}{N-1}, & \text{for } -\left(\dfrac{N-1}{2}\right) \le n \le \left(\dfrac{N-1}{2}\right) \\ 0 \qquad\quad , & \text{otherwise} \end{cases}$$

or
$$w_T(n) = \begin{cases} 1 - \dfrac{2|n-(N-1)/2|}{N-1}, & 0 \le n \le N-1 \\ 0 \qquad\qquad\qquad\quad , & \text{otherwise} \end{cases}$$

In magnitude response of triangular window, the side lobe level is smaller than that of the rectangular window being reduced from –13 dB to –25 dB. However, the main lobe width is now $8\pi/N$ or twice that of the rectangular window.

The triangular window produces a smooth magnitude response in both pass band and stop band, but it has the following disadvantages when compared to magnitude response obtained by using rectangular window:

1. The transition region is more.
2. The attenuation in stop band is less.

Because of these characteristics, the triangular window is not usually a good choice.

9.6.3 Raised Cosine Window

The raised cosine window multiplies the central Fourier coefficients by approximately unity and smoothly truncates the Fourier coefficients toward the ends of the filter. The smoother ends and broader middle section produces less distortion of $h_d(n)$ around $n = 0$. It is also called generalized Hamming window.

The window sequence is of the form:

$$w_H(n) = \begin{cases} \alpha + (1-\alpha)\cos\left(\dfrac{2\pi n}{N-1}\right), & \text{for } -\left(\dfrac{N-1}{2}\right) \le n \le \left(\dfrac{N-1}{2}\right) \\ 0, & \text{elsewhere} \end{cases}$$

9.6.4 Hanning Window

The Hanning window function is given by

$$w_{Hn}(n) = \begin{cases} 0.5 + 0.5\cos\left(\dfrac{2\pi n}{N-1}\right), & \text{for } -\left(\dfrac{N-1}{2}\right) \le n \le \left(\dfrac{N-1}{2}\right) \\ 0 & , \text{ otherwise} \end{cases}$$

or
$$w_{Hn}(n) = \begin{cases} 0.5 - 0.5\cos\dfrac{2n\pi}{N-1}, & \text{for } 0 \le n \le N-1 \\ 0 & , \text{ otherwise} \end{cases}$$

The width of main lobe is $8\pi/N$, i.e., twice that of rectangular window which results in doubling of the transition region of the filter. The peak of the first side lobe is –32 dB relative to the maximum value. This results in smaller ripples in both pass band and stop band of the low-pass filter designed using Hanning window. The minimum stop band attenuation of the filter is 44 dB. At higher frequencies the stop band attenuation is even greater. When compared to triangular window, the main lobe width is same, but the magnitude of the side lobe is reduced, hence the Hanning window is preferable to triangular window.

9.6.5 Hamming Window

The Hamming window function is given by

$$w_H(n) = \begin{cases} 0.54 + 0.46\cos\left(\dfrac{2\pi n}{N-1}\right), & \text{for } -\left(\dfrac{N-1}{2}\right) \le n \le \left(\dfrac{N-1}{2}\right) \\ 0 & , \text{ otherwise} \end{cases}$$

or
$$w_H(n) = \begin{cases} 0.54 - 0.46\cos\left(\dfrac{2n\pi}{N-1}\right), & 0 \le n \le N-1 \\ 0 & , \text{ otherwise} \end{cases}$$

In the magnitude response for $N = 31$, the magnitude of the first side lobe is down about 41 dB from the main lobe peak, an improvement of 10 dB relative to the Hanning window. But this improvement is achieved at the expense of the side lobe magnitudes at higher frequencies, which are almost constant with frequency. The width of the main lobe is $8\pi/N$. In the magnitude response of low-pass filter designed using Hamming window, the first side lobe peak is –51 dB, which is –7 dB lesser with respect to the Hanning window filter. However, at higher frequencies, the stop band attenuation is low when compared to that of Hanning window. Because the Hamming window generates lesser oscillations in the side lobes than the Hanning window for the same main lobe width, the Hamming window is generally preferred.

9.6.6 Blackman Window

The Blackman window function is another type of cosine window and given by the equation

$$w_B(n) = \begin{cases} 0.42 + 0.5\cos\dfrac{2\pi n}{N-1} + 0.08\cos\dfrac{4\pi n}{N-1}, & \text{for } -\left(\dfrac{N-1}{2}\right) \le n \le \left(\dfrac{N-1}{2}\right) \\ 0 & \text{, otherwise} \end{cases}$$

or $$w_B(n) = \begin{cases} 0.42 - 0.5\cos\dfrac{2n\pi}{N-1} + 0.08\cos\dfrac{4n\pi}{N-1}, & 0 \le n \le N-1 \\ 0 & \text{, otherwise} \end{cases}$$

In the magnitude response, the width of the main lobe is $12\pi/N$, which is highest among windows. The peak of the first side lobe is at –58 dB and the side lobe magnitude decreases with frequency. This desirable feature is achieved at the expense of increased main lobe width. However, the main lobe width can be reduced by increasing the value of N. The side lobe attenuation of a low-pass filter using Blackman window is –78 dB.

Table 9.2 gives the important frequency domain characteristics of some window functions.

TABLE 9.2 Frequency domain characteristics of some window functions.

Type of window	*Approximate transition width of main lobe*	*Minimum stop band attenuation* (dB)	*Peak of first side lobe* (dB)
Rectangular	$4\pi/N$	–21	–13
Bartlett	$8\pi/N$	–25	–25
Hanning	$8\pi/N$	– 44	–31
Hamming	$8\pi/N$	–51	– 41
Blackmann	$12\pi/N$	–78	–58

EXAMPLE 9.6 Design an ideal low-pass filter with $N = 11$ with a frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1, & \text{for } -\dfrac{\pi}{2} \le \omega \le \dfrac{\pi}{2} \\ 0, & \text{for } \dfrac{\pi}{2} \le |\omega| \le \pi \end{cases}$$

Solution: For the given desired frequency response,

$$H_d(\omega) = \begin{cases} 1, & \text{for } -\dfrac{\pi}{2} \le \omega \le \dfrac{\pi}{2} \\ 0, & \text{for } \dfrac{\pi}{2} \le |\omega| \le \pi \end{cases}$$

The filter coefficients are given by

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} (1)\, e^{j\omega n}\, d\omega \;=\; \frac{1}{2\pi} \left[\frac{e^{j\omega n}}{jn} \right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{n\pi} \left[\frac{e^{j\frac{n\pi}{2}} - e^{-j\frac{n\pi}{2}}}{2j} \right]$$

$$= \frac{1}{n\pi} \sin \frac{n\pi}{2} \quad \text{for } n \ne 0$$

and $\qquad h_d(n) = \dfrac{1}{2} \quad$ for $n = 0$ [using L'Hospital rule]

$\therefore \qquad h_d(0) = \dfrac{1}{2}, \qquad h_d(1) = \dfrac{1}{\pi} \sin \dfrac{\pi}{2} = \dfrac{1}{\pi} = h_d(-1)$

$$h_d(2) = \frac{1}{2\pi} \sin \pi = 0 = h_d(-2), \qquad h_d(3) = \frac{1}{3\pi} \sin \frac{3\pi}{2} = -\frac{1}{3\pi} = h_d(-3)$$

$$h_d(4) = \frac{1}{4\pi} \sin 2\pi = 0 = h_d(-4), \qquad h_d(5) = \frac{1}{5\pi} \sin \frac{5\pi}{2} = \frac{1}{5\pi} = h_d(-5)$$

Assuming the window function,

$$w(n) = \begin{cases} 1, & \text{for } -5 \le n \le 5 \\ 0, & \text{otherwise} \end{cases}$$

We have $\qquad h(n) = h_d(n) \cdot w(n) = h_d(n)$

Therefore, the designed filter coefficients are given as:

$$h(0) = \frac{1}{2}, h(1) = \frac{1}{\pi} = h(-1), \quad h(2) = 0 = h(-2), \quad h(3) = -\frac{1}{3\pi} = h(-3),$$

$$h(4) = 0 = h(-4), \quad h(5) = \frac{1}{5\pi} = h(-5)$$

The above coefficients correspond to a non-causal filter which is not realizable.

The realizable digital filter transfer function $H(z)$ is given by

$$H(z) = z^{-(N-1)/2}\left[h(0) + \sum_{n=1}^{(N-1)/2} h(n)\,[z^{-n} + z^{n}]\right] = z^{-5}\left[h(0) + \sum_{n=1}^{5} h(n)[z^{-n} + z^{n}]\right]$$

$$= z^{-5}\left[h(0) + h(1)\,[z + z^{-1}] + h(3)[z^{3} + z^{-3}] + h(5)[z^{5} + z^{-5}]\right]$$

$$= h(5) + h(3)\,z^{-2} + h(1)\,z^{-4} + h(0)\,z^{-5} + h(1)\,z^{-6} + h(3)\,z^{-8} + h(5)z^{-10}$$

$$= \frac{1}{5\pi} - \frac{1}{3\pi}z^{-2} + \frac{1}{\pi}z^{-4} + \frac{1}{2}z^{-5} + \frac{1}{\pi}z^{-6} - \frac{1}{3\pi}z^{-8} + \frac{1}{5\pi}z^{-10}$$

Therefore, the coefficients of the realizable digital filter are:

$$h(0) = \frac{1}{5\pi} = h(10), \;\; h(1) = 0 = h(9), \quad h(2) = -\frac{1}{3\pi} = h(8),$$

$$h(3) = 0 = h(7), \quad h(4) = \frac{1}{\pi} = h(6), \quad h(5) = \frac{1}{2}$$

EXAMPLE 9.7 A low-pass filter is to be designed with the following desired frequency response:

$$H_d(e^{j\omega}) = \begin{cases} e^{-j2\omega}, & -\frac{\pi}{4} \le \omega \le \frac{\pi}{4} \\ 0, & \frac{\pi}{4} \le |\omega| \le \pi \end{cases}$$

Determine the filter coefficients $h(n)$ if the window function is defined as:

$$w(n) = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$$

Also determine the frequency response $H(e^{j\omega})$ of the designed filter.

Solution: For the given filter with

$$H_d(\omega) = \begin{cases} e^{-j2\omega}, & -\dfrac{\pi}{4} \le \omega \le \dfrac{\pi}{4} \\ 0, & \dfrac{\pi}{4} \le |\omega| \le \pi \end{cases}$$

The filter coefficients are given by

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} e^{-j2\omega}\, e^{j\omega n}\, d\omega = \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} e^{j\omega(n-2)}\, d\omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega(n-2)}}{j(n-2)} \right]_{-\pi/4}^{\pi/4} = \frac{1}{\pi(n-2)} \left[\frac{e^{j(n-2)\frac{\pi}{4}} - e^{-j(n-2)\frac{\pi}{4}}}{2j} \right]$$

$$= \frac{1}{\pi(n-2)} \sin (n-2) \frac{\pi}{4}, \quad n \ne 2.$$

For $n = 2$, the filter coefficient can be obtained by applying L'Hospital rule to the above expression. Thus,

$$h_d(2) = \operatorname*{Lt}_{n \to 2} \frac{1}{\pi} \frac{\sin (n-2) \dfrac{\pi}{4}}{(n-2)} = \frac{1}{\pi} \cdot \frac{\pi}{4} = \frac{1}{4}$$

Since it is a linear phase filter, the other filter coefficients are given by

$$h_d(0) = \frac{1}{\pi(0-2)} \sin (0-2) \frac{\pi}{4} = \frac{1}{2\pi}$$

$$h_d(1) = \frac{1}{\pi(1-2)} \sin (1-2) \frac{\pi}{4} = \frac{1}{\sqrt{2}\pi}$$

$$h_d(3) = \frac{1}{\pi(3-2)} \sin (3-2) \frac{\pi}{4} = \frac{1}{\sqrt{2}\pi}$$

$$h_d(4) = \frac{1}{\pi(4-2)} \sin (4-2) \frac{\pi}{4} = \frac{1}{2\pi}$$

The filter coefficients of the filter using rectangular window would be then

$$h(n) = h_d(n) \cdot w(n) = h_d(n)$$

Therefore, $$h(0) = \frac{1}{2\pi} = h(4), \quad h(1) = \frac{1}{\sqrt{2}\pi} = h(3), \quad \text{and} \quad h(2) = \frac{1}{4}$$

are the coefficients of the designed digital filter.

The realizable digital filter is:

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n} = h(0) + h(1)\, z^{-1} + h(2)\, z^{-2} + h(3)\, z^{-3} + h(4)\, z^{-4}$$

$$= \frac{1}{2\pi} + \frac{1}{\sqrt{2}\pi} z^{-1} + 0.25\, z^{-2} + \frac{1}{\sqrt{2}\pi} z^{-3} + \frac{1}{2\pi} z^{-4}$$

$$= z^{-2}\left[0.25 + \frac{1}{\sqrt{2}\pi}(z + z^{-1}) + \frac{1}{2\pi}(z^2 + z^{-2})\right]$$

The frequency response $H(\omega)$ of the digital filter is given by

$$H(\omega) = \sum_{n=0}^{4} h(n) e^{-j\omega n}$$

$$= h(0) + h(1)\, e^{-j\omega} + h(2)\, e^{-j2\omega} + h(3)\, e^{-j3\omega} + h(4)\, e^{-j4\omega}$$

$$= e^{-j2\omega}\,[h(0)\, e^{j2\omega} + h(1)\, e^{j\omega} + h(2) + h(3)\, e^{-j\omega} + h(4)\, e^{-j2\omega}]$$

$$= e^{-j2\omega}\,[h(2) + h(1)\,(e^{j\omega} + e^{-j\omega}) + h(0)\,(e^{j2\omega} + e^{-j2\omega})]$$

$$= e^{-j2\omega}\left[\frac{1}{4} + \frac{\sqrt{2}}{\pi}\cos\omega + \frac{1}{\pi}\cos 2\omega\right]$$

EXAMPLE 9.8 A filter is to be designed with the following desired frequency response.

$$H_d(e^{j\omega}) = \begin{cases} 0, & -\dfrac{\pi}{2} \le \omega \le \dfrac{\pi}{2} \\ e^{-j2\omega}, & \dfrac{\pi}{2} < |\omega| \le \pi \end{cases}$$

Determine the filter coefficient $h(n)$, if the window function is defined as

$$w(n) = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$$

Also determine the frequency response $H(e^{j\omega})$ of the designed filter.

Solution: For the given high-pass filter with

$$H_d(\omega) = \begin{cases} 0, & -\dfrac{\pi}{2} \le \omega \le \dfrac{\pi}{2} \\ e^{-j2\omega}, & \dfrac{\pi}{2} \le |\omega| \le \pi \end{cases}$$

the filter coefficients are given by

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{-\pi/2} e^{-j2\omega} e^{j\omega n}\, d\omega + \frac{1}{2\pi} \int_{\pi/2}^{\pi} e^{-j2\omega} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{-\pi/2} e^{j\omega(n-2)}\, d\omega + \frac{1}{2\pi} \int_{\pi/2}^{\pi} e^{j\omega(n-2)}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-2)}}{j(n-2)}\right]_{-\pi}^{-\pi/2} + \frac{1}{2\pi}\left[\frac{e^{j\omega(n-2)}}{j(n-2)}\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{j2\pi(n-2)}\left[e^{-j(n-2)\frac{\pi}{2}} - e^{-j(n-2)\pi} + e^{j(n-2)\pi} - e^{j(n-2)\frac{\pi}{2}}\right]$$

$$= \frac{1}{\pi(n-2)}\left\{\left[\frac{e^{j(n-2)\pi} - e^{-j(n-2)\pi}}{2j}\right] - \left[\frac{e^{j(n-2)\frac{\pi}{2}} - e^{-j(n-2)\frac{\pi}{2}}}{2j}\right]\right\}$$

$$= \frac{1}{\pi(n-2)}\left[\sin(n-2)\pi - \sin(n-2)\frac{\pi}{2}\right], \qquad n \ne 2$$

For $n = 2$, $h_d(n)$ is indeterminate. So using L'Hospital rule, we have

$$h_d(2) = \operatorname*{Lt}_{n\to 2} \frac{1}{\pi}\left[\frac{\sin(n-2)\pi}{(n-2)} - \frac{\sin(n-2)\pi/2}{n-2}\right] = \frac{1}{\pi}\left[\pi - \frac{\pi}{2}\right] = 1 - \frac{1}{2} = \frac{1}{2}$$

The other filter coefficients are:

$$h_d(0) = \frac{1}{\pi(0-2)}\left[\sin(0-2)\pi - \sin(0-2)\frac{\pi}{2}\right] = 0$$

$$h_d(1) = \frac{1}{\pi(1-2)}\left[\sin(1-2)\pi - \sin(1-2)\frac{\pi}{2}\right] = -\frac{1}{\pi}$$

$$h_d(3) = \frac{1}{\pi(3-2)}\left[\sin(3-2)\pi - \sin(3-2)\frac{\pi}{2}\right] = -\frac{1}{\pi}$$

$$h_d(4) = \frac{1}{\pi(4-2)}\left[\sin(4-2)\pi - \sin(4-2)\frac{\pi}{2}\right] = 0$$

Since $h(n) = h_d(n) \cdot w(n)$, applying the window function, the new filter coefficients are:

$$h(0) = 0,\ h(1) = -\frac{1}{\pi},\ h(2) = \frac{1}{2},\ h(3) = -\frac{1}{\pi},\ \text{and } h(4) = 0$$

The transfer function of the filter is:

$$\begin{aligned} H(z) &= \sum_{n=0}^{N-1} h(n)\, z^{-n} \\ &= h(0) + h(1)\, z^{-1} + h(2)\, z^{-2} + h(3)\, z^{-3} + h(4)\, z^{-4} \\ &= 0 - \frac{1}{\pi} z^{-1} + \frac{1}{2} z^{-2} - \frac{1}{\pi} z^{-3} + 0 \\ &= z^{-2}\left[\frac{1}{2} - \frac{1}{\pi}(z + z^{-1})\right] \end{aligned}$$

The frequency response $H(\omega)$ is given by

$$\begin{aligned} H(\omega) &= \sum_{n=0}^{4} h(n) e^{-j\omega n} \\ &= h(0) + h(1) e^{-j\omega} + h(2) e^{-j2\omega} + h(3) e^{-j3\omega} + h(4) e^{-j4\omega} \\ &= -\frac{1}{\pi} e^{-j\omega} + \frac{1}{2} e^{-j2\omega} - \frac{1}{\pi} e^{-j3\omega} \\ &= e^{-j2\omega}\left[\frac{1}{2} - \frac{1}{\pi}(e^{j\omega} + e^{-j\omega})\right] \\ &= e^{-j2\omega}\left[\frac{1}{2} - \frac{2}{\pi}\cos\omega\right] \end{aligned}$$

EXAMPLE 9.9 Design a filter with

$$H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega}, & -\dfrac{\pi}{4} \le \omega \le \dfrac{\pi}{4} \\ 0 \quad , & \dfrac{\pi}{4} \le |\omega| \le \pi \end{cases}$$

using a Hamming window with $N = 7$.

Solution: For the given filter with

$$H_d(\omega) = \begin{cases} e^{-j3\omega}, & -\dfrac{\pi}{4} \le \omega \le \dfrac{\pi}{4} \\ 0 \quad , & \dfrac{\pi}{4} \le |\omega| \le \pi \end{cases}$$

the filter coefficients are given by

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{-j3\omega}\,e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{j\omega(n-3)}\,d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega(n-3)}}{j(n-3)}\right]_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{\pi(n-3)}\left[\frac{e^{j\pi(n-3)/4} - e^{-j\pi(n-3)/4}}{2j}\right]$$

$$= \frac{\sin \pi(n-3)/4}{\pi(n-3)}, \qquad n \neq 3$$

For $n = 3$, the filter coefficient can be obtained by applying L'Hospital's rule to the above expression. Thus,

$$h_d(3) = \underset{n\to 3}{\text{Lt}} \frac{\sin \frac{1}{4}(n-3)\pi}{\sin(n-3)\pi} = \frac{1}{4}$$

The other filter coefficients are given by

$$h_d(0) = \frac{\sin \pi(0-3)/4}{\pi(0-3)} = \frac{0.707}{3\pi}, \qquad h_d(1) = \frac{\sin \pi(1-3)/4}{\pi(1-3)} = \frac{1}{2\pi}$$

$$h_d(2) = \frac{\sin \pi(2-3)/4}{\pi(2-3)} = \frac{0.707}{\pi}, \qquad h_d(4) = \frac{\sin \pi(4-3)/4}{\pi(4-3)} = \frac{0.707}{\pi}$$

$$h_d(5) = \frac{\sin \pi(5-3)/4}{\pi(5-3)} = \frac{1}{2\pi}, \qquad h_d(6) = \frac{\sin \pi(6-3)/4}{\pi(6-3)} = \frac{0.707}{3\pi}$$

So the filter coefficients are:

$$h_d(0) = \frac{0.707}{3\pi}, \quad h_d(1) = \frac{1}{2\pi}, \quad h_d(2) = \frac{0.707}{\pi}, \quad h_d(3) = \frac{1}{4},$$

$$h_d(4) = \frac{0.707}{\pi}, \quad h_d(5) = \frac{1}{2\pi}, \quad h_d(6) = \frac{0.707}{3\pi}$$

The Hamming window function of a causal filter is:

$$w(n) = \begin{cases} 0.54 - 0.46 \cos \dfrac{2\pi n}{N-1}, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Therefore, with $N = 7$

$$w(0) = 0.54 - 0.46 \cos 0 = 0.08, \qquad w(1) = 0.54 - 0.46\cos \frac{2\pi \times 1}{7-1} = 0.31$$

$$w(2) = 0.54 - 0.46\cos \frac{2\pi \times 2}{7-1} = 0.77, \qquad w(3) = 0.54 - 0.46\cos \frac{2\pi \times 3}{7-1} = 1$$

$$w(4) = 0.54 - 0.46\cos \frac{2\pi \times 4}{7-1} = 0.77, \qquad w(5) = 0.54 - 0.46\cos \frac{2\pi \times 5}{7-1} = 0.31$$

$$w(6) = 0.54 - 0.46\cos \frac{2\pi \times 6}{7-1} = 0.08$$

The filter coefficients of the resultant filter are:

$$h(n) = h_d(n)\, \omega(n), \qquad n = 0, 1, 2, 3, 4, 5, 6$$

Therefore,

$$h(0) = h_d(0)\, w(0) = \frac{0.707}{3\pi} \times 0.08 = 0.006, \qquad h(1) = h_d(1)\, w(1) = \frac{1}{2\pi} \times 0.31 = 0.049$$

$$h(2) = h_d(2)\, w(2) = \frac{0.707}{\pi} \times 0.77 = 0.173, \qquad h(3) = h_d(3)\, w(3) = \frac{1}{4} \times 1 = \frac{1}{4} = 0.25$$

$$h(4) = h_d(4)\, w(4) = \frac{0.707}{\pi} \times 0.77 = 0.173, \qquad h(5) = h_d(5)\, w(5) = \frac{1}{2\pi} \times 0.31 = 0.049$$

$$h(6) = h_d(6)\, w(6) = \frac{0.707}{3\pi} \times 0.08 = 0.006$$

The frequency response of a causal filter is given by

$$H(\omega) = \sum_{n=0}^{6} h(n)\, e^{-j\omega n}$$

$$= h(0) + h(1)e^{-j\omega} + h(2)e^{-j2\omega} + h(3)e^{-j3\omega} + h(4)e^{-j4\omega} + h(5)e^{-j5\omega} + h(6)e^{-j6\omega}$$

$$= e^{-j3\omega}\{h(3) + [h(0)\, e^{j3\omega} + h(6)\, e^{-j3\omega}] + [h(1)\, e^{j2\omega} + h(5)\, e^{-j2\omega}] + [h(2)\, e^{j\omega} + h(4)\, e^{-j\omega}]\}$$

$$= e^{-j3\omega}\,[h(3) + 2h(0)\cos 3\omega + 2h(1)\cos 2\omega + 2h(2)\cos\omega]$$

$$= e^{-j3\omega}[0.25 + 0.012\cos 3\omega + 0.098\cos 2\omega + 0.346\cos\omega]$$

The transfer function of the digital FIR low-pass filter is:

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6}$$

$$= z^{-3}\left[h(3) + h(2)\,(z^{-1} + z) + h(1)\,(z^{-2} + z^{2}) + h(0)\,(z^{-3} + z^{3})\right]$$

$$= z^{-3}\left[0.25 + 0.173\,[z + z^{-1}] + 0.049\,[z^{2} + z^{-2}] + 0.006\,[z^{3} + z^{-3}]\right]$$

$$= 0.006 + 0.049\,z^{-1} + 0.173z^{-2} + 0.25\,z^{-3} + 0.173\,z^{-4} + 0.049\,z^{-5} + 0.006^{-6}$$

EXAMPLE 9.10 Design a digital FIR low-pass filter using rectangular window by taking 9 samples of $w(n)$ and with a cutoff frequency of 1.2 rad/sec.

Solution: Cutoff frequency of given low-pass filter ω_c = 1.2 rad/sec and N = 9. For a low-pass filter, the desired frequency response is

$$H_d(\omega) = \begin{cases} e^{-j\omega\alpha}, & -\omega_c \le \omega \le \omega_c \\ 0\quad, & \text{otherwise} \end{cases}$$

The desired impulse response is obtained by taking the inverse Fourier transform of $H_d(\omega)$. Therefore,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\alpha}\, e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-\alpha)}\, d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{\pi(n-\alpha)}\left[\frac{e^{j(n-\alpha)\omega_c} - e^{-j(n-\alpha)\omega_c}}{2j}\right]$$

$$= \frac{1}{\pi(n-\alpha)} \sin(n-\alpha)\omega_c \qquad \text{for } n \neq \alpha$$

and for $n = \alpha$, $\dfrac{\sin(n-\alpha)\omega_c}{\pi(n-\alpha)}$ is 0/0, which is indeterminate. Using L'hospital rule, we have

for $n = \alpha$, $H_d(\alpha) = \underset{n\to\alpha}{\text{Lt}} \dfrac{\sin(n-\alpha)\,\omega_c}{\pi(n-\alpha)} = \dfrac{\omega_c}{\pi}$.

The impulse response of FIR filter $h(n)$ is obtained by multiplying $h_d(n)$ by the window sequence. Therefore, Impulse response $h(n) = h_d(n) \cdot w_R(n)$.

Rectangular window sequence, $w_R(n) = \begin{cases} 1, & \text{for } 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$

$$\therefore \qquad h(n) = h_d(n); \qquad \text{for } 0 \leq n \leq N-1$$

Here $N = 9$; $\omega_c = 1.2$ rad/sec and $\alpha = \dfrac{N-1}{2} = \dfrac{9-1}{2} = 4$

Therefore, we have $h(n) = \dfrac{\sin(n-4) \times 1.2}{\pi(n-4)}$, for $n \neq 4$

Therefore,

$$h(0) = \frac{\sin(0-4)1.2}{\pi(0-4)} = -0.0793, \quad h(1) = \frac{\sin(1-4)1.2}{\pi(1-4)} = -0.0470$$

$$h(2) = \frac{\sin(2-4)1.2}{\pi(2-4)} = 0.1075, \quad h(3) = \frac{\sin(3-4)1.2}{\pi(3-4)} = 0.2967$$

$$h(4) = \frac{\omega_c}{\pi} = \frac{1.2}{\pi} = 0.382, \quad h(5) = \frac{\sin(5-4)1.2}{\pi(5-4)} = 0.2967$$

$$h(6) = \frac{\sin(6-4)1.2}{\pi(6-4)} = 0.1075, \quad h(7) = \frac{\sin(7-4)1.2}{\pi(7-4)} = -0.047$$

$$h(8) = \frac{\sin(8-4)1.2}{\pi(8-4)} = -0.0793$$

Here we can observe that $h(0) = h(8)$, $h(1) = h(7)$, $h(2) = h(6)$ and $h(3) = h(5)$, i.e., the impulse response is satisfying the symmetry condition $h(N-1-n) = h(n)$.

The transfer function of the filter is given by

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

$$= h(0) + h(1)\,z^{-1} + h(2)\,z^{-2} + h(3)\,z^{-3} + h(4)\,z^{-4} + h(5)\,z^{-5} + h(6)\,z^{-6} + h(7)\,z^{-7} + h(8)\,z^{-8}$$

$$= z^{-4}\left[h(4) + h(3)[z + z^{-1}] + h(2)[z^{2} + z^{-2}] + h(1)[z^{3} + z^{-3}]\right] + h(0)[z^{4} + z^{-4}]$$

$$= z^{-4}\left[0.382 + 0.2967[z + z^{-1}] + 0.1075[z^{2} + z^{-2}] - 0.4070[z^{3} + z^{-3}]\right] - 0.0793[z^{4} + z^{-4}]$$

$$= -0.0793 - 0.4070z^{-1} + 0.1075z^{-2} + 0.2967z^{-3} + 0.382z^{-4} + 0.2967z^{-5} + 0.1075z^{-6}$$
$$- 0.4070z^{-7} - 0.0793z^{-8}$$

$$H(\omega) = e^{-j\omega(N-1)/2}\left[h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{(N-1)/2} 2h\left(\frac{N-1}{2} - n\right)\cos\omega n\right]$$

$$= e^{-j4\omega}\left[h(4) + \sum_{n=1}^{4} 2h(4-n)\cos\omega n\right]$$

$$= e^{-j4\omega}\,\{h(4) + 2h(3)\cos\omega + 2h(2)\cos 2\omega + 2h(1)\cos 3\omega + 2h(0)\cos 4\omega\}$$

$$= e^{-j4\omega}\,\{0.382 + 2\times 0.2967\cos\omega + 2\times 0.1075\cos 2\omega + 2\times(-0.047)\cos 3\omega$$
$$+ 2\times(-0.0793)\cos 4\omega\}$$

$$= e^{-j4\omega}\{0.382 + 0.5934\cos\omega + 0.215\cos 2\omega - 0.094\cos 3\omega - 0.1586\cos 4\omega\}$$

The magnitude response is:

$$|H(\omega)| = 0.382 + 0.5934\cos\omega + 0.215\cos 2\omega - 0.094\cos 3\omega - 0.1586\cos 4\omega$$

EXAMPLE 9.11 Design a high-pass filter using Hamming window, with a cutoff frequency of 1.2 rad/sec and $N = 9$.

Solution: The desired frequency response $H_d(\omega)$ for a high-pass filter is

$$H_d(\omega) = \begin{cases} 0 & , \quad -\omega_c \le \omega \le \omega_c \\ e^{-j\omega\alpha} & , \quad \omega_c < |\omega| \le \pi \end{cases}$$

The desired impulse response $h_d(n)$ is obtained by taking the inverse Fourier transform of $H_d(\omega)$.

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega\alpha}.e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{j\omega(n-\alpha)}\, d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{j\omega(n-\alpha)}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\pi}^{-\omega_c} + \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{\omega_c}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j(n-\alpha)\omega_c} - e^{-j(n-\alpha)\pi} + e^{j(n-\alpha)\pi} - e^{j(n-\alpha)\omega_c}}{j(n-\alpha)}\right]$$

$$= \frac{1}{\pi(n-\alpha)}\left[\frac{e^{j(n-\alpha)\pi} - e^{-j(n-\alpha)\pi}}{2j} - \frac{e^{j(n-\alpha)\omega_c} - e^{-j(n-\alpha)\omega_c}}{2j}\right]$$

$$= \frac{1}{\pi(n-\alpha)}\{\sin(n-\alpha)\pi - \sin(n-\alpha)\,\omega_c\}$$

When $n = \alpha$, the terms $\frac{\sin(n-\alpha)\pi}{\pi(n-\alpha)}$ and $\frac{\sin(n-\alpha)\omega_c}{\pi(n-\alpha)}$ become 0/0 which is indeterminate. Using L'Hospital rule, we have for $n = \alpha$.

$$h_d(n) = \operatorname*{Lt}_{n\to\alpha} \frac{1}{\pi}\left[\frac{\sin(n-\alpha)\pi}{(n-\alpha)} - \frac{\sin(n-\alpha)\omega_c}{(n-\alpha)}\right]$$

$$= \frac{1}{\pi}[\pi - \omega_c] = 1 - \frac{\omega_c}{\pi}$$

For $n \neq \alpha$,

$$h_d(n) = \frac{\sin(n-\alpha)\pi - \sin(n-\alpha)\omega_c}{\pi(n-\alpha)}$$

Here $\alpha = \frac{N-1}{2} = \frac{9-1}{2} = 4$ is an integer and since n is also an integer, $(n - \alpha)$ is also an integer and so $\sin(n - \alpha)\pi = 0$.

$$\therefore \qquad h_d(n) = -\frac{\sin(n-\alpha)\omega_c}{\pi(n-\alpha)} = -\frac{\sin(n-4)\,1.2}{\pi(n-4)}$$

Therefore,

$$h_d(0) = -\frac{\sin(0-4)\,1.2}{\pi(0-4)} = 0.0792, \qquad h_d(1) = -\frac{\sin(1-4)\,1.2}{\pi(1-4)} = 0.0469$$

$$h_d(2) = -\frac{\sin(2-4)\,1.2}{\pi(2-4)} = -0.1075, \qquad h_d(3) = -\frac{\sin(3-4)\,1.2}{\pi(3-4)} = -0.2966$$

$$h_d(4) = 1 - \frac{\omega_c}{\pi} = 1 - \frac{1.2}{\pi} = 0.618, \quad h_d(5) = -\frac{\sin(5-4)\,1.2}{\pi(5-4)} = -0.2966$$

$$h_d(6) = -\frac{\sin(6-4)\,1.2}{\pi(6-4)} = -0.1075, \qquad h_d(7) = -\frac{\sin(7-4)\,1.2}{\pi(7-4)} = -0.0469$$

$$h_d(8) = -\frac{\sin(8-4)\,1.2}{\pi(8-4)} = -0.0792$$

The window sequence for Hamming window is given by

$$w_H(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right); \qquad \text{for } n = 0 \text{ to } N-1$$

Therefore,

$$w_H(0) = 0.54 - 0.46\cos\left(\frac{2\pi \times 0}{9-1}\right) = 0.08$$

$$w_H(1) = 0.54 - 0.46\cos\left(\frac{2\pi \times 1}{9-1}\right) = 0.2147$$

$$w_H(2) = 0.54 - 0.46\cos\left(\frac{2\pi \times 2}{9-1}\right) = 0.54$$

$$w_H(3) = 0.54 - 0.46\cos\left(\frac{2\pi \times 3}{9-1}\right) = 0.8652$$

$$w_H(4) = 0.54 - 0.46\cos\left(\frac{2\pi \times 4}{9-1}\right) = 1$$

$$w_H(5) = 0.54 - 0.46\cos\left(\frac{2\pi \times 5}{9-1}\right) = 0.8652$$

$$w_H(6) = 0.54 - 0.46\cos\left(\frac{2\pi \times 6}{9-1}\right) = 0.54$$

$$w_H(7) = 0.54 - 0.46 \cos\left(\frac{2\pi \times 7}{9 - 1}\right) = 0.2147$$

$$w_H(8) = 0.54 - 0.46 \cos\left(\frac{2\pi \times 8}{9 - 1}\right) = 0.08$$

The filter coefficients are $h(n) = h_d(n)w_H(n)$

$$\therefore \qquad h(0) = h_d(0)\, w_H(0) = 0.0792 \times 0.08 = 0.0063$$

$$h(1) = h_d(1)\, w_H(1) = 0.0469 \times 0.2147 = 0.0100$$

$$h(2) = h_d(2)\, w_H(2) = -0.1075 \times 0.54 = -0.0580$$

$$h(3) = h_d(3)\, w_H(3) = -0.2966 \times 0.8652 = -0.2566$$

$$h(4) = h_d(4)\, w_H(4) = 0.618 \times 1 = 0.618$$

$$h(5) = h_d(5)\, w_H(5) = -0.2966 \times 0.8652 = -0.2566$$

$$h(6) = h_d(6)\, w_H(6) = -0.1075 \times 0.54 = -0.0580$$

$$h(7) = h_d(7)\, w_H(7) = 0.0469 \times 0.2147 = 0.0100$$

$$h(8) = h_d(8)\, w_H(8) = 0.0792 \times 0.08 = 0.0063$$

From the above calculations, we can observe that $h(N-1-n) = h(n)$, i.e., the impulse response is symmetrical with centre of symmetry at $n = 4$. The frequency response of the filter is:

$$H(\omega) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$$

$$= \begin{bmatrix} h(0) + h(1)e^{-j\omega} + h(2)e^{-j2\omega} + h(3)e^{-j3\omega} + h(4)e^{-j4\omega} + h(5)e^{-j5\omega} \\ + h(6)e^{-j6\omega} + h(7)e^{-j7\omega} + h(8)e^{-j8\omega} \end{bmatrix}$$

$$= e^{-j4\omega}\,[h(4) + 2h(3)\cos\omega + 2h(2)\cos 2\omega + 2h(1)\cos 3\omega + 2h(0)\cos 4\omega]$$

$$= e^{-j4\omega} \begin{bmatrix} 0.618 + 2(-0.256)\cos\omega + 2(-0.058)\cos 2\omega + 2(0.010)\cos 3\omega \\ + 2(0.0063)\cos 4\omega \end{bmatrix}$$

$$= e^{-j4\omega}\,[0.618 - 0.5132\cos\omega - 0.116\cos 2\omega + 0.020\cos 3\omega + 0.012\cos 4\omega]$$

The magnitude response is given by

$$|H(\omega)| = 0.618 - 0.5132\cos\omega - 0.116\cos 2\omega + 0.020\cos 3\omega + 0.012\cos 4\omega$$

The transfer function of the filter is given by

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n} = \sum_{n=0}^{8} h(n)\, z^{-n}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7} + h(8)z^{-8}$$

$$= 0.0063 + 0.010z^{-1} - 0.0580z^{-2} - 0.2566z^{-3} + 0.618z^{-4} - 0.2566z^{-5}$$

$$- 0.0580z^{-6} + 0.0100z^{-7} + 0.0063z^{-8}$$

$$= z^{-4}[0.618 - 0.2566(z + z^{-1}) - 0.0580(z^2 + z^{-2}) + 0.0100(z^3 + z^{-3}) + 0.0063(z^4 + z^{-4})]$$

EXAMPLE 9.12 Design a band-pass filter to pass frequencies in the range 1 to 2 rad/sec. using Hanning window, with $N = 5$.

Solution: The desired frequency response $H_d(\omega)$ for band pass filter is:

$$H_d(\omega) = \begin{cases} e^{-j\omega\alpha}, & -\omega_{c_2} \le \omega \le -\omega_{c_1} \text{ and } \omega_{c_1} \le \omega \le \omega_{c_2} \\ 0, & \text{otherwise} \end{cases}$$

The band-pass filter has to pass frequencies in the range 1 to 2 rad/sec.

Therefore, $\omega_{c1} = 1$ and $\omega_{c2} = 2$. The desired impulse response $h_d(n)$ is obtained by taking inverse Fourier transform of $H_d(\omega)$.

$$\therefore \qquad h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_{c_2}}^{-\omega_{c_1}} e^{-j\omega\alpha} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\omega_{c_1}}^{\omega_{c_2}} e^{-j\omega\alpha}\, e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{-2}^{-1} e^{j\omega(n-\alpha)} d\omega + \frac{1}{2\pi}\int_{1}^{2} e^{j\omega(n-\alpha)} d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-2}^{-1} + \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{1}^{2}$$

$$= \frac{1}{\pi(n-\alpha)}\left[\frac{e^{j2(n-\alpha)} - e^{-j2(n-\alpha)}}{2j} - \frac{e^{j(n-\alpha)} - e^{-j(n-\alpha)}}{2j}\right]$$

$$= \frac{1}{\pi(n-\alpha)}\left[\sin 2(n-\alpha) - \sin(n-\alpha)\right]$$

Hence, $$h_d(n) = \frac{1}{\pi(n-\alpha)}\,[\sin 2(n-\alpha) - \sin(n-\alpha)]\,;\ \text{for } n \neq \alpha$$

When $n = \alpha$, the terms $\dfrac{\sin 2(n-\alpha)}{\pi(n-\alpha)}$ and $\dfrac{\sin(n-\alpha)}{\pi(n-\alpha)}$ become 0/0 which is indeterminate.

Hence using L' Hospital rule, we get when $n = \alpha$

$$h_d(n) = \underset{n\to\alpha}{\text{Lt}}\ \frac{1}{\pi}\left[\frac{\sin 2(n-\alpha)}{(n-\alpha)} - \frac{\sin(n-\alpha)}{(n-\alpha)}\right] = \frac{1}{\pi}\,[2-1] = \frac{1}{\pi}$$

Here, $$\alpha = \frac{N-1}{2} = \frac{5-1}{2} = 2$$

Therefore, we have

$$h_d(0) = \frac{\sin 2(0-2) - \sin(0-2)}{\pi(0-2)} = -0.2651\,, \quad h_d(1) = \frac{\sin 2(1-2) - \sin(1-2)}{\pi(1-2)} = 0.0215$$

$$h_d(2) = \frac{1}{\pi} = 0.3183, \quad h_d(3) = \frac{\sin 2(3-2) - \sin(3-2)}{\pi(3-2)} = 0.0215$$

$$h_d(4) = \frac{\sin 2(4-2) - \sin(4-2)}{\pi(4-2)} = -0.2651$$

The Hanning window sequence is given by $w(n) = 0.5 - 0.5\cos\left(\dfrac{2\pi n}{N-1}\right)$; for $n = 0$ to $N-1$.

Therefore, we have

$$w(0) = 0.5 - 0.5\cos\left(\frac{2\pi\times 0}{5-1}\right) = 0\,, \quad w(1) = 0.5 - 0.5\cos\left(\frac{2\pi\times 1}{5-1}\right) = 0.5$$

$$w(2) = 0.5 - 0.5\cos\left(\frac{2\pi\times 2}{5-1}\right) = 1\,, \quad w(3) = 0.5 - 0.5\cos\left(\frac{2\pi\times 3}{5-1}\right) = 0.5$$

$$w(4) = 0.5 - 0.5\cos\left(\frac{2\pi\times 4}{5-1}\right) = 0$$

Therefore, the filter coefficients are $h(n) = h_d(n)\ w(n)$. Hence, we have

$$h(0) = h_d(0)\ w(0) = -0.2651 \times 0 = 0$$

$$h(1) = h_d(1)\ w(1) = 0.0215 \times 0.5 = 0.0108$$

$$h(2) = h_d(2)\ w(2) = 0.3183 \times 1 = 0.3183$$

$$h(3) = h_d(3)\ w(3) = 0.0215 \times 0.5 = 0.0108$$

$$h(4) = h_d(4)\ w(4) = -0.2651 \times 0 = 0$$

i.e., $h(0) = 0,\ h(1) = 0.0108, h(2) = 0.3183, h(3) = 0.0108$, and $h(4) = 0$

From the above filter coefficients, it can observed that the impulse response is symmetrical with centre of symmetry at $n = 2$. The frequency response is given by

$$H(\omega) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$$

$$= h(0) + h(1)\, e^{-j\omega} + h(2)\, e^{-j2\omega} + h(3)\, e^{-j3\omega} + h(4)\, e^{-j4\omega}$$

$$= 0.0108 e^{-j\omega} + 0.3183\, e^{-j2\omega} + 0.0108\, e^{-j3\omega}$$

$$= e^{-j2\omega}[0.3183 + 0.0108\, e^{j\omega} + 0.0108\, e^{-j\omega}]$$

$$= e^{-j2\omega}[0.3183 + 0.0216 \cos \omega]$$

The magnitude response is given by

$$|H(\omega)| = 0.3183 + 0.0216 \cos \omega$$

The transfer function of the digital FIR band-pass filter is

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n} = \sum_{n=0}^{4} h(n)\, z^{-n}$$

$$= h(0) + h(1)\, z^{-1} + h(2)\, z^{-2} + h(3)\, z^{-3} + h(4)\, z^{-4}$$

$$= 0.0108\, z^{-1} + 0.3183\, z^{-2} + 0.0108\, z^{-3}$$

EXAMPLE 9.13 Design a band-stop filter to reject frequencies in the range 1 to 2 rad/sec using rectangular window, with $N = 7$.

Solution: The desired frequency response for band stop filter is:

$$H_d(\omega) = \begin{cases} e^{-j\omega\alpha}, & -\pi \le \omega \le -\omega_{c2} \text{ and } -\omega_{c1} \le \omega \le \omega_{c1} \text{ and } \omega_{c2} \le \omega \le \pi \\ 0, & \text{otherwise} \end{cases}$$

The desired impulse response $h_d(n)$ is obtained by taking inverse Fourier transform of $H_d(\omega)$.

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \left[\int_{-\pi}^{-\omega_{c2}} e^{-j\omega\alpha} e^{j\omega n} d\omega + \int_{-\omega_{c1}}^{\omega_{c1}} e^{-j\omega\alpha} e^{j\omega n} d\omega + \int_{\omega_{c2}}^{\pi} e^{-j\omega\alpha} e^{j\omega n} d\omega \right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\omega_{c2}} e^{j\omega(n-\alpha)}d\omega + \frac{1}{2\pi}\int_{-\omega_{c1}}^{\omega_{c1}} e^{j\omega(n-\alpha)}d\omega + \frac{1}{2\pi}\int_{\omega_{c2}}^{\pi} e^{j\omega(n-\alpha)}d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\pi}^{-\omega_{c2}} + \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\omega_{c1}}^{\omega_{c1}} + \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{\omega_{c2}}^{\pi}$$

$$= \frac{1}{2\pi j(n-\alpha)}\left[e^{-j(n-\alpha)}\omega_{c_2} - e^{-j(n-\alpha)}\pi + e^{j(n-\alpha)}\omega_{c_1} - e^{-j(n-\alpha)}\omega_{c_1} + e^{j(n-\alpha)}\pi - e^{j(n-\alpha)}\omega_{c_2}\right]$$

$$= \frac{1}{\pi(n-\alpha)}\left[\frac{e^{j(n-\alpha)}\omega_{c_1} - e^{-j(n-\alpha)}\omega_{c_1}}{2j} + \frac{e^{j(n-\alpha)}\pi - e^{-j(n-\alpha)}\pi}{2j} - e^{j(n-\alpha)}\omega_{c_2} - e^{-j(n-\alpha)}\omega_{c_2}\right]$$

$$= \frac{1}{\pi(n-\alpha)}[\sin(n-\alpha)\omega_{c1} + \sin(n-\alpha)\pi - \sin(n-\alpha)\omega_{c2}]$$

When $n = \alpha$, the terms $\frac{\sin(n-\alpha)\omega_{c1}}{n-\alpha}$, $\frac{\sin(n-\alpha)\pi}{n-\alpha}$ and $\frac{\sin(n-\alpha)\omega_{c2}}{n-\alpha}$ become 0/0, which is indeterminate.

Hence, $h_d(n) = \frac{1}{\pi(n-\alpha)}[\sin(n-\alpha)\omega_{c1} + \sin(n-\alpha)\pi - \sin(n-\alpha)\omega_{c2}]$ for $n \neq \alpha$

When $n = \alpha$, using L'Hospital rule, we have

$$h_d(n) = \operatorname*{Lt}_{n\to\alpha} \frac{1}{\pi}\left[\frac{\sin(n-\alpha)\omega_{c1}}{(n-\alpha)} + \frac{\sin(n-\alpha)\pi}{(n-\alpha)} - \frac{\sin(n-\alpha)\omega_{c2}}{(n-\alpha)}\right]$$

$$= \frac{1}{\pi}[\omega_{c1} + \pi - \omega_{c2}] = 1 - \left(\frac{\omega_{c2}-\omega_{c1}}{\pi}\right)$$

$$\therefore \qquad h_d(n) = 1 - \left(\frac{\omega_{c2}-\omega_{c1}}{\pi}\right) \text{ for } n = \alpha$$

The rectangular window sequence $w_R(n)$ is given by

$$w_R(n) = \begin{cases} 1, & \text{for } 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Therefore, the filter coefficients are $h(n) = h_d(n) \cdot w_R(n) = h_d(n)$. Given that the order of the filter $N = 7$.

$$\therefore \qquad \alpha = \frac{N-1}{2} = \frac{7-1}{2} = 3$$

Since both n and α are integers, $(n - \alpha)$ is also an integer, so $\sin (n - \alpha)\pi = 0$. Also we are given that ω_{c2} = 2 rad/sec and ω_{c1} = 1 rad/sec. Therefore, we have for n = 3.

$$h(n) = h_d(n) = 1 - \frac{(\omega_{c2} - \omega_{c1})}{\pi} = 1 - \frac{(2-1)}{\pi} = 1 - \frac{1}{\pi}$$

and for $n \neq 3$

$$h(n) = h_d(n) = \frac{1}{(n-\alpha)\pi}\,[\sin (n-\alpha)\omega_{c1} - \sin (n-\alpha)\omega_{c2}] = \frac{\sin (n-3) - \sin 2(n-3)}{(n-3)\pi}$$

Therefore, the designed filter coefficients are:

$$h(0) = \frac{\sin (0-3) - \sin 2(0-3)}{(0-3)\pi} = 0.0446, \qquad h(1) = \frac{\sin (1-3) - \sin 2(1-3)}{(1-3)\pi} = 0.2652$$

$$h(2) = \frac{\sin (2-3) - \sin 2(2-3)}{(2-3)\pi} = -0.0216, \quad h(3) = 1 - \frac{1}{\pi} = 0.6817$$

$$h(4) = \frac{\sin (4-3) - \sin 2(4-3)}{(4-3)\pi} = -0.0216, \quad h(5) = \frac{\sin (5-3) - \sin 2(5-3)}{(5-3)\pi} = 0.2652$$

$$h(6) = \frac{\sin (6-3) - \sin 2(6-3)}{(6-3)\pi} = 0.0446$$

That is $h(0) = 0.0446$, $h(1) = 0.2652$, $h(2) = -0.216$, $h(3) = 0.6817$, $h(4) = -0.0216$, $h(5) = 0.2652$, $h(6) = 0.0446$.

Observing the values of $h(n)$, we can note that

$$h(0) = h(6),\ h(1) = h(3) \text{ and } h(2) = h(4)$$

So the impulse response is symmetrical with centre of symmetry at n = 3. The frequency response of the filter is:

$$H(\omega) = \sum_{n=0}^{N-1} h(n)e^{-j\omega n} = \sum_{n=0}^{6} h(n)e^{-j\omega n}$$

$$= h(0) + h(1)e^{-j\omega} + h(2)e^{-j2\omega} + h(3)e^{-j3\omega} + h(4)e^{-j4\omega} + h(5)e^{-j5\omega} + h(6)e^{-j6\omega}$$

$$= e^{-j3\omega}[h(0)e^{j3\omega} + h(6)e^{-j3\omega} + h(1)e^{j2\omega} + h(5)e^{-j2\omega} + h(2)e^{j\omega} + h(4)e^{-j\omega} + h(3)]$$

$$= e^{-j3\omega}\,[h(3) + 2h(0)\cos 3\omega + 2h(1)\cos 2\omega + 2h(2)\cos \omega]$$

$$= e^{-j3\omega}\,[0.6817 + 0.0892\cos 3\omega + 0.5304\cos 2\omega - 0.0432\cos \omega]$$

The magnitude response is:

$$|H(\omega)| = 0.6817 - 0.0432\cos \omega + 0.5304\cos 2\omega + 0.0892\cos 3\omega$$

The transfer function of the digital FIR filter is:

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n} = \sum_{n=0}^{6} h(n)\, z^{-n}$$

$$= h(0) + h(1)\, z^{-1} + h(2)z^{-2} + h(3)\, z^{-3} + h(4)\, z^{-4} + h(5)\, z^{-5} + h(6)\, z^{-6}$$

$$= 0.0446 + 0.2652\, z^{-1} - 0.0216\, z^{-2} + 0.6817\, z^{-3} - 0.0216\, z^{-4} + 0.2652\, z^{-5} + 0.0446\, z^{-6}$$

$$= z^{-3}[0.6817 - 0.0216(z + z^{-1}) + 0.2652\,(z^2 + z^{-2}) + 0.0446(z^3 + z^{-3})]$$

EXAMPLE 9.14 Design a high-pass FIR filter for the following specifications:

Cutoff frequency = 500 Hz
Sampling frequency = 2000 Hz
$N = 11$

Solution: Given f_c = 500 Hz and f_s = 2000 Hz
Normalized cutoff frequency is:

$$\omega_c = \frac{2\pi f_c}{f_s} = \frac{2\pi \times 500}{2000} = \frac{\pi}{2} \text{ rad/sec}$$

The desired frequency response of the filter is:

$$H_d(\omega) = \begin{cases} 1, & \dfrac{\pi}{2} \le |\,\omega\,| \le \pi, \text{ i.e. for } -\pi \le \omega \le -\dfrac{\pi}{2} \text{ and } \dfrac{\pi}{2} \le \omega \le \pi \\ 0, & \text{otherwise, i.e. for } -\dfrac{\pi}{2} \le \omega \le \dfrac{\pi}{2} \end{cases}$$

The desired impulse response is:

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\pi/2} H_d(\omega)\, e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\pi/2}^{\pi} H_d(\omega)\, e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\pi/2} (1)\, e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\pi/2}^{\pi} (1)\, e^{j\omega n} d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-\pi/2} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\frac{n\pi}{2}} - e^{-jn\pi}}{jn} + \frac{e^{jn\pi} - e^{j\frac{n\pi}{2}}}{jn}\right]$$

$$= \frac{1}{n\pi}\left[\frac{e^{jn\pi} - e^{-jn\pi}}{2j} - \frac{e^{j\frac{n\pi}{2}} - e^{-j\frac{n\pi}{2}}}{2j}\right]$$

$$= \frac{\sin n\pi}{n\pi} - \frac{\sin\frac{n\pi}{2}}{n\pi} = -\frac{\sin\frac{n\pi}{2}}{n\pi}$$

Knowing $h_d(n)$, using standard procedure, we can design the required high-pass FIR filter with $N = 11$.

EXAMPLE 9.15 Design a band-pass FIR filter for the following specifications:

Cutoff frequencies = 400 Hz and 800 Hz

Sampling frequency = 2000 Hz

$N = 11$

Solution: Given cutoff frequencies are:

$$f_{c1} = 400\text{ Hz} \quad \text{and} \quad f_{c2} = 800\text{ Hz}$$

Sampling frequency is $f_s = 2000$ Hz

The normalized cutoff frequencies are:

$$\omega_{c1} = \frac{2\pi f_{c1}}{f_s} = \frac{2\pi \times 400}{2000} = 0.4\pi$$

and $$\omega_{c2} = \frac{2\pi f_{c2}}{f_s} = \frac{2\pi \times 800}{2000} = 0.8\pi$$

The desired frequency response is:

$$H_d(\omega) = \begin{cases} 1, & -\omega_{c2} \le \omega \le -\omega_{c1} \text{ and } \omega_{c1} \le \omega \le \omega_{c2} \\ 0, & \text{otherwise (i.e. } -\pi \le \omega \le -\omega_{c2}, -\omega_{c1} \le \omega \le \omega_{c1} \text{ and } \omega_{c2} \le \omega \le \pi) \end{cases}$$

i.e. $$H_d(\omega) = \begin{cases} 1, & -0.8\pi \le \omega \le -0.4\pi \text{ and } 0.4\pi \le \omega \le 0.8\pi \\ 0, & -\pi \le \omega \le -0.8\pi, -0.4\pi \le \omega \le 0.4\pi, 0.8\pi \le \omega \le \pi \end{cases}$$

The desired impulse response of the filter is:

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{-0.8\pi}^{-0.4\pi} (1)\, e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{0.4\pi}^{0.8\pi} (1)\, e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-0.8\pi}^{-0.4\pi} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{0.4\pi}^{0.8\pi}$$

$$= \frac{1}{n\pi}\left[\frac{e^{-j0.4\pi n} - e^{-j0.8\pi n}}{2j} + \frac{e^{j0.8\pi n} - e^{j0.4\pi n}}{2j}\right]$$

$$= \frac{1}{n\pi}\left[\frac{e^{j0.8n\pi} - e^{-j0.8n\pi}}{2j} - \frac{e^{j0.4\pi n} - e^{-j0.4\pi n}}{2j}\right]$$

$$= \frac{\sin(0.8n\pi)}{n\pi} - \frac{\sin(0.4n\pi)}{n\pi}$$

Knowing $h_d(n)$ using standard procedure we can design the required band-pass FIR filter with $n = 11$.

EXAMPLE 9.16 Design an FIR band-stop (band reject or band elimination or notch) filter for the following specifications.

Cutoff frequencies = 400 Hz and 800 Hz

Sampling frequency = 2000 Hz

$N = 11$

Solution: Given cutoff frequencies are

$$f_{c1} = 400 \text{ Hz} \quad \text{and} \quad f_{c2} = 800 \text{ Hz}$$

Sampling frequency is $f_s = 2000$ Hz

The normalized cutoff frequencies are:

$$\omega_{c1} = \frac{2\pi f_{c1}}{f_s} = \frac{2\pi \times 400}{2000} = 0.4\pi$$

and

$$\omega_{c2} = \frac{2\pi f_{c2}}{f_s} = \frac{2\pi \times 800}{2000} = 0.8\pi$$

The desired frequency response is:

$$H_d(\omega) = \begin{cases} 1, & -\pi \le \omega \le -\omega_{c2}, -\omega_{c1} \le \omega \le \omega_{c1} \text{ and } \omega_{c2} \le \omega \le \pi \\ 0, & \text{otherwise (i.e } -\omega_{c2} \le \omega \le -\omega_{c1} \text{ and } \omega_{c1} \le \omega \le \omega_{c2}) \end{cases}$$

$$\therefore \quad H_d(\omega) = \begin{cases} 1, & -\pi \le \omega \le -0.8\pi, -0.4\pi \le \omega \le 0.4\pi \text{ and } 0.8\pi \le \omega \le \pi \\ 0, & \text{otherwise} \end{cases}$$

The desired impulse response of the filter is:

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-0.8\pi} (1)\, e^{j\omega n} d\omega + \int_{-0.4\pi}^{0.4\pi} (1)\, e^{j\omega n} d\omega + \int_{0.8\pi}^{\pi} (1)\, e^{j\omega n} d\omega\right]$$

$$= \frac{1}{2\pi}\left[\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-0.8\pi} + \left[\frac{e^{j\omega n}}{jn}\right]_{-0.4\pi}^{0.4\pi} + \left[\frac{e^{j\omega n}}{jn}\right]_{0.8\pi}^{\pi}\right]$$

$$= \frac{1}{2\pi jn}\left[e^{-j0.8n\pi} - e^{-jn\pi} + e^{j0.4n\pi} - e^{-j0.4n\pi} + e^{jn\pi} - e^{j0.8n\pi}\right]$$

$$= \frac{1}{n\pi}\left[\frac{e^{jn\pi} - e^{-jn\pi}}{2j} - \frac{e^{j0.8n\pi} - e^{-j0.8n\pi}}{2j} + \frac{e^{j0.4n\pi} - e^{-j0.4n\pi}}{2j}\right]$$

$$= \frac{1}{n\pi}[\sin n\pi - \sin(0.8n\pi) + \sin(0.4n\pi)]$$

$$= \frac{1}{n\pi}[\sin(0.4n\pi) - \sin(0.8n\pi)]$$

Knowing $h_d(n)$, using standard procedure, we can design the required band-reject filter with $N = 11$.

9.6.7 Kaiser Window

From the frequency domain characteristics of the window functions listed in Table 9.1, it can be seen that a trade off exists between the main lobe width and side lobe amplitude. The main lobe width is inversely proportional to N. As the length of the filter is increased, the width of the main lobe becomes narrower and narrower, and the transition band is reduced considerably. However, the minimum stop band attenuation is independent of N and is a function of the selected window. Thus, in order to achieve prescribed minimum stop band attenuation and pass band ripple, the designer must find a window with an appropriate side lobe level and then choose N to achieve the prescribed transition width. In this process, the designer may often have to settle for a window with undesirable design specifications.

A desirable property of the window function is that the function is of finite duration in the time domain and that the Fourier transform has maximum energy in the main lobe or a given peak side lobe amplitude. The prolate spheroidal functions have this desirable property; however these functions are complicated and difficult to compute. A simple approximation to these functions have been developed by Kaiser in terms of zeroth order modified Bessel functions of the first kind. In a Kaiser window, the side lobe level can be

controlled with respect to the main lobe peak by varying a parameter α. The width of the main lobe can be varied by adjusting the length of the filter. The Kaiser window function is given by

$$w_k(n) = \begin{cases} \dfrac{I_0(\beta)}{I_0(\alpha)}, & \text{for } |n| \le \dfrac{N-1}{2} \\ 0 & \text{, otherwise} \end{cases}$$

where α is an independent variable determined by Kaiser. The parameter β is expressed by

$$\beta = \alpha\left[1 - \left(\frac{2n}{N-1}\right)^2\right]^{\frac{1}{2}}$$

The modified Bessel function of the first kind, $I_0(x)$ can be computed from its power series expansion given by

$$I_0(x) = 1 + \sum_{k=1}^{\infty}\left[\frac{1}{k!}\left(\frac{x}{2}\right)^k\right]^2$$

$$= 1 + \frac{0.25x^2}{(1!)^2} + \frac{(0.25x^2)^2}{(2!)^2} + \frac{(0.25x^2)^3}{(3!)^3} + \cdots$$

Figure 9.8 shows the idealized frequency responses of different filters with their pass band and stop band specifications. Considering the design specifications of the filters in Figure 9.8, the actual pass band ripple (α_p) and minimum stop band attenuation (α_s) are given by

$$\alpha_p = 20 \log_{10} \frac{1+\delta_p}{1-\delta_p} \text{ dB}$$

and

$$\alpha_s = -20 \log_{10} \alpha_s \text{ dB}$$

The transition bandwidth is

$$\Delta F = f_s - f_p$$

Let α'_p and α'_s be the specified pass band ripple and minimum stop band attenuation, respectively and

$$\alpha_p \le \alpha'_p$$

$$\alpha_s \le \alpha'_s$$

where α_p and α_s are the actual pass band peak to peak ripple and minimum stop band attenuation respectively.

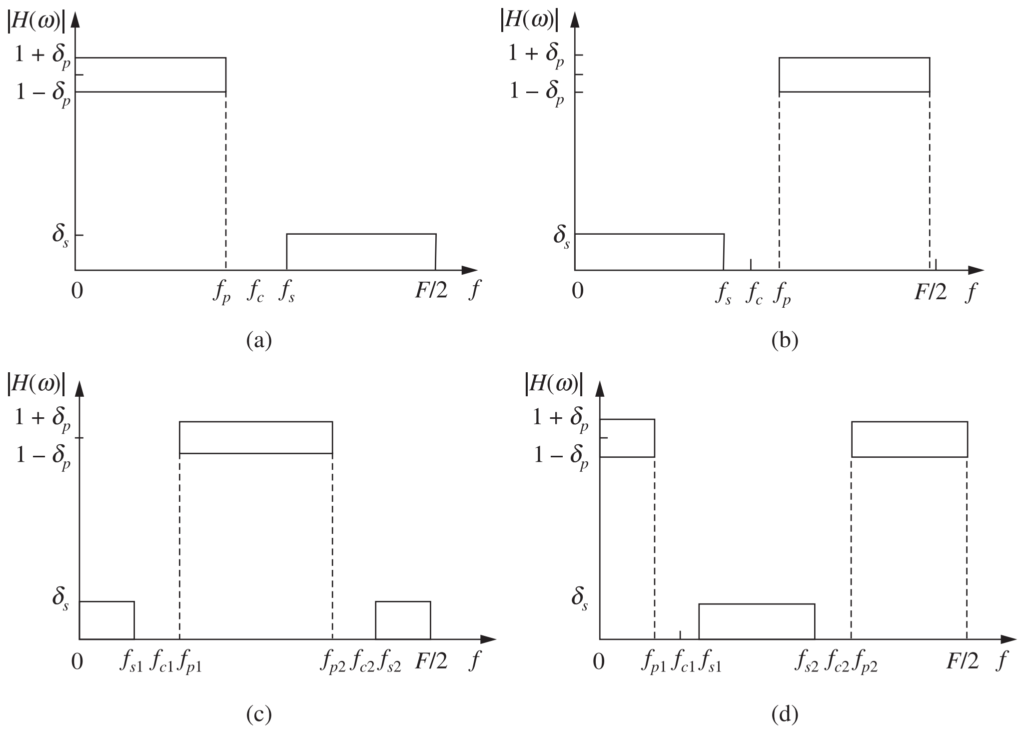


Figure 9.8 Idealized frequency responses: (a) low-pass filter, (b) high-pass filter, (c) band pass filter and (d) band stop filter.

Design specifications

1. Filter type; low-pass, high-pass, band pass or band stop
2. Pass band and stop band frequencies in hertz
   For low-pass/high-pass: f_p and f_s
   For band pass/band stop: f_{p1}, f_{p2}, f_{s1}, f_{s2}
3. Pass band ripple and minimum stop band attenuation in positive dB; α'_p and α'_s
4. Sampling frequency in hertz: F
5. Filter order N-odd

Design procedure

1. Determine $h_d(n)$ for an ideal frequency response $H(\omega)$
2. Choose δ according to equations

$$\alpha_p = 20 \log_{10} \frac{1+\delta_p}{1-\delta_p} \text{ dB and } \alpha_s = -20 \log \delta_s \text{ dB}$$

and $\alpha_p \le \alpha'_p$ and $\alpha_s \ge \alpha'_s$

where the actual design parameter can be determined from

$$\delta = \min(\delta_p,\ \delta_s)$$

where $\delta_s = 10\,e^{-0.05\alpha'_s}$ and $\delta_p = \dfrac{10\,e^{0.05\alpha'_p} - 1}{10\,e^{0.05\alpha'_p} + 1}$

3. Calculate α_s using the formula

$$\alpha_s = -20\ \log_{10}\ \delta_s$$

4. Determine the parameter α from the following equation for

$$\alpha = \begin{cases} 0, & \text{for } \alpha_s < 21 \\ 0.5842\,(\alpha_s - 21)^{0.4} + 0.07886\,(\alpha_s + 21), & \text{for } 21 < \alpha_s \le 50 \\ 0.1102(\alpha_s - 8.7), & \text{for } \alpha_s > 50 \end{cases}$$

5. Determine the parameter D from the following Kaiser's design equation

$$D = \begin{cases} 0.9222, & \text{for } \alpha_s \le 21 \\ \dfrac{\alpha_s - 7.95}{14.36}, & \text{for } \alpha_s > 21 \end{cases}$$

6. Choose the filter order for the lowest odd value of N

$$N \ge \frac{\omega_{sf} D}{B} + 1$$

7. Compute the window sequence using equation

$$w_k(n) = \begin{cases} \dfrac{I_0\left[\alpha\sqrt{1 - \left(\dfrac{2n}{N-1}\right)^2}\,\right]}{I_0(\alpha)}, & \text{for } |n| \le \dfrac{N-1}{2} \\ 0, & \text{otherwise} \end{cases}$$

8. Compute the modified impulse response using

$$h(n) = w_k(n)h_d(n)$$

9. The transfer function is given by

$$H(z) = z^{-\left(\frac{N-1}{2}\right)}\left[h(0) + 2\sum_{n=1}^{(N-1)/2} h(n)\,(z^n + z^{-n})\right]$$

10. The magnitude response can be obtained using

$$\tilde{H}(\omega) = \sum_{n=0}^{(N-1)/2} a(n) \cos \omega n$$

where
$$a(0) = h\left(\frac{N-1}{2}\right)$$

$$a(n) = 2h\left(\frac{N-1}{2} - n\right)$$

The design equations for the low-pass, high-pass, band-pass and band-stop filters are given below:

Low-pass FIR filter

$$h_d(n) = \begin{cases} \left(\dfrac{2f_c}{F}\right)\dfrac{\sin(2\pi n f_c/F)}{2\pi n f_c/F}, & \text{for } n > 0 \\ \dfrac{2f_c}{F}, & \text{for } n = 0 \end{cases}$$

where
$$f_c = \frac{1}{2}(f_p + f_s) \text{ and } \Delta F = f_s - f_p$$

High-pass FIR filter

$$h_d(n) = \begin{cases} -\left(\dfrac{2f_c}{F}\right)\dfrac{\sin(2\pi n f_c/F)}{2\pi n f_c/F}, & \text{for } n > 0 \\ 1 - \dfrac{2f_c}{F}, & \text{for } n = 0 \end{cases}$$

where
$$f_c = \frac{1}{2}(f_p + f_s) \text{ and } \Delta F = f_p - f_s$$

Band-pass FIR filter

$$h_d(n) = \begin{cases} \dfrac{1}{n\pi}[\sin(2\pi n f_{c2}/F) - \sin(2\pi n f_{c1}/F)], & \text{for } n > 0 \\ \dfrac{2}{F}(f_{c2} - f_{c1}), & \text{for } n = 0 \end{cases}$$

where

$$f_{c1} = f_{p1} - \frac{\Delta F}{2} \qquad f_{c2} = f_{p2} + \frac{\Delta F}{2}$$

$$\Delta F_1 = f_{p1} - f_{s1} \qquad \Delta F_h = f_{s2} - f_{p2}$$

$$\Delta F = \min[\Delta F_l, \Delta F_h]$$

Band-stop FIR filter

$$h_d(n) = \begin{cases} \dfrac{1}{n\pi}[\sin(2\pi n f_{c1}/F) - \sin(2\pi n f_{c2}/F)], & \text{for } n > 0 \\ \dfrac{2}{F}(f_{c1} - f_{c2}) + 1, & \text{for } n = 0 \end{cases}$$

where

$$f_{c1} = f_{p1} + \frac{\Delta F}{2} \qquad f_{c2} = f_{p2} - \frac{\Delta F}{2}$$

$$\Delta F_1 = f_{s1} - f_{p1} \qquad \Delta F_h = f_{p2} - f_{s2}$$

$$\Delta F = \min[\Delta F_l, \Delta F_h]$$

Summary of windows

The different windows parameters are compared in Table 9.3. Looking at the parameters for rectangular and triangular window, it can be noted that the triangular window has a transition width twice that of rectangular window. However, the attenuation in stop band for triangular window is less. Therefore, it is not very popular for FIR filter design. The Hanning and Hamming windows have same transition width. But the Hamming window is most widely used because it generates less ringing in the side lobes. The Blackman window reduces the side lobe level at the cost of increase in transition width. The Kaiser window is superior to other windows because for given specifications its transition width is always small. By varying the parameter α, the desired side lobe level and main lobe peak can be achieved. Further the main lobe width can be varied by varying the length N. That is why Kaiser window is the favourite window for many digital filter designers.

The window design for FIR filter has certain advantages and disadvantages.

Advantages

1. The filter coefficients can be obtained with minimum computation effort.
2. The window functions are readily available in closed-form expression.
3. The ripples in both stop band and pass band are almost completely eliminated.

Disadvantages

1. It is not always possible to obtain a closed form expression for the Fourier series coefficients $h(n)$.
2. Windows provide little flexibility in design.
3. It is somewhat difficult to determine, in advance, the type of window and duration N required to meet a given prescribed frequency specification.

Table 9.3 shows the different window sequences.

TABLE 9.3 Window sequences (functions) for FIR filter design

Name of window	*Window sequence*
Rectangular window	$w_R(n) = \begin{cases} 1, & -\dfrac{(N-1)}{2} \le n \le \dfrac{N-1}{2} \\ 0, & \text{otherwise} \end{cases}$ $w_R(n) = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$
Triangular window	$w_T(n) = \begin{cases} 1 - \dfrac{2\|n\|}{N-1}, & -\dfrac{(N-1)}{2} \le n \le \dfrac{(N-1)}{2} \\ 0, & \text{otherwise} \end{cases}$ (or) $w_T(n) = \begin{cases} 1 - \dfrac{2\|n-(N-1)/2\|}{N-1}, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$
Hanning window	$w_{hn}(n) = \begin{cases} 0.5 + 0.5 \cos \dfrac{2n\pi}{N-1}, & -\dfrac{(N-1)}{2} \le n \le \dfrac{(N-1)}{2} \\ 0, & \text{otherwise} \end{cases}$ (or) $w_{hn}(n) = \begin{cases} 0.5 - 0.5 \cos \dfrac{2n\pi}{N-1}, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$
Hamming window	$w_H(n) = \begin{cases} 0.54 + 0.46 \cos \dfrac{2n\pi}{N-1}, & -\dfrac{(N-1)}{2} \le n \le \dfrac{N-1}{2} \\ 0, & \text{otherwise} \end{cases}$ (or) $w_H(n) = \begin{cases} 0.54 - 0.46 \cos \dfrac{2n\pi}{N-1}, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$

(*Contd.*)

TABLE 9.3 Window sequences (functions) for FIR filter design (*Contd.*)

Name of window	*Window sequence*
Blackman window	$w_B(n) = \begin{cases} 0.42 + 0.5\cos\dfrac{2n\pi}{N-1} + 0.08\cos\dfrac{4n\pi}{N-1}, & -\dfrac{(N-1)}{2} \le 0 \le \left(\dfrac{N-1}{2}\right) \\ 0 & \text{, otherwise} \end{cases}$ (or) $w_B(n) = \begin{cases} 0.42 - 0.5\cos\dfrac{2n\pi}{N-1} + 0.08\cos\dfrac{4n\pi}{N-1}, & 0 \le n \le N-1 \\ 0 & \text{, otherwise} \end{cases}$
Kaiser	$w_k(n) = \begin{cases} \dfrac{I_0\left[\alpha\sqrt{1-\left(\dfrac{2n}{N-1}\right)^2}\right]}{I_0(\alpha)}, & -\dfrac{(N-1)}{2} \le n \le \dfrac{(N-1)}{2} \\ 0 & \text{, otherwise} \end{cases}$ (or) $w_k(n) = \begin{cases} \dfrac{I_0\left[\alpha\sqrt{\left(\dfrac{N-1}{2}\right)^2 - \left(n - \dfrac{N-1}{2}\right)^2}\right]}{I_0\left(\alpha\dfrac{N-1}{2}\right)}, & 0 \le n \le \dfrac{(N-1)}{2} \\ 0 & \text{, otherwise} \end{cases}$

EXAMPLE 9.17 Design an FIR low-pass filter satisfying the following specifications:

$$\alpha_p \le 0.1 \text{ dB}, \quad \alpha_s \ge 38 \text{ dB}$$

$$\omega_p = 15 \text{ rad/sec}, \quad \omega_s = 25 \text{ rad/sec}, \quad \omega_{sf} = 80 \text{ rad/sec}$$

Solution: From the given specifications,

$$B = \omega_s - \omega_p = 25 - 15 = 10 \text{ rad/sec}$$

$$\omega_c = \frac{1}{2}(\omega_p + \omega_s) = \frac{1}{2}(15 + 25) = 20 \text{ rad/sec}$$

$$\omega_c \text{(in radians)} = \omega_c T = \omega_c \times \frac{2\pi}{\omega_{sf}} = \frac{20 \times 2\pi}{80} = \frac{\pi}{2}$$

Step 1:

$$H(\omega) = \begin{cases} 1, & \text{for } |\omega| \le \dfrac{\pi}{2} \\ 0, & \text{for } \dfrac{\pi}{2} < |\omega| \le 2\pi \end{cases}$$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{j\omega n}\, d\omega = \frac{\sin n\frac{\pi}{2}}{n\pi}$$

Step 2:

$$\delta_s = 10^{-0.05(38)} = 12.59 \times 10^{-3}$$

$$\delta_p = \frac{10^{0.05(0.1)} - 1}{10^{0.05(0.1)} + 1} = 5.7564 \times 10^{-3}$$

$$\delta = \min(\delta_s, \delta_p) = 5.7564 \times 10^{-3}$$

Step 3:

$$\alpha_s = -20 \log_{10}\delta = 44.797 \text{ dB}$$

Step 4: For

$$\alpha_s = 44.797$$

$$\alpha = 0.5842\,(\alpha_s - 21)^{0.4} + 0.07886\,(\alpha_s - 21) = 3.9524$$

Step 5: For

$$\alpha_s = 44.797$$

$$D = \frac{\alpha_s - 7.95}{14.36} = 2.566$$

Step 6:

$$N \ge \frac{\omega_{sf} D}{B} + 1$$

$$\ge \frac{80(2.566)}{10} + 1$$

$$\ge 21.52$$

Hence $N = 23$

Step 7: The window sequence

$$w_k(n) = \frac{I_0\left[\alpha\sqrt{1 - \left(\dfrac{2n}{N-1}\right)^2}\right]}{I_0(\alpha)} \quad \text{for } |n| \le \frac{N-1}{2}$$

For α = 3.9524, from equation

$$I_0(x) = 1 + \sum_{k=1}^{\infty}\left[\frac{1}{k!}\left(\frac{x}{2}\right)^k\right]^2$$

We can find

$$I_0(\alpha) = 1 + \sum_{k=1}^{\infty}\left[\frac{1}{k!}\left(\frac{\alpha}{2}\right)^k\right]^2 = 1 + \frac{0.25(3.9524)^2}{(1!)^2} + \frac{(0.25\times 3.9524^2)^2}{(2!)^2} + \frac{(0.25\times 3.9524^2)^3}{(3!)^2} + \cdots$$

$$= 10.8468$$

$$w_k(0) = \frac{I_0(\alpha)}{I_0(\alpha)} = 1$$

$$w_k(1) = w_k(-1) = \frac{I_0(3.9360)}{10.8468} = \frac{10.23}{10.8468} = 0.9431$$

$$w_k(2) = w_k(-2) = \frac{I_0(3.8860)}{10.8468} = \frac{9.828}{10.8468} = 0.906$$

$$w_k(3) = w_k(-3) = \frac{I_0(3.8025)}{10.8468} = \frac{9.1932}{10.8468} = 0.8475$$

$$w_k(4) = w_k(-4) = \frac{I_0(3.6818)}{10.8468} = \frac{8.3411}{10.8468} = 0.7689$$

$$w_k(5) = w_k(-5) = \frac{I_0(3.5204)}{10.8468} = \frac{7.3242}{10.8468} = 0.6752$$

$$w_k(6) = w_k(-6) = \frac{I_0(3.3126)}{10.8468} = \frac{6.1981}{10.8468} = 0.5714$$

$$w_k(7) = w_k(-7) = \frac{I_0(3.048)}{10.8468} = \frac{5.018}{10.8468} = 0.4626$$

$$w_k(8) = w_k(-8) = \frac{I_0(2.7127)}{10.8468} = \frac{3.8971}{10.8468} = 0.3592$$

$$w_k(9) = w_k(-9) = \frac{I_0(2.2724)}{10.8468} = \frac{2.7672}{10.8468} = 0.2551$$

$$w_k(10) = w_k(-10) = \frac{I_0(1.6465)}{10.8468} = \frac{1.8011}{10.8468} = 0.1660$$

$$w_k(11) = w_k(-11) = \frac{I_0(0)}{10.8468} = \frac{1}{10.8468} = 0.0922$$

We have $$h_d(n) = \frac{\sin(n\pi/2)}{n\pi}$$

For $n = 0$, $h_d(0) = 0.5$, $\therefore h(0) = h_d(0) \times w_k(0) = 0.5 \times 1 = 0.5$

For $n = 1$, $h_d(1) = 0.318$, $\therefore h(1) = h_d(1) \times w_k(1) = 0.318 \times 0.9431 = 0.2999$

For $n = 2$, $h_d(2) = 0$, $\therefore h(2) = h_d(2) \times w_k(2) = 0 \times 0.906 = 0$

For $n = 3$, $h_d(3) = -0.106$, $\therefore h(3) = h_d(3) \times w_k(3) = -0.106 \times 0.8475 = -0.0898$

For $n = 4$, $h_d(4) = 0$, $\therefore h(4) = h_d(4) \times w_k(4) = 0 \times 0.7689 = 0$

For $n = 5$, $h_d(5) = 0.0636$, $\therefore h(5) = h_d(5) \times w_k(5) = 0.0636 \times 0.6752 = 0.0429$

For $n = 6$, $h_d(6) = 0$, $\therefore h(6) = h_d(6) \times w_k(6) = 0 \times 0.5714 = 0$

For $n = 7$, $h_d(7) = -0.0454$, $\therefore h(7) = h_d(7) \times w_k(7) = -0.0454 \times 0.4626 = -0.0210$

For $n = 8$, $h_d(8) = 0$, $\therefore h(8) = h_d(8) \times w_k(8) = 0 \times 0.3592 = 0$

For $n = 9$, $h_d(9) = 0.03536$, $\therefore h(9) = h_d(9) \times w_k(9) = 0.03536 \times 0.2551 = 0.00902$

For $n = 10$, $h_d(10) = 0$, $\therefore h(10) = h_d(10) \times w_k(10) = 0 \times 0.1660 = 0$

For $n = 11$, $h_d(11) = -0.0289$, $\therefore h(11) = h_d(11) \times w_k(11) = -0.0289 \times 0.0922 = -0.00266$

The transfer function is given by

$$H(z) = z^{-11}\left[h(0) + \sum_{n=1}^{11} h(n)\,(z^{n} + z^{-n})\right]$$

9.7 DESIGN OF FIR FILTERS BY FREQUENCY SAMPLING TECHNIQUE

In this method, the ideal frequency response is sampled at sufficient number of points (i.e. N-points). These samples are the DFT coefficients of the impulse response of the filter. Hence the impulse response of the filter is determined by taking IDFT.

Let $H_d(\omega)$ = Ideal (desired) frequency response

$\tilde{H}(k)$ = The DFT sequence obtained by sampling $H_d(\omega)$

$h(n)$ = Impulse response of FIR filter

The impulse response $h(n)$ is obtained by taking IDFT of $\tilde{H}(k)$. For practical realizability, the samples of impulse response should be real. This can happen if all the complex terms appear in complex conjugate pairs. This suggests that the terms can be matched by comparing the exponentials. The terms $\tilde{H}(k)\,e^{j(2\pi nk/N)}$ should be matched by the term that has the exponential $e^{-j(2\pi nk)/N}$ as a factor.

Two design techniques are available, viz., type-I design and type-II design. In the type-I design, the set of frequency samples includes the sample at frequency $\omega = 0$. In some cases, it may be desirable to omit the sample at $\omega = 0$ and use some other set of samples. Such a design procedure is referred to as the type-II design.

Procedure for type-I design

1. Choose the ideal (desired) frequency response $H_d(\omega)$.
2. Sample $H_d(\omega)$ at N-points by taking $\omega = \omega_k = \frac{2\pi k}{N}$, where $k = 0, 1, 2, 3, ..., (N-1)$ to generate the sequence $\tilde{H}(k)$.

   $\therefore \qquad \tilde{H}(k) = H_d(\omega)|_{\omega=(2\pi k)/N}; \quad \text{for } k = 0, 1, 2, \ldots, (N-1)$
3. Compute the N samples of $h(n)$ using the following equations:

   When N is odd, $h(n) = \frac{1}{N}\left[\tilde{H}(0) + 2\sum_{k=1}^{(N-1)/2} \text{Re}\left(\tilde{H}(k)\, e^{j\frac{2\pi nk}{N}}\right)\right]$

   When N is even, $h(n) = \frac{1}{N}\left[\tilde{H}(0) + 2\sum_{k=1}^{\left(\frac{N}{2}-1\right)} \left(\tilde{H}(k)\, e^{j\frac{2\pi nk}{N}}\right)\right]$

   where 'Re' stands for 'real part of'.
4. Take Z-transform of the impulse response $h(n)$ to get the transfer function $H(z)$.

   $$\therefore \qquad H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

Procedure for type-II design

1. Choose the ideal (desired) frequency response $H_d(\omega)$.
2. Sample $H_d(\omega)$ at N-points by taking $\omega = \omega_k = \frac{2\pi(2k+1)}{2N}$, where $k = 0, 1, 2, ..., (N-1)$ to generate the sequence $\tilde{H}(k)$.

   $\therefore \qquad \tilde{H}(k) = H_d(\omega)|_{\omega=\frac{2\pi(2k+1)}{2N}}; \quad \text{for } k = 0, 1, 2, \ldots, (N-1)$
3. Compute the N samples of $h(n)$ using the following equations:

   When N is odd, $h(n) = \frac{2}{N}\sum_{k=0}^{(N-3)/2} \text{Re}\left[\tilde{H}(k)\, e^{jn\pi(2k+1)/N}\right]$

   When N is even, $h(n) = \frac{2}{N}\sum_{k=0}^{\left(\frac{N}{2}-1\right)} \text{Re}\left[\tilde{H}(k)\, e^{jn\pi(2k+1)/N}\right]$

4. Take Z-transform of the impulse response $h(n)$ to get the transfer function $H(z)$.

$$\therefore \qquad H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

Important formulae

The following formulae can be used for calculation of $h(n)$ while designing FIR filter by frequency sampling method.

1. $$\sum_{k=0}^{N-1} \cos k\theta = \frac{\sin \dfrac{N\theta}{2} \cos \dfrac{(N-1)}{2}\theta}{\sin \dfrac{\theta}{2}}$$

2. $$\sum_{k=0}^{N-1} \sin k\theta = \frac{\sin \dfrac{N\theta}{2} \sin \dfrac{(N-1)}{2}\theta}{\sin \dfrac{\theta}{2}}$$

The procedure for FIR filter design by frequency sampling method is:

1. Choose the desired frequency response $H_d(\omega)$.
2. Take N samples of $H_d(\omega)$ to generate the sequence $\tilde{H}(k)$.
3. Take inverse DFT of $\tilde{H}(k)$ to get the impulse response $h(n)$.
4. The transfer function $H(z)$ of the filter is obtained by taking Z-transform of the impulse response $h(n)$.

EXAMPLE 9.18 Design a linear phase low-pass FIR filter with a cutoff frequency of $\pi/2$ rad/sec using frequency sampling technique. Take $N = 13$.

Solution: The frequency response of desired linear phase low-pass filter with a cutoff frequency $\omega_c = \pi/2$ rad/sec can be written as:

$$H_d(\omega) = \begin{cases} e^{-j\alpha\omega}, & 0 \le |\omega| \le \pi/2 \\ 0 & , \text{ otherwise} \end{cases}$$

where $\alpha = \dfrac{N-1}{2} = \dfrac{13-1}{2} = 6$ (for linear phase filter)

Let us choose type-I design, therefore,

$$\omega_k = \frac{2\pi k}{N} = \frac{2\pi k}{13}$$

The sequence $\tilde{H}(k)$ is obtained by sampling $H_d(\omega)$ at 13 equidistant points in a period of 2π.

i.e., $$\tilde{H}(k) = H_d(\omega)\Big|_{\omega=\omega_k=\frac{2\pi k}{N}=\frac{2\pi k}{13}}$$

When $k = 0$, $\omega_k = \omega_0 = \dfrac{2\pi\times 0}{13} = 0$, When $k = 1$, $\omega_k = \omega_1 = \dfrac{2\pi\times 1}{13} = \dfrac{2\pi}{13}$

When $k = 2$, $\omega_k = \omega_2 = \dfrac{2\pi\times 2}{13} = \dfrac{4\pi}{13}$, When $k = 3$, $\omega_k = \omega_3 = \dfrac{2\pi\times 3}{13} = \dfrac{6\pi}{13}$

When $k = 4$, $\omega_k = \omega_4 = \dfrac{2\pi\times 4}{13} = \dfrac{8\pi}{13}$, When $k = 5$, $\omega_k = \omega_5 = \dfrac{2\pi\times 5}{13} = \dfrac{10\pi}{13}$

When $k = 6$, $\omega_k = \omega_6 = \dfrac{2\pi\times 6}{13} = \dfrac{12\pi}{13}$, When $k = 7$, $\omega_k = \omega_7 = \dfrac{2\pi\times 7}{13} = \dfrac{14\pi}{13}$

When $k = 8$, $\omega_k = \omega_8 = \dfrac{2\pi\times 8}{13} = \dfrac{16\pi}{13}$, When $k = 9$, $\omega_k = \omega_9 = \dfrac{2\pi\times 9}{13} = \dfrac{18\pi}{13}$

When $k = 10$, $\omega_k = \omega_{10} = \dfrac{2\pi\times 10}{13} = \dfrac{20\pi}{13}$, When $k = 11$, $\omega_k = \omega_{11} = \dfrac{2\pi\times 11}{13} = \dfrac{22\pi}{13}$

When $k = 12$, $\omega_k = \omega_{12} = \dfrac{2\pi\times 12}{13} = \dfrac{24\pi}{13}$

From the above calculations, the following observations can be made:

For $k = 0$ to 3, the samples lie in the range $0 \le \omega \le \dfrac{\pi}{2}$.

For $k = 4$ to 9, the samples lie in the range $\dfrac{\pi}{2} \le \omega \le \dfrac{3\pi}{2}$.

For $k = 10$ to 12, the samples lie in the range $\dfrac{3\pi}{2} \le \omega \le 2\pi$.

The sampling points on the ideal response are shown in Figure 9.9 and the magnitude spectrum of $\tilde{H}(k)$ is shown in Figure 9.10.

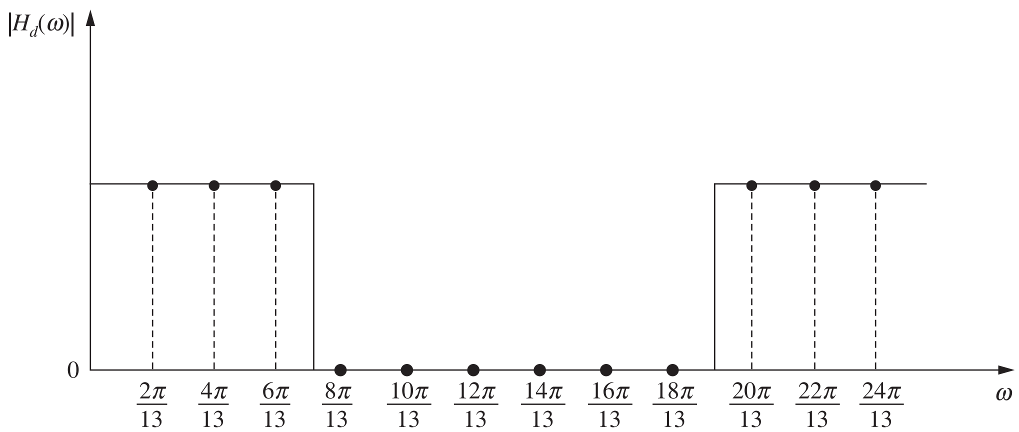


Figure 9.9 Sampling points in $H_d(\omega)$.

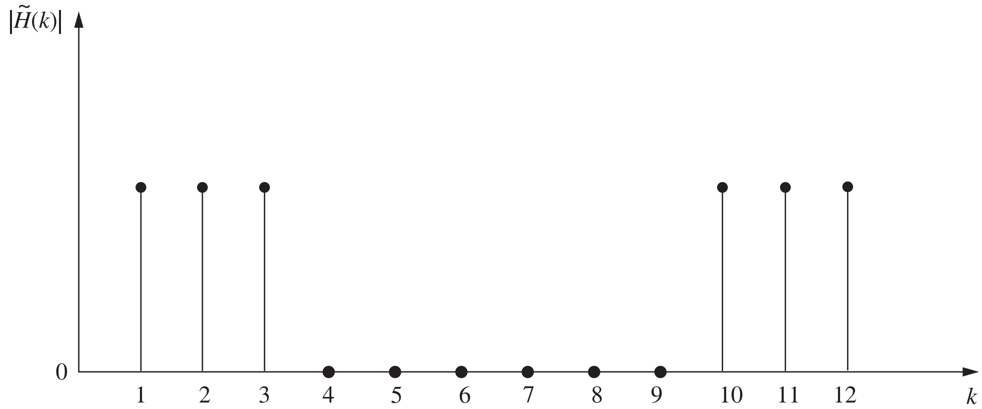


Figure 9.10 Magnitude spectrum of the sequence $\tilde{H}(k)$.

$$\therefore \qquad \tilde{H}(k) = \begin{cases} e^{-j6\times\frac{2\pi k}{13}}, & \text{for } k = 0, 1, 2, 3 \\ 0 \qquad\quad , & \text{for } k = 4, 5, 6, 7, 8, 9 \\ e^{-j6\times\frac{2\pi k}{13}}, & \text{for } k = 10, 11, 12 \end{cases}$$

The samples of impulse response are given by

$$h(n) = \frac{1}{N}\left\{\tilde{H}(0) + 2\sum_{k=1}^{(N-1)/2} \text{Re}[\tilde{H}(k)\, e^{j2\pi nk/N}]\right.$$

$$= \frac{1}{13}\left\{\tilde{H}(0) + 2\sum_{k=1}^{6} \text{Re}[\tilde{H}(k)\, e^{j2\pi nk/13}]\right.$$

Here $\tilde{H}(0) = 1$ and $\tilde{H}(k) = 1$, for $k = 1, 2, 3$ and $\tilde{H}(k) = 0$ for $k = 4, 5, 6$
Hence $h(n)$ can be written as:

$$h(n) = \frac{1}{13}\left\{1 + 2\sum_{k=1}^{3} \text{Re}\left[e^{\frac{-j12\pi k}{13}}\, e^{\frac{j2\pi nk}{13}}\right]\right\}$$

$$= \frac{1}{13}\left\{1 + 2\sum_{k=1}^{3} \text{Re}\left[e^{\frac{-j2\pi k(6-n)}{13}}\right]\right\}$$

$$= \frac{1}{13}\left\{1 + 2\left[\sum_{k=1}^{3} \cos\frac{2\pi k}{13}\,(6-n)\right]\right\}$$

$$= \frac{1}{13}\left\{1 + 2\left[\sum_{k=0}^{3} \cos\frac{2\pi k}{13}(6-n) - 1\right]\right\}$$

$$= \frac{1}{13}\left[2\sum_{k=0}^{3} \cos\frac{2\pi k}{13}(6-n) - 1\right]$$

We know that, $$\sum_{k=0}^{M-1}\cos k\theta = \frac{\sin\frac{M\theta}{2}\cos\left(\frac{M-1}{2}\right)\theta}{\sin\frac{\theta}{2}}$$

Here, $M - 1 = 3$, and $\theta = \frac{2\pi}{13}(6-n), \quad \therefore M = 4$

Using the above formula, $h(n)$ can be written as:

$$h(n) = \frac{1}{13}\left\{\frac{2\sin\frac{4}{2}\times\frac{2\pi}{13}(6-n)\cos\frac{3}{2}\times\frac{2\pi}{13}(6-n)}{\sin\frac{\pi(6-n)}{13}} - 1\right\}$$

$$= \frac{1}{13}\left\{\frac{2\sin\frac{4\pi}{13}(6-n)\cos\frac{3\pi}{13}(6-n) - \sin\frac{\pi(6-n)}{13}}{\sin\frac{\pi(6-n)}{13}}\right\}$$

$$= \frac{1}{13}\left\{\frac{\sin\left[\frac{4\pi(6-n)}{13} + \frac{3\pi(6-n)}{13}\right] + \sin\left[\frac{4\pi(6-n)}{13} - \frac{3\pi(6-n)}{13}\right] - \sin\frac{\pi(6-n)}{13}}{\sin\frac{\pi(6-n)}{13}}\right\}$$

$$= \frac{1}{13}\left\{\frac{\sin\left[\frac{7\pi(6-n)}{13}\right] + \sin\left[\frac{\pi(6-n)}{13}\right] - \sin\frac{\pi(6-n)}{13}}{\sin\frac{\pi(6-n)}{13}}\right\}$$

$$\therefore \quad h(n) = \frac{1}{13}\left\{\frac{\sin\left[\frac{7\pi(6-n)}{13}\right]}{\sin\frac{\pi(6-n)}{13}}\right\}; \text{ for } n = 0, 1, 2, \ldots, 12 \text{ except when } n = 6$$

When $n = 6$, the $h(n)$ can be evaluated using L'Hospital rule.

$$h(6) = \underset{n\to 6}{\mathrm{Lt}} \frac{1}{13} \frac{\sin \dfrac{7\pi(6-n)}{13}}{\sin \dfrac{\pi(6-n)}{13}} = \frac{1}{13} \underset{n\to 6}{\mathrm{Lt}} \frac{\sin \dfrac{7\pi(6-n)}{13}}{\sin \dfrac{\pi(6-n)}{13}} = \frac{7}{13} = 0.5384$$

Substituting the values of n, we have

$$h(0) = \frac{1}{13} \frac{\sin 7\pi \times 6/13}{\sin 6\pi/13} = -0.0513, \qquad h(1) = \frac{1}{13} \frac{\sin 7\pi \times 5/13}{\sin 5\pi/13} = 0.0677$$

$$h(2) = \frac{1}{13} \frac{\sin 7\pi \times 4/13}{\sin 4\pi/13} = 0.0434, \qquad h(3) = \frac{1}{13} \frac{\sin 7\pi \times 3/13}{\sin 3\pi/13} = -0.1084$$

$$h(4) = \frac{1}{13} \frac{\sin 7\pi \times 2/13}{\sin 2\pi/13} = -0.0396, \qquad h(5) = \frac{1}{13} \frac{\sin 7\pi/13}{\sin 2\pi/13} = 0.3190$$

$$h(6) = \frac{7}{13} = 0.538$$

For linear phase FIR filters, the condition $h(N-1-n) = h(n)$ will be satisfied when $\alpha = (N-1)/2$. Therefore,

$h(7) = h(13-1-7) = h(5) = 0.3190$, $\quad h(8) = h(13-1-8) = h(4) = -0.0396$

$h(9) = h(13-1-9) = h(3) = -0.1084$, $\quad h(10) = h(13-1-10) = h(2) = 0.0434$

$h(11) = h(13-1-11) = h(1) = 0.0677$, $\quad h(12) = h(13-1-12) = h(0) = -0.0513$

The transfer function of the filter $H(z)$ is given by the Z-transform of $h(n)$. Therefore,

$$\therefore \qquad H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n} = \sum_{n=0}^{12} h(n)\, z^{-n}$$

$$= \sum_{n=0}^{5} h(n)\, z^{-n} + h(6)\, z^{-6} + \sum_{n=7}^{12} h(n)\, z^{-n}$$

$$= \sum_{n=0}^{5} h(n)\, z^{-n} + h(6)\, z^{-6} + \sum_{n=0}^{5} h(12-n)\, z^{-(12-n)}$$

$$= \sum_{n=0}^{5} h(n)\, [z^{-n} + z^{-(12-n)}] + h(6)\, z^{-6}$$

Therefore the transfer function $H(z)$ is:

$$H(z) = h(0)[1 + z^{-12}] + h(1)\,[z^{-1} + z^{-11}] + h(2)\,[z^{-2} + z^{-10}] + h(3)\,[z^{-3} + z^{-9}]$$
$$+ h(4)\,[z^{-4} + z^{-8}] + h(5)\,[z^{-5} + z^{-7}] + h(6)\, z^{-6}$$

$$= -0.0513[1 + z^{-12}] + 0.0677[z^{-1} + z^{-11}] + 0.0434[z^{-2} + z^{-10}]$$
$$-0.1084[z^{-3} + z^{-9}] - 0.0396[z^{-4} + z^{-8}] + 0.3190[z^{-5} + z^{-7}] + 0.538z^{-6}$$

EXAMPLE 9.19 Design a linear phase FIR filter of length $N = 11$ which has a symmetric unit sample response and a frequency response that satisfies the conditions:

$$H\left(\frac{2\pi k}{11}\right) = \begin{cases} 1\ , & \text{for } k = 0, 1, 2 \\ 0.5\ , & \text{for } k = 3 \\ 0\ , & \text{for } k = 4, 5 \end{cases}$$

Solution: For linear-phase FIR filter, the phase function, $\theta(\omega) = -\alpha\omega$, where $\alpha = (N-1)/2$.

Here $N = 11$, $\quad \therefore \quad \alpha = (11 - 1)/2 = 5$.

Also, here $\omega = \omega_k = \dfrac{2\pi k}{N} = \dfrac{2\pi k}{11}$.

Hence we can go for type-I design. In this problem, the samples of the magnitude response of the ideal (desired) filter are directly given for various values of k. Therefore,

$$\tilde{H}(k) = H_d(\omega)|_{\omega=\omega_k} = \begin{cases} 1\, e^{-j\alpha\omega_k}\ , & k = 0, 1, 2 \\ 0.5\, e^{-j\alpha\omega_k}\ , & k = 3 \\ 0\ , & k = 4, 5 \end{cases}$$

where, $\omega_k = \dfrac{2\pi k}{11}$

When $k = 0$, $\quad \tilde{H}(0) = e^{-j\alpha\omega_0} = e^{-j5\times\left(\frac{2\pi\times 0}{11}\right)} = 1$

When $k = 1$, $\quad \tilde{H}(1) = e^{-j\alpha\omega_1} = e^{-j5\times\left(\frac{2\pi\times 1}{11}\right)} = e^{-j\left(\frac{10\pi}{11}\right)}$

When $k = 2$, $\quad \tilde{H}(2) = e^{-j\alpha\omega_2} = e^{-j5\times\left(\frac{2\pi\times 2}{11}\right)} = e^{-j\left(\frac{20\pi}{11}\right)}$

When $k = 3$, $\quad \tilde{H}(3) = e^{-j\alpha\omega_3} = e^{-j5\times\left(\frac{2\pi\times 3}{11}\right)} = e^{-j\left(\frac{30\pi}{11}\right)}$

When $k = 4$, $\quad \tilde{H}(4) = 0$

When $k = 5$, $\quad \tilde{H}(5) = 0$

The samples of impulse response are given by

$$h(n) = \frac{1}{N}\left\{\tilde{H}(0) + 2\sum_{k=1}^{(N-1)/2} \operatorname{Re}\left[\tilde{H}(k)\, e^{j2\pi nk/N}\right]\right\}$$

$$= \frac{1}{11}\left\{\tilde{H}(0) + 2\sum_{k=1}^{5} \operatorname{Re}\left[\tilde{H}(k)\, e^{j2\pi nk/N}\right]\right\}$$

$$= \frac{1}{11}\left\{\tilde{H}(0) + 2\sum_{k=1}^{2} \operatorname{Re}\left[\tilde{H}(k)\, e^{j2\pi nk/11}\right] + 2\operatorname{Re}\left[\tilde{H}(3)\, e^{j2\pi n3/11}\right]\right\}$$

$$= \frac{1}{11}\left\{1 + 2\sum_{k=1}^{2} \operatorname{Re}\left[e^{-j5\left(\frac{2\pi k}{11}\right)} \times e^{j\left(\frac{2\pi nk}{11}\right)}\right] + 2\operatorname{Re}\left[0.5^{-j5\left(\frac{2\pi\times 3}{11}\right)} \times e^{j\left(\frac{2\pi n3}{11}\right)}\right]\right\}$$

$$= \frac{1}{11}\left\{1 + 2\sum_{k=1}^{2} \operatorname{Re}\left[e^{j\frac{2\pi k}{11}(n-5)}\right] + 2\operatorname{Re}\left[0.5\, e^{j\frac{6\pi}{11}(n-5)}\right]\right\}$$

$$= \frac{1}{11} + \frac{2}{11}\cos\frac{2\pi}{11}(n-5) + \frac{2}{11}\cos\frac{4\pi}{11}(n-5) + \cos\frac{6\pi}{11}(n-5)$$

$$n=0,\ h(0) = \frac{1}{11} + \frac{2}{11}\cos\left(\frac{-10\pi}{11}\right) + \frac{2}{11}\cos\left(\frac{-20\pi}{11}\right) + \cos\left(\frac{-30\pi}{11}\right) = -0.5854$$

$$n=1,\ h(1) = \frac{1}{11} + \frac{2}{11}\cos\left(\frac{-8\pi}{11}\right) + \frac{2}{11}\cos\left(\frac{-16\pi}{11}\right) + \cos\left(\frac{-24\pi}{11}\right) = 0.787$$

$$n=2,\ h(2) = \frac{1}{11} + \frac{2}{11}\cos\left(\frac{-6\pi}{11}\right) + \frac{2}{11}\cos\left(\frac{-12\pi}{11}\right) + \cos\left(\frac{-18\pi}{11}\right) = 0.3059$$

$$n=3,\ h(3) = \frac{1}{11} + \frac{2}{11}\cos\left(\frac{-4\pi}{11}\right) + \frac{2}{11}\cos\left(\frac{-8\pi}{11}\right) + \cos\left(\frac{-12\pi}{11}\right) = -0.9120$$

$$n=4,\ h(4) = \frac{1}{11} + \frac{2}{11}\cos\left(\frac{-2\pi}{11}\right) + \frac{2}{11}\cos\left(\frac{-4\pi}{11}\right) + \cos\left(\frac{-6\pi}{11}\right) = 0.1770$$

$$n=5,\ h(5) = \frac{1}{11} + \frac{2}{11}\cos 0 + \frac{2}{11}\cos 0 + \cos 0 = \frac{5}{11} + 1 = 1.4545$$

For linear-phase FIR filters, the condition $h(N-1-n) = h(n)$ will be satisfied when $\alpha = (N-1)/2$.

When $n = 6$, $\quad h(6) = h(11 - 1 - 6) = h(4) = 0.1770$

When $n = 7$, $\quad h(7) = h(11 - 1 - 7) = h(3) = -0.9120$

When $n = 8$, $h(8) = h(11 - 1 - 8) = h(2) = 0.3059$

When $n = 9$, $h(9) = h(11 - 1 - 9) = h(1) = 0.787$

When $n = 10$, $h(10) = h(11 - 1 - 10) = h(0) = -0.5854$

The transfer function of the filter $H(z)$ is given by Z-transform of $h(n)$. Therefore,

$$H(z) = \sum_{n=0}^{N-1} h(n)\,z^{-n} = \sum_{n=0}^{10} h(n)\,z^{-n}$$

$$= \sum_{n=0}^{4} h(n)\,z^{-n} + h(5)\,z^{-5} + \sum_{n=6}^{10} h(n)\,z^{-n}$$

$$= \sum_{n=0}^{4} h(n)\,z^{-n} + h(5)\,z^{-5} + \sum_{n=0}^{4} h(10-n)\,z^{-(10-n)}$$

$$= \sum_{n=0}^{4} h(n)\left[z^{-n} + z^{-(10-n)}\right] + h(5)\,z^{-5}$$

$$= h(0)\left[1 + z^{-10}\right] + h(1)\left[z^{-1} + z^{-9}\right] + h(2)\left[z^{-2} + z^{-8}\right] + h(3)\left[z^{-3} + z^{-7}\right]$$
$$+ h(4)\left[z^{-4} + z^{-6}\right] + h(5)\,z^{-5}$$

$$= -0.5854\left[1 + z^{-10}\right] + 0.787\left[z^{-1} + z^{-9}\right] + 0.3059\left[z^{-2} + z^{-8}\right] - 0.9120\left[z^{-3} + z^{-7}\right]$$
$$+ 0.1770\left[z^{-4} + z^{-6}\right] + 1.4545\,z^{-5}$$

SHORT QUESTIONS WITH ANSWERS

1. What are the different types of filters based on impulse response?

Ans. Based on impulse response, filters are of two types:

(i) IIR filters and (ii) FIR filters

The IIR filters are designed using infinite number of samples of impulse response. They are of recursive type, whereby the present output depends on the present input, past input and past output samples.

The FIR filters are designed using only a finite number of samples of impulse response. They are non-recursive type whereby the present output depends on the present input and past input samples.

2. What are the different types of filters based on frequency response?

Ans. Based on frequency response, filters are of four types:

(i) Low-pass filter, (ii) High-pass filter, (iii) Band-pass filter, and (iv) Band-stop filter.

3. How phase distortion and delay distortion are introduced?
Ans. The phase distortion is introduced when the phase characteristics of a filter is not linear within the desired frequency band. The delay distortion is introduced when the delay is not a constant within the desired frequency range.

4. What are FIR filters?
Ans. The filters designed by considering only a finite number of samples of impulse response are called FIR filters.

5. What are the advantages and disadvantages of FIR filters?
Ans. The advantages of FIR filters are as follows:
(i) FIR filters have exact linear phase.
(ii) FIR filters can be realized in both recursive and non-recursive structures.
(iii) FIR filters realized non-recursively are always stable.
(iv) The round off noise can be made small in non-recursive realization.

The disadvantages of FIR filters are as follows:

(i) For the same filter specifications, the order of the filter to be designed is much higher than that of IIR.
(ii) Large storage requirements and powerful computational facilities required.
(iii) The non-integral delay can lead to problems in some signal processing applications.

6. Compare FIR and IIR filters.
Ans. The IIR and FIR filters are compared as follows:

IIR filters	*FIR filters*
1. All the infinite samples of impulse response are considered.	1. Only a finite number of samples of impulse response are considered.
2. The impulse response cannot be directly converted to digital filter transfer function.	2. The impulse response can be directly converted to digital filter transfer function.
3. Linear phase characteristics cannot be achieved.	3. Linear phase filters can be easily designed.
4. IIR filters are easily realized recursively.	4. FIR filters can be realized recursively and non-recursively.
5. The specifications include the desired characteristics for magnitude response only.	5. The specifications include the desired characteristics for both magnitude and phase response.
6. The design involves design of analog filter and then transforming analog filter to digital filter.	6. The digital filter can be directly designed to achieve the desired specifications.
7. The round off noise in IIR filters is more.	7. Errors due to round off noise are less severe in FIR filters, mainly because feedback is not used.

7. What is the necessary and sufficient condition for the linear phase characteristic of a FIR filter?

Ans. The necessary and sufficient condition for the linear phase characteristic of FIR filter is that the phase function should be a linear function of ω, which in turn requires constant phase delay or constant phase and group delay.

or

Ans. The necessary and sufficient condition for linear phase characteristic in FIR filter is, the impulse response $h(n)$ of the system should have the symmetrical property, i.e., $h(n) = h(N-1-n)$, where N is the duration of the sequence.

8. What is the reason that FIR filter is always stable?

Ans. FIR filter is always stable because all its poles are at the origin.

9. What conditions on the FIR sequence $h(n)$ are to be imposed in order that this filter can be called a linear phase filter?

Ans. For an FIR filter to be a linear phase filter the conditions to be imposed on the sequence are:

(i) Symmetric condition $h(n) = h(N-1-n)$

(ii) Antisymmetric condition $h(n) = -h(N-1-n)$

10. Under what conditions a finite duration sequence $h(n)$ will yield constant group delay in its frequency response characteristics and not the phase delay?

Ans. If the impulse response is antisymmetrical satisfying the condition

$$h(n) = -h(N-1-n)$$

then frequency response of FIR filter will have constant group delay and not constant phase delay.

11. State the condition for a digital filter to be causal and stable.

Ans. The condition for a digital filter to be causal is its impulse response $h(n) = 0$ for $n < 0$. The condition for a digital filter to be stable is its impulse response is absolutely summable, i.e., $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$.

12. Write the steps involved in FIR filter design.

Ans. The steps involved in FIR filter design are:

(i) Choose the desired (ideal) frequency response $H_d(\omega)$.

(ii) Take inverse Fourier transform of $H_d(\omega)$ to get $h_d(n)$.

(iii) Convert the infinite duration $h_d(n)$ to finite duration sequence $h(n)$.

(iv) Take Z-transform of $h(n)$ to get the transfer function $H(z)$ of the FIR filter.

13. What are the possible types of impulse response for linear phase FIR filters?

Ans. There are four possible types of impulse response for linear phase FIR filters:

(i) Symmetric impulse response when N is odd

(ii) Symmetric impulse response when N is even

(iii) Antisymmetric impulse response when N is odd

(iv) Antisymmetric impulse response when N is even

14. List the three well known design techniques for linear phase FIR filters.
Ans. The three well known design techniques for linear phase FIR filters are:
(i) Fourier series method and window method
(ii) Frequency sampling method
(iii) Optimal filter design method

15. What are the two concepts that lead to the Fourier series or window method of designing FIR filters?
Ans. The two concepts that lead to the design of FIR filter by Fourier series are:
(i) The frequency response of a digital filter is periodic with period equal to sampling frequency.
(ii) Any periodic function can be expressed as linear combination of complex exponentials.

16. What is the basis for Fourier series method of design? Why truncation is necessary?
Ans. The frequency response $H(\omega)$ of any digital filter is periodic in frequency and can be expanded in a Fourier series

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n}$$

where the Fourier coefficients are:

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)\, e^{j\omega n} d\omega$$

$h(n)$ is of infinite duration, hence, the filter resulting from a Fourier series representation of $H(\omega)$ is an unrealizable IIR filter. To get an FIR filter that approximates $H(\omega)$ would be to truncate the infinite Fourier series at $n = \pm (N-1)/2$.

17. Explain briefly the method of designing FIR filter using Fourier series method.
Ans:. FIR filter is designed using Fourier series method as follows:

Step 1: For the desired frequency response $H_d(\omega)$, find the impulse response $h_d(n)$ using the equation:

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n} d\omega$$

Step 2: Truncate $h_d(n)$ at $n = \pm (N-1)/2$ to get the finite duration sequence $h(n)$.
Step 3: Find $H(z)$ using the equation:

$$H(z) = z^{-(N-1)/2}\left[h(0) + \sum_{n=1}^{(N-1)/2} h(n)\,(z^{n} + z^{-n})\right]$$

18. How causality is brought-in in the Fourier series method of filter design?
Ans. The transfer function obtained in Fourier series method of filter design will represent an unrealizable non-causal system. The non-causal transfer function is multiplied by $z^{-(N-1)/2}$ to convert it to a causal transfer function.

19. What are the disadvantages of Fourier series method?

Ans. In designing FIR filter using Fourier series method, the infinite duration impulse response is truncated at $n = \pm (N-1)/2$. Direct truncation of the series will lead to fixed percentage overshoots and undershoots before and after an approximated discontinuity in the frequency response.

20. What is Gibbs phenomenon?

Ans. One possible way of finding an FIR filter that approximates $H(\omega)$ would be to truncate the infinite series at $n = \pm (N-1)/2$. Abrupt truncation of the series will lead to oscillations both in pass band and in stop band. This phenomenon is known as Gibbs phenomenon.

21. What is window and why it is necessary?

Ans. A finite weighing sequence $w(n)$ with which the infinite impulse response is multiplied to obtain a finite impulse response is called a window. This is necessary because abrupt truncation of the infinite impulse response will lead to oscillations in the pass band and stop band, and these oscillations can be reduced through the use of less abrupt truncation of the Fourier series.

22. Explain the procedure for designing FIR filters using windows.

Ans. The procedure for designing FIR filters using windows is:

Step 1: For the desired frequency response $H(\omega)$, find the impulse response $h_d(n)$ using the equation:

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega$$

Step 2: Multiply the infinite impulse response with a chosen window sequence $w(n)$ of length N to obtain filter coefficients $h(n)$, i.e.,

$$h(n) = \begin{cases} h_d(n)\, w(n), & \text{for } |n| \le \dfrac{N-1}{2} \\ 0, & \text{otherwise} \end{cases}$$

Step 3: Find the transfer function of the realizable filter

$$H(z) = z^{-(N-1)/2}\left[h(0) + \sum_{n=0}^{(N-1)/2} h(n)\,[z^n + z^{-n}]\right]$$

23. What are the desirable characteristics of the window?

Ans. The desirable characteristics of the window are:

(i) The central lobe of the frequency response of the window should contain most of the energy and should be narrow.

(ii) The highest side lobe level of the frequency response should be small.

(iii) The side lobes of the frequency response should decrease in energy rapidly as ω tends to π.

24. What is the principle of designing FIR filter using windows?

Ans. One possible way of obtaining FIR filter is to truncate the infinite Fourier series at $n = \pm (N-1)/2$, where N is the length of the desired sequence. But abrupt truncation of the Fourier series results in oscillation in the pass band and stop band. These oscillations are due to slow convergence of the Fourier series. To reduce these oscillations, the Fourier coefficients of the filter are modified by multiplying the infinite impulse response by a finite weighing sequence $w(n)$ called a window sequence, where

$$w(n) = \begin{cases} w(-n) \neq 0, & \text{for } |n| \leq \dfrac{N-1}{2} \\ 0, & \text{for } |n| \geq \dfrac{N-1}{2} \end{cases}$$

After multiplying window sequence $w(n)$ by $h_d(n)$, we get a finite duration sequence $h(n)$ that satisfies the desired magnitude response

$$h(n) = \begin{cases} h_d(n)\, w(n), & \text{for all } |n| \leq \dfrac{N-1}{2} \\ 0, & \text{for } \quad |n| > \dfrac{N-1}{2} \end{cases}$$

25. Write the characteristic features of rectangular window.

Ans. The characteristic features of rectangular window are:

(i) The main lobe width is equal to $4\pi/N$.
(ii) The maximum side lobe magnitude is –13 dB.
(iii) The side lobe magnitude does not decrease significantly with increasing ω.

26. List the features of FIR filter designed using rectangular window.

Ans. The features of FIR filter designed using rectangular window are:

(i) The width of the transition region is related to the width of the main lobe of window spectrum.
(ii) Gibbs oscillations are noticed in the pass band and stop band.
(iii) The attenuation in the stop band is constant and cannot be varied.

27. How the transition width of the FIR filter can be reduced in design using windows?

Ans. In FIR filters designed using windows, the width of the transition region is related to the width of the main lobe in the window function. The width of the main lobe is inversely proportional to the length of the window sequence. Hence the width of the main lobe can be reduced by increasing the value of N, which in turn reduces the width of the transition region in the FIR filter.

28. What are the advantages of Kaiser window?

Ans. The advantages of Kaiser window are:

(i) It provides flexibility for the designer to select the side lobe level and N.
(ii) It has the attractive property that the side lobe level can be varied continuously from the low value in the Blackman window to the high value in the rectangular window.

29. What is the principle of designing FIR filter using frequency sampling method?

Ans. In frequency sampling method of design of FIR filters, the desired magnitude response is sampled and a linear phase response is specified. The samples of desired frequency response are identified as DFT coefficients. The filter coefficients are then determined as the IDFT of this set of samples.

30. For what type of filters frequency sampling method is suitable?

Ans. Frequency sampling method is attractive for narrow band frequency selective filters where only a few of the samples of the frequency response are zero.

31. Write the procedure for FIR filter design by frequency sampling method.

Ans. The procedure for FIR filter design by frequency sampling method is

Step 1: Choose the desired frequency response $H_d(\omega)$.

Step 2: Take N-samples of $H_d(\omega)$ to generate the sequence $\tilde{H}(k)$.

Step 3: Take IDFT of $\tilde{H}(k)$ to get the impulse response $h(n)$.

Step 4: The transfer function $H(z)$ of the filter is obtained by taking Z-transform of impulse response.

32. Write the characteristic features of triangular window.

Ans. The characteristic features of triangular window are:

(i) The main lobe width is equal to $8\pi/N$.

(ii) The maximum side lobe magnitude is –25 dB.

(iii) The side lobe magnitude slightly decreases with increasing ω.

33. Why triangular window is not a good choice for designing FIR filters?

Ans. The triangular window is not a good choice for designing FIR filters because in FIR filters designed using triangular window the transition from pass band to stop band is not sharp and the attenuation in stop band is less when compared to filters designed with rectangular window.

34. List the features of Hanning window spectrum.

Ans. The features of Hanning window spectrum are:

(i) The main lobe width is equal to $8\pi/N$.

(ii) The maximum side lobe magnitude is –31 dB.

(iii) The side lobe magnitude decreases with increasing ω.

35. List the features of Hamming window spectrum.

Ans. The features of Hamming window spectrum are:

(i) The main lobe width is equal to $8\pi/N$.

(ii) The maximum side lobe magnitude is – 41 dB.

(iii) The side lobe magnitude remains constant for increasing ω.

36. List the features of Blackman window spectrum.

Ans. The features of Blackman window spectrum are:

(i) The main lobe width is equal to $12\pi/N$.

(ii) The maximum side lobe magnitude is –58 dB.

(iii) The side lobe magnitude decreases with increasing ω.

(iv) The side lobe attenuation in Blackman window is the highest among windows, which is achieved at the expense of increased main lobe width. However, the main lobe width can be reduced by increasing the value of N.

37. List the desirable features of Kaiser window spectrum.
Ans. The desirable features of Kaiser window spectrum are:
(i) The width of the main lobe and peak side lobe are variable.
(ii) The parameter α in the Kaiser window function is an independent variable that can be varied to control the side lobe levels with respect to main lobe peak.
(iii) The width of the main lobe in the window spectrum (and so the transition region in the FIR filter) can be varied by varying the length N of the window sequence.

38. What is the main advantage of windowing?
Ans. The main advantage of windowing is that it is reasonably straight forward to obtain the filter impulse response with minimal computational effort.

39. What are the major reasons for the success of window?
Ans. The major reasons for the relative success of windows are their simplicity and ease of use and the fact that closed from expressions are often available for the window coefficients.

40. List the characteristics of FIR filters designed using windows.
Ans. The characteristics of FIR filters designed using windows are:
(i) The width of the transition region depends on the type of the window.
(ii) The width of the transition region can be made narrow by increasing the value of N where N is the length of the window sequence.
(iii) The attenuation in the stop band is fixed for a given window, except in case of Kaiser window where it is variable.

41. Compare the rectangular window and Hanning window.
Ans. The rectangular window and Hanning window are compared as follows:

Rectangular window	*Hanning window*
1. The width of main lobe in window spectrum is $4\pi/N$.	1. The width of main lobe in window spectrum is $8\pi/N$.
2. The maximum side lobe magnitude in window spectrum is –13 dB.	2. The maximum side lobe magnitude in window spectrum is –31 dB.
3. In window spectrum the side lobe magnitude slightly decreases with increasing ω.	3. In window spectrum the side lobe magnitude decreases with increasing ω.
4. In FIR filter designed using rectangular window the minimum stop band attenuation is 22 dB.	4. In FIR filter designed using Hanning window the minimum stop band attenuation is 44 dB.

42. Compare the rectangular window and Hamming window.

Ans. The rectangular window and Hamming window are compared as follows:

Rectangular window	*Hamming window*
1. The width of main lobe in window spectrum is $4\pi/N$.	1. The width of main lobe in window spectrum is $8\pi/N$.
2. The maximum side lobe magnitude in window spectrum is –13 dB.	2. The maximum side lobe magnitude in window spectrum is – 41 dB.
3. In window spectrum the side lobe magnitude slightly decreases with increasing ω.	3. In window spectrum the side lobe magnitude remains constant.
4. In FIR filter designed using rectangular window the minimum stop band attenuation is 22 dB.	4. In FIR filter designed using Hamming window the minimum stop band attenuation is 51 dB.

43. Compare the Hanning and Hamming windows.

Ans. The Hanning and Hamming windows are compared as follows:

Hanning window	*Hamming window*
1. The width of main lobe in window spectrum is $8\pi/N$.	1. The width of main lobe in window spectrum is $8\pi/N$.
2. The maximum side lobe magnitude in window spectrum is –31 dB.	2. The maximum side lobe magnitude in window spectrum is – 41 dB.
3. In window spectrum the side lobe magnitude decreases with increasing ω.	3. In window spectrum the side lobe magnitude remains constant. Here the increased side lobe attenuation is achieved at the expense of constant attenuation at higher frequencies.
4. In FIR filter designed using Hanning window the minimum stop band attenuation is 44 dB.	4. In FIR filter designed using Hamming window the minimum stop band attenuation is 51 dB.

44. Compare the Hamming and Blackman windows.

Ans. The Hamming and Blackman windows are compared as follows:

Hamming window	*Blackman window*
1. The width of main lobe in window spectrum is $8\pi/N$.	1. The width of main lobe in window spectrum is $12\pi/N$.
2. The maximum side lobe magnitude in window spectrum is – 41 dB.	2. The maximum side lobe magnitude in window spectrum is –58 dB.
3. In window spectrum the side lobe magnitude remains constant with	3. In window spectrum, the side lobe magnitude decreases rapidly with increasing ω.

Hamming window	*Blackman window*
increasing ω. The higher value of side lobe attenuation is achieved at the expense of constant attenuation at higher frequencies.	The higher value of side lobe attenuation is achieved at the expense of increased main lobe width.
4. In FIR filter designed using Hamming window the minimum stop band attenuation is 51 dB.	4. In FIR filter designed using Blackman window the minimum stop band attenuation is 78 dB.

45. Compare the Hamming and Kaiser windows.

Ans. The Hamming and Kaiser windows are compared as follows:

Hamming window	*Kaiser window*
1. The width of main lobe in window spectrum is $8\pi/N$.	1. The width of main lobe in window spectrum depends on the values of α and N.
2. The maximum side lobe magnitude in window spectrum is – 41 dB.	2. The maximum side lobe magnitude with respect to peak of main lobe is variable using the parameter α.
3. In window spectrum the side lobe magnitude remains constant with increasing ω.	3. In window spectrum, the side lobe magnitude decreases with increasing ω.
4. In FIR filter designed using Hamming window the minimum stop band attenuation is fixed at 51 dB.	4. In FIR filter designed using Kaiser window the minimum stop band attenuation is variable and depends on the value of α.

REVIEW QUESTIONS

1. What is an FIR filter? Compare an FIR filter with an IIR filter.

2. Write the steps in the design of FIR filters.

3. Show that the magnitude response $|H(\omega)|$ of FIR filter is symmetric when impulse response is symmetric and N is odd.

4. Show that the magnitude response $|H(\omega)|$ of FIR filter is antisymmetric when impulse response is symmetric and N is even.

5. Show that the magnitude response $|H(\omega)|$ of FIR filter is antisymmetric when impulse response is antisymmetric and N is odd.

6. Show that the magnitude response $|H(\omega)|$ of FIR filter is symmetric when impulse response is antisymmetric and N is even.

7. Explain FIR filter design using windowing method.

8. Find the frequency response of a rectangular window.

9. Discuss the frequency sampling method of FIR filter design.
10. Discuss the methods of designing FIR filters.

FILL IN THE BLANKS

1. An LTI system modifies the input spectrum $X(\omega)$ according to its _______ to yield an output spectrum $Y(\omega)$.
2. $H(\omega)$ acts as a _______ or _______ function to the different frequency components in the input signal.
3. For a linear phase filter the delay is a _______.
4. The phase distortion is due to _______ phase characteristics of the filter.
5. In FIR filters _______ is a linear function of ω.
6. Delay distortion is _______ with phase distortion.
7. The $H_d(\omega)$ is periodic with periodicity of _______.
8. The filters are classified according to their _______ response.
9. Based on impulse response filters are classified _______ as _______ or filters.
10. Based on frequency response filters are classified as _______, _______, _______ and _______ filters.
11. In FIR filters with constant phase delay the impulse response is _______.
12. In FIR filters with constant group and phase delay the impulse response is _______.
13. The ideal filters are _______ and hence physically unrealizable.
14. The transition of the frequency response from pass band to stop band defines the _______ of the filter.
15. Linear phase filter $[\theta(\omega) = -\alpha\omega]$ requires the filter to have both constant _______ and constant _______.
16. The _______ response of the filter is the Fourier transform of the impulse response of the filter.
17. In linear phase filters when impulse response is antisymmetric and N is odd, the magnitude function is _______.
18. In linear phase filters when impulse response is antisymmetric and N is even, the magnitude function is _______.
19. In linear phase filters when impulse response is symmetric and N is odd, the magnitude function is _______.
20. In linear phase filters when impulse response is symmetrical and N is even, the magnitude function is _______.
21. The abrupt truncation of the impulse response leads to _______ in pass band and stop band.

22. The generation of oscillations due to slow convergence of the Fourier series near the points of discontinuity is called _______.

23. In Fourier series method of FIR filter design, the causality is brought about by multiplying the transfer function with _______.

24. The width of the main lobe in window spectrum can be reduced by increasing the length of _______.

25. The width of the transition region of FIR filter directly depends on the width of the _______ in window spectrum.

26. The _______ can be eliminated by replacing the sharp transitions in window sequence by gradual transition.

27. In rectangular window the width of main lobe is equal to _______.

28. In _______ window spectrum, the width of main lobe is double that of rectangular window for the same value of N.

29. In _______ window spectrum, the width of the main lobe is triple that of rectangular window for same value of N.

30. In _______ window spectrum, the side lobe magnitude is variable.

31. The _______ window spectrum has the highest attenuation for side lobes.

32. In _______ window spectrum the increase in side lobe attenuation is achieved at the expense of constant attenuation at high frequencies.

33. In _______ window spectrum the higher side lobe attenuation is achieved at the expense of increased main lobe width.

OBJECTIVE TYPE QUESTIONS

1. The ideal filters are:
(a) causal (b) non-causal
(c) may be causal or may not be causal (d) none of these

2. In Fourier series method to get the transfer function of realizable filter, $H(z)$ is to be multiplied by
(a) $z^{-(N-1)/2}$ (b) $z^{(N-1)/2}$ (c) $z^{-(N-1)}$ (d) $z^{(N-1)}$

3. The abrupt truncation of Fourier series results in oscillations in
(a) stop band (b) pass band
(c) both pass band and stop band (d) none of these

4. The frequency response of a digital filter is
(a) periodic (b) non periodic
(c) may be periodic or non periodic (d) none of these

5. For rectangular window the main lobe width is equal to
(a) $2\pi/N$ (b) $4\pi/N$ (c) $8\pi/N$ (d) $12\pi/N$

6. For Hanning window, the width of the main lobe is equal to
(a) $2\pi/N$ (b) $4\pi/N$ (c) $8\pi/N$ (d) $12\pi/N$

7. For Hamming window, the width of the main lobe is equal to
(a) $2\pi/N$ (b) $4\pi/N$ (c) $8\pi/N$ (d) $12\pi/N$

8. For Blackman window, the width of the main lobe is equal to
(a) $2\pi/N$ (b) $4\pi/N$ (c) $8\pi/N$ (d) $12\pi/N$

9. For Kaiser window, the width of the main lobe is
(a) $4\pi/N$ (b) $8\pi/N$ (c) $12\pi/N$ (d) Adjustable.

10. For rectangular window, the peak side lobe magnitude in dB is
(a) –13 (b) –31 (c) –41 (d) –58

11. For Hanning window, the peak side lobe magnitude in dB is
(a) –13 (b) –31 (c) –41 (d) –58

12. For Hamming window, the peak side lobe magnitude in dB is
(a) –13 (b) –31 (c) –41 (d) –58

13. For Blackman window, the peak side lobe magnitude in dB is
(a) –13 (b) –31 (c) –41 (d) –58

PROBLEMS

1. Design an FIR low-pass digital filter with $N = 7$, with a cutoff frequency of $\pi/3$ rad/sec.

2. Design a FIR digital low-pass filter with a cutoff frequency of 1 kHz and a sampling rate of 4 kHz with 7 samples using Fourier series method.

3. The desired response of a digital low-pass filter is:

$$H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega}, & \dfrac{-3\pi}{4} \le \omega \le \dfrac{3\pi}{4} \\ 0 \quad, & \dfrac{3\pi}{4} \le |\omega| \le \pi \end{cases}$$

Determine the filter coefficients and frequency response for $N = 5$ using a Hanning window.

4. Design a digital low-pass filter with a cutoff frequency of 1 rad/sec using rectangular window with $N = 7$.

5. Design a digital high-pass filter with a cutoff frequency of 1 rad/sec using Hamming window with $N = 7$.

6. Design a digital band pass filter to pass frequencies in the range 1 to 2 rad/sec using Blackman window with $N = 7$.

7. Design a digital band stop filter to reject frequencies in the range 1 to 2 rad/sec using Hanning window with $N = 7$.

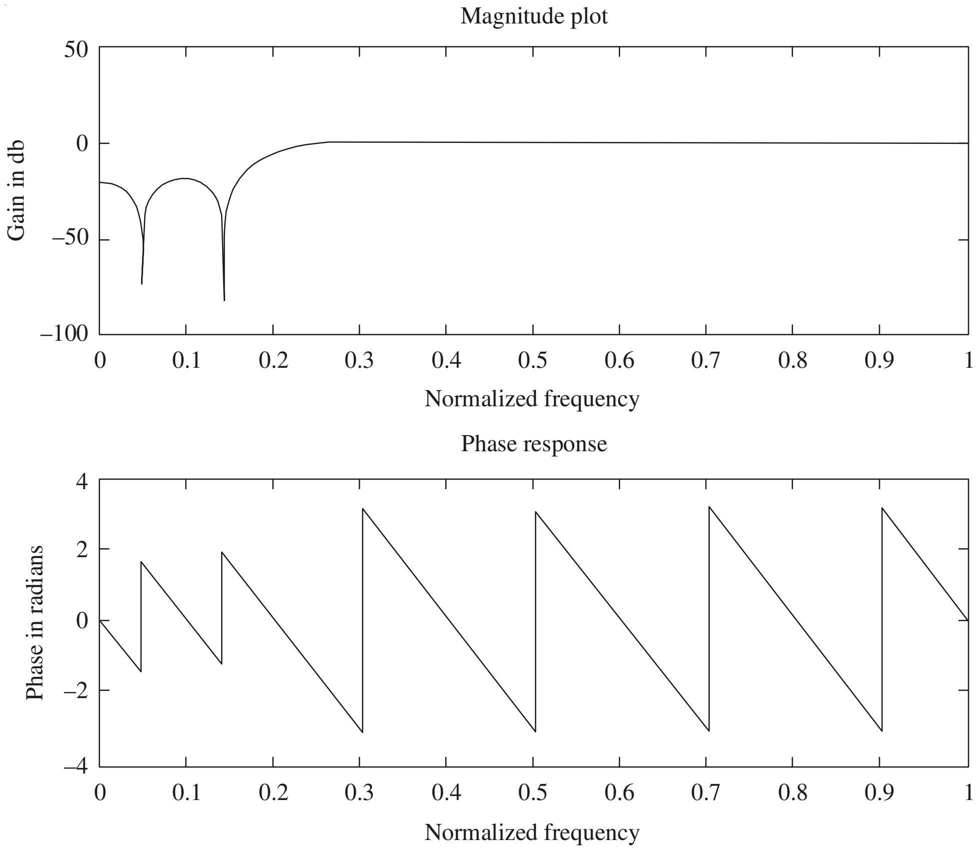
MATLAB PROGRAM

Program 9.1

% Response of high-pass FIR filter using Rectangular window

```
clc; clear all; close all;
n=20;
fp=100;
fq=300;
fs=1000;
fn=2*fp/fs;
window=rectwin(n+1);
b=fir1(n,fn,'high',window);
w=0:0.001:pi;
[h,om]=freqz(b,1,w);
a=20*log10(abs(h));
b=angle(h);
subplot(2,1,1);plot(w/pi,a);
xlabel('Normalized frequency')
ylabel('Gain in db')
title('magnitude plot')
subplot(2,1,2);plot(w/pi,b);
xlabel('Normalized frequency')
ylabel('Phase in radians')
title('Phase Response')
```

Output:



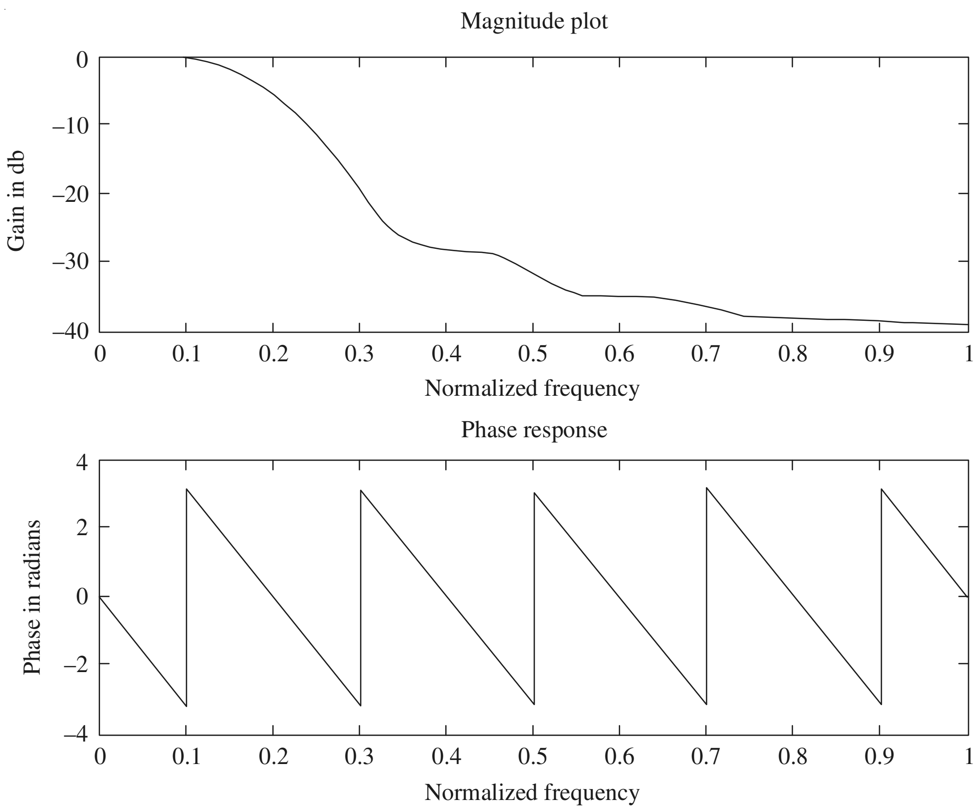
Program 9.2

% Response of low-pass FIR filter using Bartlett(Triangular) window

```
clc; clear all; close all;
n=20;
fp=100;
fq=300;
fs=1000;
fn=2*fp/fs;
window=bartlett(n+1);
b=fir1(n,fn,window);
w=0:0.001:pi;
[h,om]=freqz(b,1,w);
a=20*log10(abs(h));
```

```
b=angle(h);
subplot(2,1,1);plot(w/pi,a);
xlabel('Normalized frequency')
ylabel('Gain in db')
title('magnitude plot')
subplot(2,1,2);plot(w/pi,b);
xlabel('Normalized frequency')
ylabel('Phase in radians')
title('Phase Response')
```

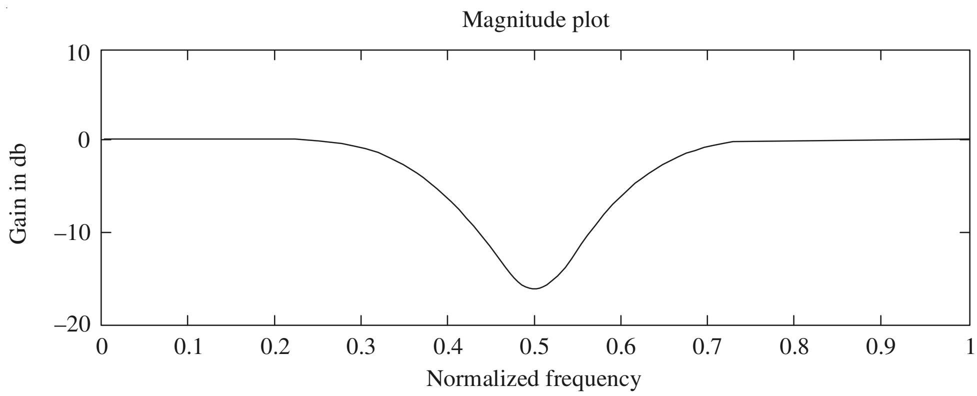
Output:

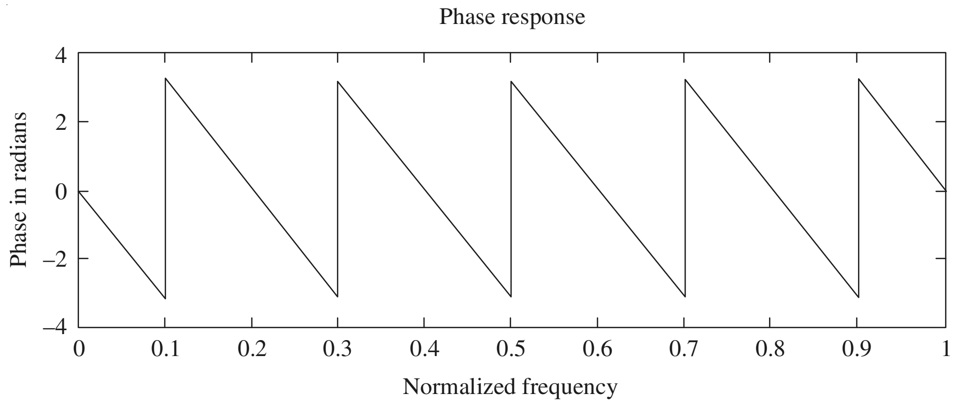


Program 9.3

```
% Response of band stop FIR filter using Hamming window

clc; clear all; close all;
n=20;
fp=200;
fq=300;
fs=1000;
wp=2*fp/fs;
ws=2*fq/fs;
wn=[wp ws];
window=hamming(n+1);
b=fir1(n,wn,'stop',window);
w=0:0.001:pi;
[h,om]=freqz(b,1,w);
a=20*log10(abs(h));
b=angle(h);
subplot(2,1,1);plot(w/pi,a);
xlabel('Normalized frequency')
ylabel('Gain in db')
title('magnitude plot')
subplot(2,1,2);plot(w/pi,b);
xlabel('Normalized frequency')
ylabel('Phase in radians')
title('Phase Response')
```

Output:



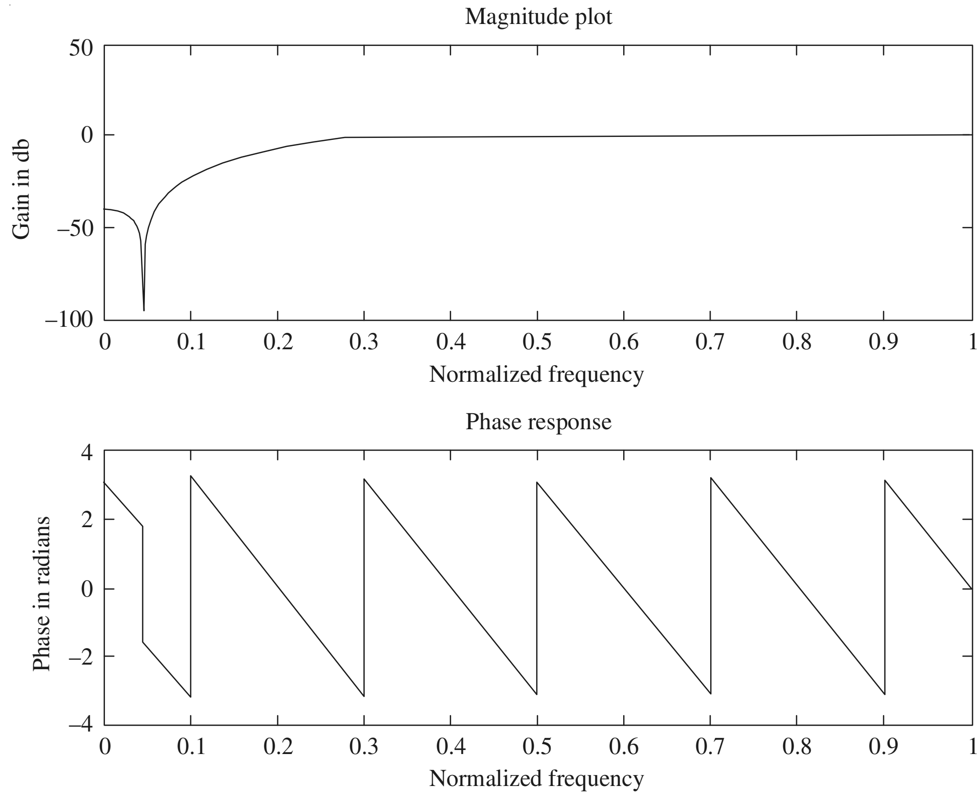


Program 9.4

% Response of low-pass FIR filter using Hanning window

```
clc; clear all; close all
n=20;
fp=100;
fq=300;
fs=1000;
fn=2*fp/fs;
window=hanning(n+1);
b=fir1(n,fn,'high',window);
w=0:0.001:pi;
[h,om]=freqz(b,1,w);
a=20*log10(abs(h));
b=angle(h);
subplot(2,1,1);plot(w/pi,a);
xlabel('Normalized frequency')
ylabel('Gain in db')
title('magnitude plot')
subplot(2,1,2);plot(w/pi,b);
xlabel('Normalized frequency')
ylabel('Phase in radians')
title('Phase Response')
```

Output:

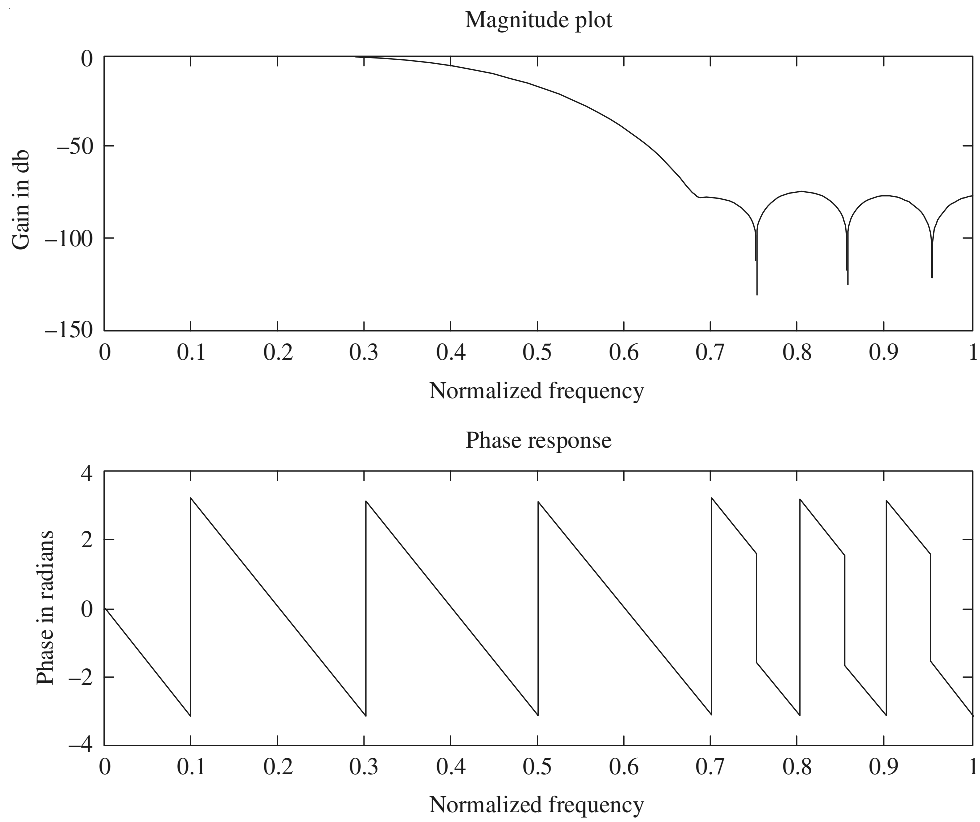


Program 9.5

```
% Response of low-pass FIR filter using Blackman window
clc; clear all; close all;
n=20;
fp=200;
fq=300;
fs=1000;
fn=2*fp/fs;
window=blackman(n+1);
b=fir1(n,fn,window);
w=0:0.001:pi;
[h,om]=freqz(b,1,w);
a=20*log10(abs(h));
```

```
b=angle(h);
subplot(2,1,1);plot(w/pi,a);
xlabel('Normalized frequency')
ylabel('Gain in db')
title('magnitude plot')
subplot(2,1,2);plot(w/pi,b);
xlabel('Normalized frequency')
ylabel('Phase in radians')
title('Phase Response')
```

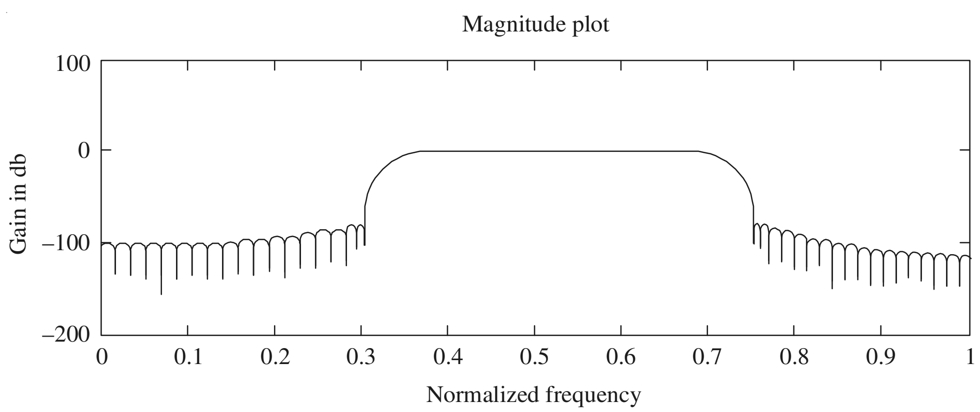
Output:

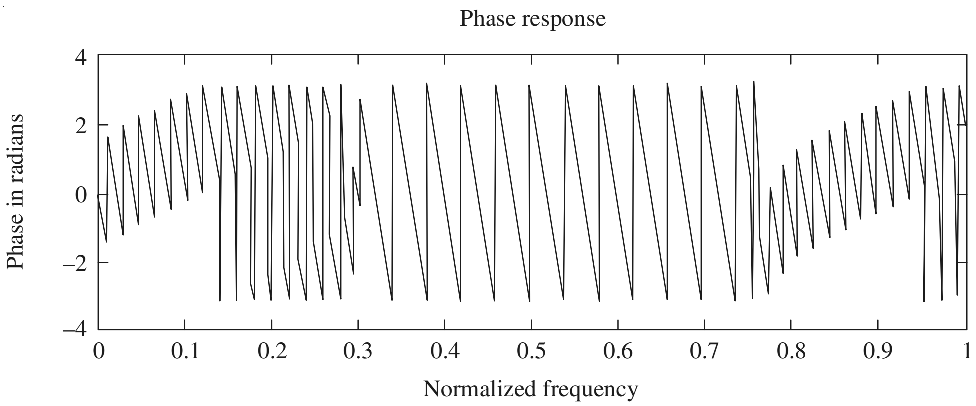


Program 9.6

```
% Response of band pass FIR filter using Kaiser window
clc; clear all; close all;
fs = 20000;   % sampling rate
F = [3000 4000 6000 8000];  % band limits
A = [0 1 0];                 % band type: 0='stop', 1='pass'
dev = [0.0001 10^(0.1/20)-1 0.0001]; % ripple/attenuation specifications
[M,Wn,beta,typ] = kaiserord(F,A,dev,fs);  % window parameters
b = fir1(M,Wn,typ,kaiser(M+1,beta),'noscale'); % filter design
w=0:0.001:pi;
[h,om]=freqz(b,1,w);
a=20*log10(abs(h));
b=angle(h);
subplot(2,1,1),plot(w/pi,a);
xlabel('Normalized frequency')
ylabel('Gain in db')
title('magnitude plot')
subplot(2,1,2),plot(w/pi,b);
xlabel('Normalized frequency')
ylabel('Phase in radians')
title('Phase Response')
```

Output:





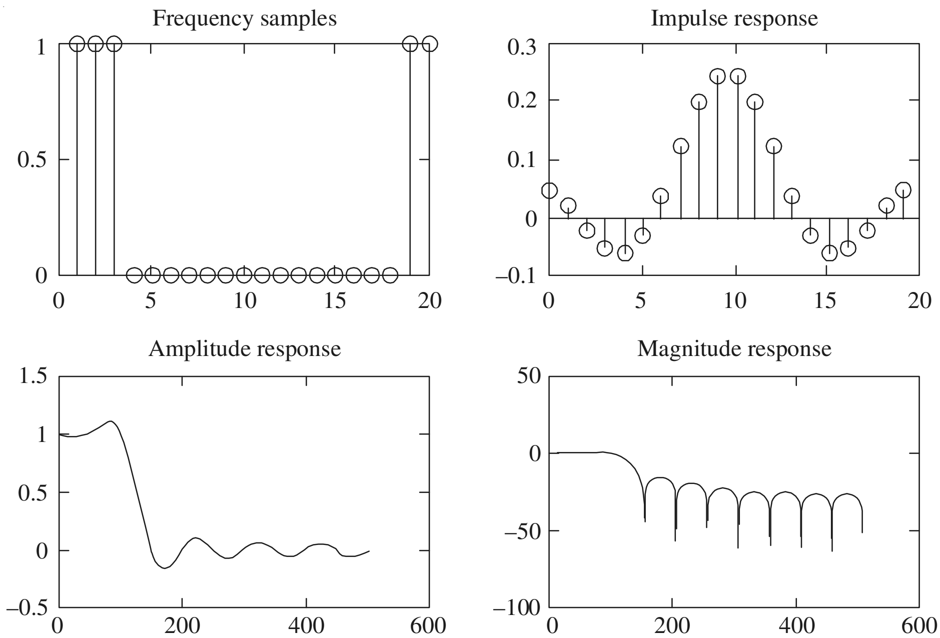
Program 9.7

% Response of low-pass FIR filter using frequency sampling method

```
clc; clear all; close all
N=20;
alpha=(N–1)/2;
k=0:N–1;
wk = (2*pi/N)*k;
Hr = [1,1,1,zeros(1,15),1,1]; %Ideal Amp Res sampled
Hd = [1,1,0,0]; wdl = [0,0.25,0.25,1]; %Ideal Amp Res for plotting
k1 = 0:floor((N-1)/2);
k2 = floor((N–1)/2)+1:N–1;
angH = [–alpha*(2*pi)/N*k1, alpha*(2*pi)/N*(N–k2)];
H = Hr.*exp(j*angH);
h = real(ifft(H,N));
[h1,w] = freqz(h,1);
M = length(h);
L = M/2;
b = 2*[h(L:–1:1)];
n = [1:1:L];
n = n–0.5;
w = [0:1:500]'*pi/500;
Hr1 = cos(w*n)*b';
[h1,w] = freqz(h,1);
subplot(2,2,1),stem(Hr);
title('frequency samples')
```

```
subplot(2,2,2),stem(k,h);
title('impulse response')
subplot(2,2,3),plot(Hr1);
title('amplitude response')
subplot(2,2,4),plot(20*log10(abs(h1)));
title('magnitude response')
```

Output:



Program 9.8

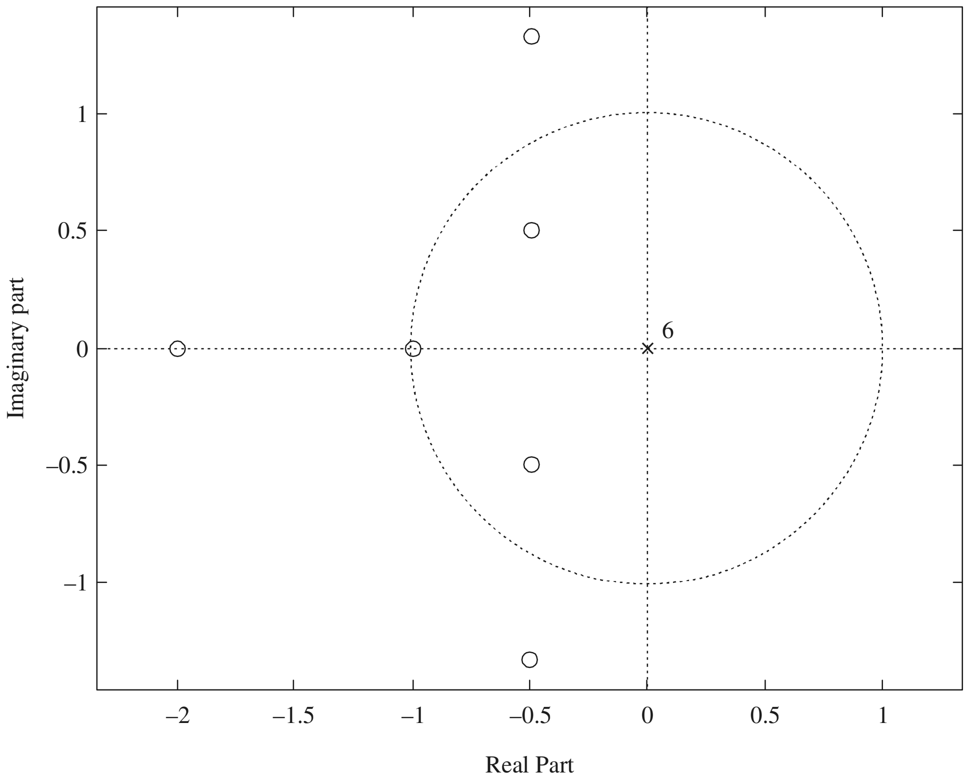
% Pole-Zero plot of FIR Filter

H(z)=2+10Z^(–1)+23Z^(–2)+34Z^(–3)+31Z^(–4)+16Z^(–5)+4Z^(–6)

```
clc; clear all; close all;
num=[2 10 23 34 31 16 4];
den=[1];
[a,b]=eqtflength(num,den);
[z,p,k]=tf2zp(a,b);
zplane(z,p);
title('pole Zero plot of FIR filter')
```

Output:

Pole zero plot of FIR filter



10

Multi-rate Digital Signal Processing

10.1 INTRODUCTION

Discrete-time systems may be single-rate systems or multi-rate systems. The systems that use single sampling rate from A/D converter to D/A converter are known as single-rate systems and the discrete-time systems that process data at more than one sampling rate are known as multi-rate systems. In digital audio, the different sampling rates used are 32 kHz for broadcasting, 44.1 kHz for compact disc and 48 kHz for audio tape. In digital video, the sampling rates for composite video signals are 14.3181818 MHz and 17.734475 MHz for NTSC and PAL respectively. But the sampling rates for digital component of video signals are 13.5 MHz and 6.75 MHz for luminance and colour difference signal. Different sampling rates can be obtained using an up sampler and down sampler. The basic operations in multirate processing to achieve this are decimation and interpolation. Decimation is for reducing the sampling rate and interpolation is for increasing the sampling rate. There are many cases where multi-rate signal processing is used. Few of them are as follows:

1. In high quality data acquisition and storage systems
2. In audio signal processing
3. In video
4. In speech processing
5. In transmultiplexers
6. For narrow band filtering

The various advantages of multirate signal processing are as follows:

1. Computational requirements are less.
2. Storage for filter coefficients is less.
3. Finite arithmetic effects are less.
4. Filter order required in multirate application is low.
5. Sensitivity to filter coefficient lengths is less.

While designing multi-rate systems, effects of aliasing for decimation and pseudoimages for interpolators should be avoided.

10.2 SAMPLING

A continuous-time signal $x(t)$ can be converted into a discrete-time signal $x(nT)$ by sampling it at regular intervals of time with sampling period T. The sampled signal $x(nT)$ is given by

$$x(nT) = x(t)\big|_{t=nT}, \quad -\infty < n < \infty$$

A sampling process can also be interpreted as a modulation or multiplication process.

Sampling Theorem

Sampling theorem states that a band limited signal $x(t)$ having finite energy, which has no spectral components higher than f_h hertz can be completely reconstructed from its samples taken at the rate of $2f_h$ or more samples per second.

The sampling rate of $2f_h$ samples per second is the Nyquist rate and its reciprocal $1/2f_h$ is the Nyquist period.

10.3 DOWN SAMPLING

Reducing the sampling rate of a discrete-time signal is called down sampling. The sampling rate of the discrete-time signal can be reduced by a factor D by taking every Dth value of the signal. Mathematically, down sampling is represented by

$$y(n) = x(Dn)$$

and the symbol for the down sampler is shown in Figure 10.1.

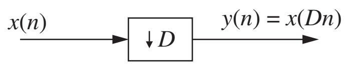


Figure 10.1 A down sampler.

If	$x(n) = \{1, 2, 3, 1, 2, 3, 1, 2, 3, ...\}$
Then,	$x(2n) = \{1, 3, 2, 1, 3, ...\}$
and	$x(3n) = \{1, 1, 1, 1, ...\}$

$x(2n)$ is obtained by keeping every second sample of $x(n)$ and $x(3n)$ is obtained by keeping every 3rd sample of $x(n)$ and removing other samples.

If the input signal $x(n)$ is not band limited, then there will be overlapping of spectra at the output of the down sampler. This overlapping of spectra is called aliasing which is undesirable. This aliasing problem can be eliminated by band limiting the input signal by inserting a low-pass filter called anti-aliasing filter before the down sampler. The anti-aliasing filter and the down sampler together is called decimator. The decimator is also

known as sub sampler, down sampler or under sampler. Decimation (sampling rate compression) is the process of decreasing the sampling rate by an integer factor D by keeping every Dth sample and removing $D - 1$ in between samples.

Figure 10.2 shows the signal $x(n)$ and its down sampled versions by a factor of 2 and 3.

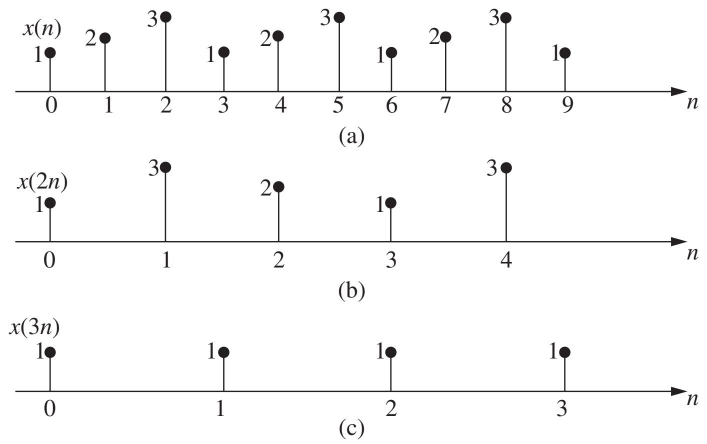


Figure 10.2 Plots of (a) $x(n)$, (b) $x(2n)$ and (c) $x(3n)$.

The block diagram of the decimator is shown in Figure 10.3. The decimator comprises two blocks such as anti-aliasing filter and down sampler. Here the anti-aliasing filter is a low-pass filter to band limit the input signal so that aliasing problem is eliminated and the down sampler is used to reduce the sampling rate by keeping every Dth sample and removing $D - 1$ in between samples.

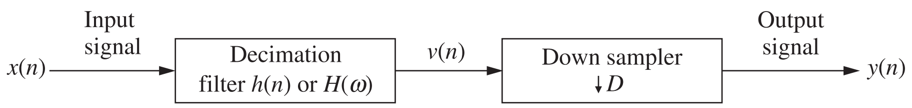


Figure 10.3 Block diagram of decimator.

Spectrum of down sampled signal

Let T be sampling period of input signal $x(n)$, and let F be its sampling rate or frequency. When the signal is down sampled by D, let T' be its new sampling period and F' be its sampling frequency, then

$$\frac{T'}{T} = D$$

$$F' = \frac{1}{T'} = \frac{1}{TD} = \frac{F}{D}$$

Let us derive the spectrum of a down sampled signal $x(Dn)$ and compare it with the spectrum of input signal $x(n)$. The Z-transform of the signal $x(n)$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

The down sampled signal $y(n)$ is obtained by multiplying the sequence $x(n)$ with a periodic train of impulses $p(n)$ with a period D and then leaving out the $D-1$ zeros between each pair of samples. The periodic train of impulses is given by

$$p(n) = \begin{cases} 1, & n = 0,\ \pm D,\ \pm 2D, \dots \\ 0, & \text{otherwise} \end{cases}$$

The discrete Fourier series representation of the signal $p(n)$ is given by

$$p(n) = \frac{1}{D} \sum_{k=0}^{D-1} e^{j2\pi kn/D}, \quad -\infty < n < \infty$$

Multiplying the sequence $x(n)$ with $p(n)$ yields

$$x'(n) = x(n)p(n)$$

That is
$$x'(n) = \begin{cases} x(n), & n = 0,\ \pm D,\ \pm 2D, \dots \\ 0\quad, & \text{otherwise} \end{cases}$$

If we leave $D-1$ zeros between each pair of samples, we get the output of down sampler

$$\begin{aligned} y(n) &= x'(nD) = x(nD)\ p(nD) \\ &= x(nD) \end{aligned}$$

The Z-transform of the output sequence is given by

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} y(n) z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x'(nD) z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x'(n) z^{-n/D} \end{aligned}$$

where $x'(n) = 0$ except at multiple of D. Since $x'(n) = x(n)\,p(n)$, we get

$$Y(z) = \sum_{n=-\infty}^{\infty} x(n)\ p(n) z^{-n/D}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left[\frac{1}{D} \sum_{k=0}^{D-1} e^{j2\pi kn/D} \right] z^{-n/D}$$

$$= \frac{1}{D} \sum_{k=0}^{D-1} \sum_{n=-\infty}^{\infty} x(n)\, (e^{-j2\pi k/D} z^{1/D})^{-n}$$

$$= \frac{1}{D} \sum_{k=0}^{D-1} X[e^{-j2\pi k/D} z^{1/D}]$$

Substituting $z = e^{j\omega}$, we get the frequency response

$$Y(\omega) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} e^{j\omega/D}) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{j(\omega-2\pi k)/D})$$

i.e.,

$$Y(\omega) = \frac{1}{D} \sum_{k=0}^{D-1} X\left[\frac{(\omega - 2\pi k)}{D} \right]$$

From the above relation we find that if the Fourier transform of the input signal $x(n)$ of a down sampler is $X(\omega)$, then the Fourier transform $Y(\omega)$ of the output signal $y(n)$ is a sum of D uniformly shifted and stretched versions of $X(\omega)$ scaled by a factor $1/D$.

If the spectrum of the original signal $X(\omega)$ is band limited to $\omega = \pi/d$, as shown in Figure 10.4(a), the spectrum being periodic with period 2π, the spectrum of the down sampled signal $Y(\omega)$ is the sum of all the uniformly shifted and stretched versions of $X(\omega)$ scaled by a factor $1/D$ as shown in Figure 10.4(b). In every interval of 2π in addition to the original spectrum we find $D-1$ equally spaced replica. In Figure 10.4(b), the frequency variable ω_x is related to the original sampling rate. In Figure 10.4(c), the frequency variable ω_y is normalized with respect to reduced sampling rate.

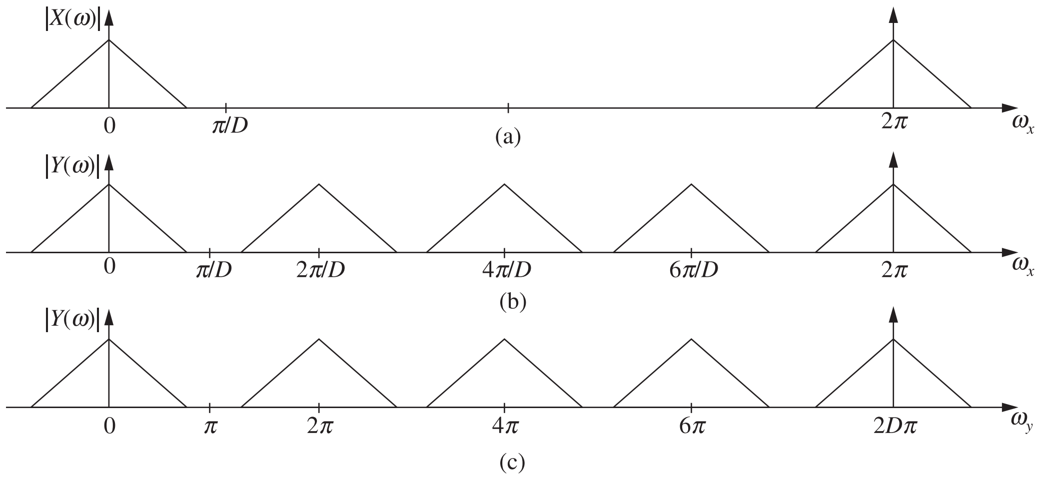


Figure 10.4 Spectrum of (a) input, (b) output, and (c) normalized output.

Aliasing effect and Anti-aliasing filter

From Figure 10.5, we can find that the spectrum obtained after down sampling will overlap if the original spectrum is not band limited to $\omega = \pi/D$. This overlapping of spectra is called aliasing. Therefore, aliasing due to down sampling a signal by a factor of D is absent if and only if the signal $x(n)$ is band limited to $\pm\pi/D$. If the signal $x(n)$ is not band limited to $\pm\pi/D$, then a low-pass filter with a cutoff frequency π/D is used prior to down sampling. This low-pass filter which is connected before the down sampler to prevent the effect of aliasing by band limiting the input signal is called the anti-aliasing filter.

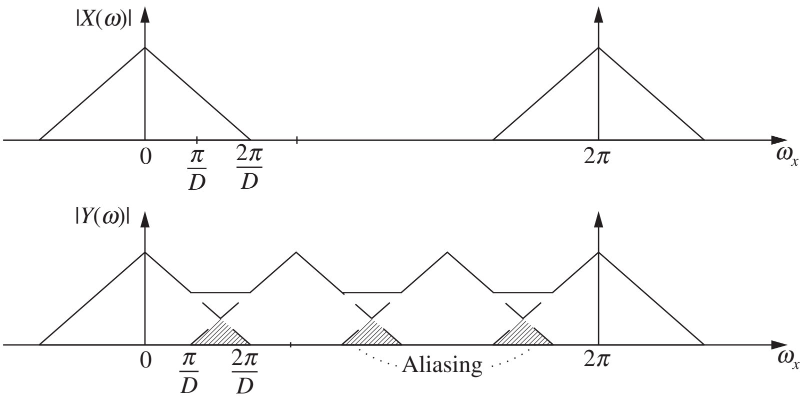


Figure 10.5 (a) Input spectrum, (b) aliased output spectrum.

The signal obtained after filtering is given by

$$v(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

and
$$y(n) = v(nD) = \sum_{k=-\infty}^{\infty} h(k)\, x(nD-k)$$

For example, consider a factor of D down sampler, then

$$Y(\omega) = \frac{1}{2}\sum_{k=0}^{1} X\left(\frac{\omega - 2\pi k}{2}\right)$$

$$= \frac{1}{2}\left[X\left(\frac{\omega}{2}\right) + X\left(\frac{\omega - 2\pi}{2}\right)\right]$$

$$= \frac{1}{2}\left[X\left(\frac{\omega}{2}\right) + X\left(-\frac{\omega}{2}\right)\right]$$

The second term $X(-\omega/2)$ is simply obtained by shifting the first term $X(\omega)$ to the right by an amount of 2π.

10.4 UP SAMPLING

Increasing the sampling rate of a discrete-time signal is called up sampling. The sampling rate of a discrete-time signal can be increased by a factor I by placing $I - 1$ equally spaced zeros between each pair of samples.

Mathematically, up sampling is represented by

$$y(n) = \begin{cases} x\left(\dfrac{n}{I}\right), & n = 0, \pm I, \pm 2I, \ldots \\ 0, & \text{otherwise} \end{cases}$$

and the symbol for up sampler is shown in Figure 10.6.

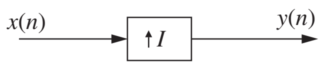


Figure 10.6 Up sampler.

If $x(n) = \{1, 2, 3, 1, 2, 3, \ldots\}$

Then, $y(n) = x\left(\dfrac{n}{2}\right)$ $\{1, 0, 2, 0, 3, 0, 1, 0, 2, 0, 3, 0, \ldots\}$ for an up-sampling factor of $I = 2$.

and $y(n) = x\left(\dfrac{n}{3}\right)$ $\{1, 0, 0, 2, 0, 0, 3, 0, 0, 1, \ldots\}$ for an up-sampling factor of $I = 3$.

Usually an anti-imaging filter is to be kept after the up sampler to remove the unwanted images developed due to up sampling. The anti-imaging filter and the up sampler together is called interpolator. Interpolation is the process of increasing the sampling rate by an integer factor I by interpolating $I - 1$ new samples between successive values of the signal.

Figure 10.7 shows the signal $x(n)$ and its two-fold up-sampled signal $y_1(n)$ and the interpolated signal $y_2(n)$.

The block diagram of the interpolator is shown in Figure 10.8. The interpolator comprises two blocks such as up sampler and anti-imaging filter. Here up sampler is used to increase the sampling rate by introducing zeros between successive input samples and the interpolation filter, also known as anti-imaging filter, is used to remove the unwanted images that are yielded by up sampling.

Expression for output of interpolator

Let I be an integer interpolating factor of the signal. Let T be sampling period and $F = 1/T$ be the sampling frequency (sampling rate) of the input signal. After up sampling, let T' be the new sampling period and F' be the new sampling frequency, then

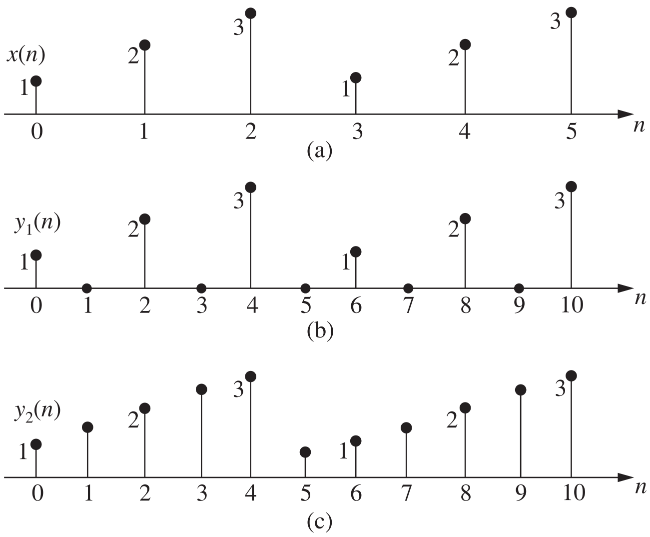


Figure 10.7 (a) Input signal $x(n)$, (b) Output of 2 fold up sampler $y_1(n) = x(n/2)$, (c) Output of interpolator $y_2(n) = x(n/2)$.

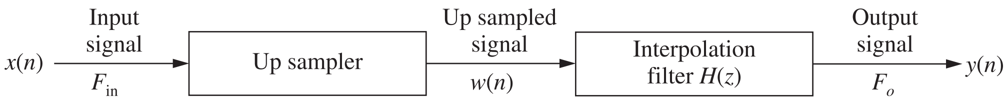


Figure 10.8 Block diagram of an interpolator.

$$\frac{T'}{T} = \frac{1}{I}$$

The sampling rate is given by

$$F' = \frac{1}{T'} = \frac{I}{T} = IF$$

Let $w(n)$ be the signal obtained by interpolating $I-1$ samples between each pair of samples of $x(n)$.

$$w(n) = \begin{cases} x\left(\dfrac{n}{I}\right), & n = 0, \pm I, \pm 2I, \ldots \\ 0, & \text{otherwise} \end{cases}$$

The Z-transform of the signal $w(n)$ is given by

$$W(z) = \sum_{n=-\infty}^{\infty} w(n)\, z^{-n} = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{I}\right) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, z^{-nI}$$

$$= X(z^I)$$

When considered over the unit circle $z = e^{j\omega'}$.

$$W(e^{j\omega'}) = X(e^{j\omega' I}) \text{ i.e. } W(\omega') = X(I\omega')$$

where $\omega' = 2\pi fT'$. The spectra of the signal $w(n)$ contains the images of base band placed at the harmonics of the sampling frequency $\pm 2\pi/I$, $\pm 4\pi/I$. To remove the images an anti-imaging filter is used. The ideal characteristics of low-pass filter is given by

$$H(e^{j\omega'}) = \begin{cases} G, & |\omega'| \le 2\pi fT'/2 = \pi/I \\ 0, & \text{otherwise} \end{cases}$$

where G is the gain of the filter and it should be I in the pass band. The frequency response of the output signal is given by

$$Y(e^{j\omega'}) = H(e^{j\omega'})X(e^{j\omega' I})$$

$$= \begin{cases} GX(e^{j\omega' I}), & |\omega'| \le \pi/I \\ 0 \quad , & \text{otherwise} \end{cases}$$

The output signal $y(n)$ is given by

$$y(n) = \sum_{k=-\infty}^{\infty} h(n-k)\, w(k)$$

$$= \sum_{k=-\infty}^{\infty} h(n-k)\, x(k/I), \qquad k/I \text{ is an integer}$$

Figure 10.9(a) shows the spectrum $X(\omega)$. The spectrum $X(I\omega)$ is sketched for $I = 3$ in Figure 10.9(b). Note that the frequency spectrum $X(3\omega)$ is three-fold repetition of $X(\omega)$. That is, inserting $I - 1$ zeros between successive values of $x(n)$ results in a signal whose spectrum $X(I\omega)$ is an I fold periodic repetition of the input spectrum $X(\omega)$. These additional spectra created are called image spectra and the phenomenon is known as imaging.

Anti-imaging Filter

The low-pass filter placed after the up sampler to remove the images created due to up sampling is called the anti-imaging filter.

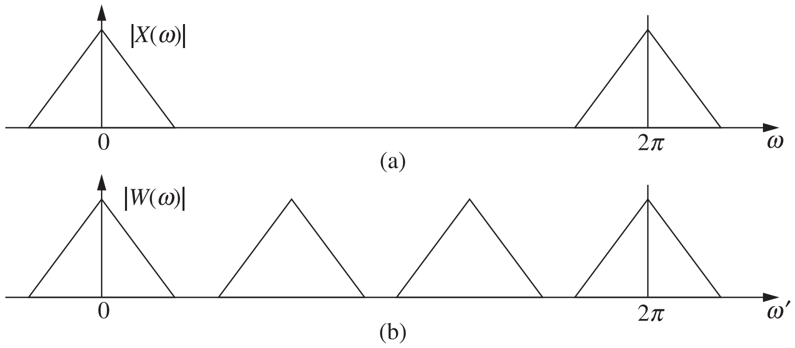


Figure 10.9 Spectrum of (a) $X(\omega)$ and (b) $X(3\omega)$.

EXAMPLE 10.1 Show that the up sampler and down sampler are time-variant systems.

Solution: Consider a factor of I up sampler defined by

$$y(n) = x\left(\frac{n}{I}\right)$$

The output due to delayed input is given by

$$y(n, k) = x\left(\frac{n}{I} - k\right)$$

The delayed output is given by

$$y(n-k) = x\left(\frac{n-k}{I}\right)$$

Therefore, $\qquad y(n, k) \neq y(n-k)$

So the up sampler is a time-variant system.

Consider a factor of D down sampler defined by

$$y(n) = x(Dn)$$

The output due to delayed input is given by

$$y(n, k) = x(Dn - k)$$

The delayed output is given by

$$y(n-k) = x[D(n-k)]$$

Therefore, $\qquad y(n, k) \neq y(n-k)$

So the down sampler is a time-variant system.

EXAMPLE 10.2 Consider a signal $x(n) = u(n)$.

(i) Obtain a signal with a decimation factor 3.
(ii) Obtain a signal with an interpolation factor 3.

Solution: Given that $x(n) = u(n)$ is the unit step sequence and is defined as:

$$u(n) = \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

The graphical representation of unit step sequence is shown in Figure 10.10(a).

(i) Signal with a decimation factor 3.
The decimated signal is given by

$$y(n) = x(Dn) = x(3n)$$

It is obtained by considering only every third sample of $x(n)$. The output signal $y(n)$ is shown in Figure 10.10(b).

(ii) Signal with interpolation factor 3.
The interpolated signal is given by

$$y(n) = x\left(\frac{n}{I}\right) = x\left(\frac{n}{3}\right)$$

The output signal $y(n)$ is shown in Figure 10.10(c). It is obtained by inserting two zeros between two consecutive samples.

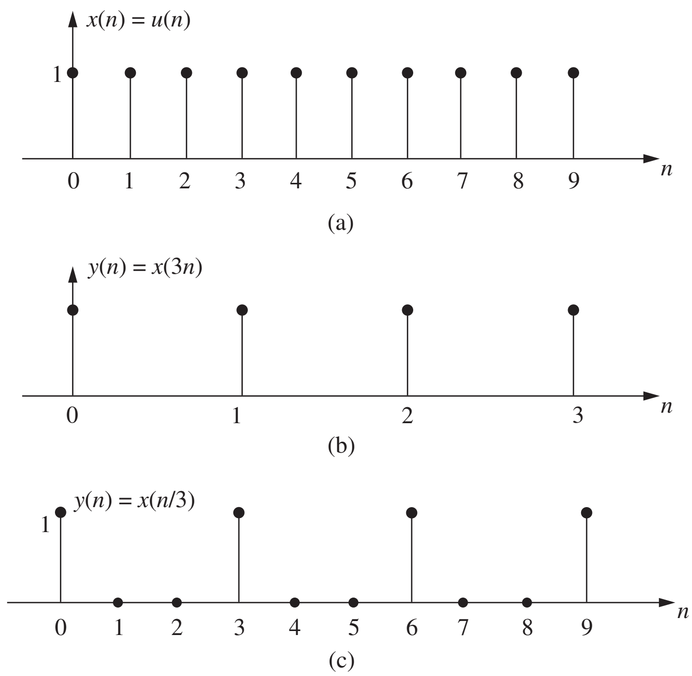


Figure 10.10 Plots of (a) $x(n) = u(n)$, (b) $x(3n)$ and (c) $x(n/3)$.

EXAMPLE 10.3 Consider a ramp sequence and sketch its interpolated and decimated versions with a factor of 3.

Solution: The ramp sequence is denoted as $r(n)$ and defined as

$$r(n) = \begin{cases} nu(n), & \text{for } n \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

The graphical representation of unit ramp signal is shown in Figure 10.11(a). The decimated signal is given by

$$y(n) = r(Dn) = r(3n)$$

The output signal $y(n) = r(3n)$ is shown in Figure 10.11(b). It is obtained by skipping 2 samples between every two successive sampling instants.

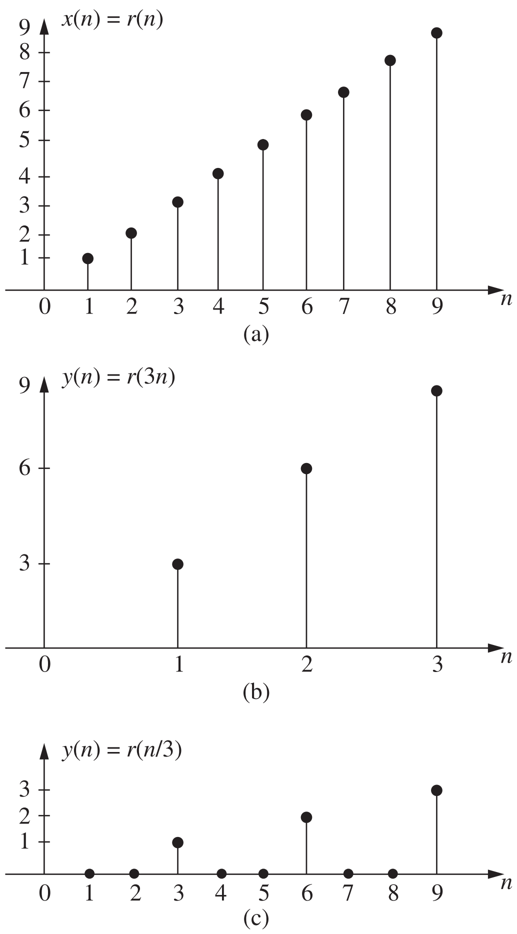


Figure 10.11 Plots of (a) $r(n) = nu(n)$, (b) $y(n) = r(3n)$ and (c) $y(n) = r(n/3)$.

The interpolated signal is given by

$$y(n) = r\left(\frac{n}{I}\right) = r\left(\frac{n}{3}\right)$$

The output signal $y(n) = r\left(\frac{n}{3}\right)$ is shown in Figure 10.11(c). It is obtained by inserting two zeros between every two successive sampling instants.

EXAMPLE 10.4 Consider a signal $x(n) = \sin \pi n u(n)$.

(i) Obtain a signal with a decimation factor 2.

(ii) Obtain a signal with an interpolation factor 2.

Solution: The given signal is $x(n) = \sin \pi n\, u(n)$. It is as shown in Figure 10.12(a).

(i) Signal with decimation factor 2. The signal $x(n)$ with a decimation factor 2 is given by

$$y(n) = x(2n) = \sin 2\pi n \cdot u(n), \quad n = 0, 1, 2, \ldots$$

Figure 10.12(b) shows the plot of $x(n)$ decimated by a factor of 2, i.e., $x(2n)$ versus n.

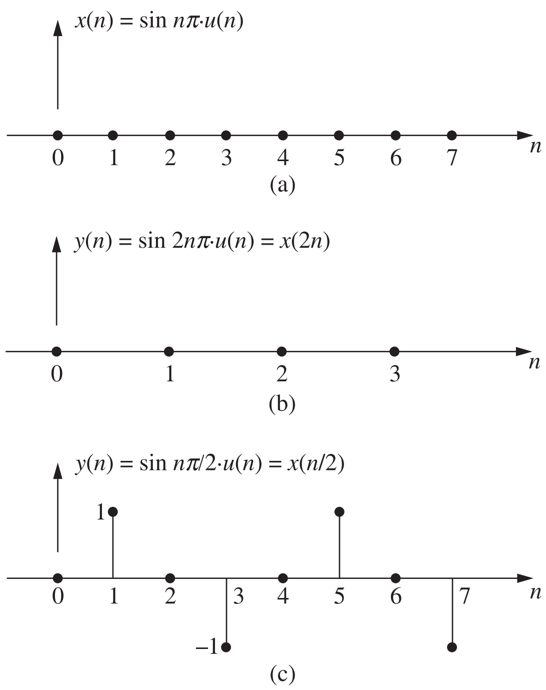


Figure 10.12 Plots of (a) $x(n) = \sin n\pi u(n)$, (b) $y(n) = \sin 2n\pi u(n)$ and (c) $y(n) = \sin (n\pi/2)u(n)$.

(ii) Signal with interpolation factor 2. The signal $x(n)$ with an interpolation factor 2 is given by

$$y(n) = x\left(\frac{n}{2}\right) = \sin\frac{n\pi}{2}\,u(n), \quad n = 0, 1, 2, \ldots$$

The plot of interpolated signal $x(n/2)$ is shown in Figure 10.12(c).

EXAMPLE 10.5 Consider the signal $x(n) = nu(n)$.

(i) Determine the spectrum of the signal.

(ii) The signal is applied to a decimator that reduces the sampling rate by a factor 3. Determine the output spectrum.

(iii) Show that the spectrum in part (ii) is simply Fourier transform of $x(3n)$.

Solution: Given that $x(n) = nu(n) = r(n)$.

From the given data, the sequence $x(n)$ is a ramp sequence. Figure 10.13 shows the graphical representation of ramp signal.

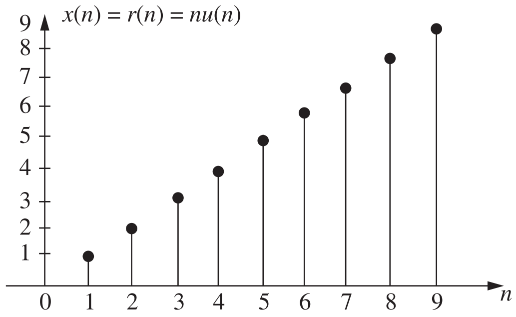


Figure 10.13 Ramp signal.

Taking Z-transform of the above equation for $x(n)$, we have

$$X(z) = Z[x(n)] = Z[nu(n)]$$

$$= \sum_{n=0}^{\infty} nz^{-n} = z^{-1} + 2z^{-2} + 3z^{-3} + \cdots$$

$$= z^{-1}[1 + 2z^{-1} + 3z^{-2} + \cdots]$$

$$= z^{-1}[1 - z^{-1}]^{-2} = z^{-1}\left[\frac{1}{1 - z^{-1}}\right]^2$$

$$= \frac{z}{(z-1)^2}$$

(i) The frequency spectrum of the signal $x(n)$, i.e., $X(\omega)$ is obtained by substituting $z = e^{j\omega}$ in $X(z)$. Therefore,

$$X(\omega) = \frac{z}{(z-1)^2}\Big|_{z=e^{j\omega}} = \frac{e^{j\omega}}{(e^{j\omega}-1)^2}$$

$$= \frac{\cos\omega + j\sin\omega}{(\cos\omega + j\sin\omega - 1)^2}$$

$$= \frac{\cos\omega + j\sin\omega}{(\cos 2\omega - 2\cos\omega + 1) + j(\sin 2\omega - 2\sin\omega)}$$

$$\therefore \qquad |X(\omega)| = \frac{1}{\sqrt{(\cos 2\omega - 2\cos\omega + 1)^2 + (\sin 2\omega - 2\sin\omega)^2}}$$

$$= \frac{1}{\sqrt{1+4+1-8\cos\omega + 2\cos 2\omega}}$$

$$= \frac{1}{\sqrt{6-8\cos\omega + 2\cos 2\omega}}$$

For different values of ω, the values of $|X(\omega)|$ are tabulated as follows:

ω	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
$\lvert X(\omega)\rvert$	∞	1.707	0.5	0.293	0.25	0.293	0.5	1.707	∞

The frequency spectrum $X(\omega)$ of $x(n)$ obtained using the values in the above table is shown in Figure 10.14.

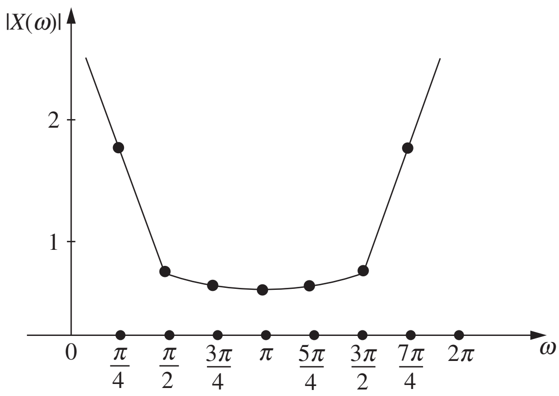


Figure 10.14 Magnitude spectrum of $x(n) = nu(n)$.

(ii) The sequence $x(n)$ is applied to a decimator that reduces sampling rate by a factor 3. So the respective decimated signal is given by $y(n) = x(3n) = 3nu(n)$. Then, the representation of the output sequence $y(n) = x(3n) = 3nu(n)$ is shown in Figure 10.15.

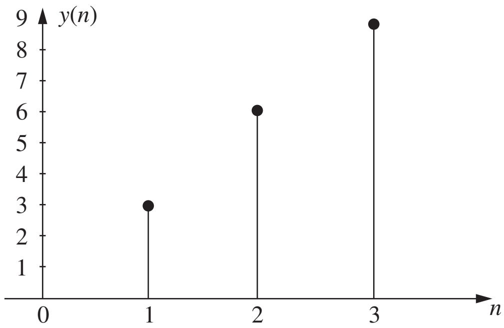


Figure 10.15 Plot of $y(n) = x(3n) = 3nu(n)$.

The Z-transform of $y(n) = 3nu(n)$ is:

$$Y(z) = 3\frac{z}{(z-1)^2}$$

Therefore, the frequency response is:

$$Y(\omega) = 3\frac{e^{j\omega}}{(e^{j\omega}-1)^2}$$

The magnitude spectrum $Y(\omega)$ of output sequence $y(n)$ is:

$$|Y(\omega)| = \frac{3}{\sqrt{6-8\cos\omega + 2\cos 2\omega}}$$

For different values of ω, the values of $|Y(\omega)|$ are tabulated as follows:

ω	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
$\lvert Y(\omega)\rvert$	∞	5.121	1.5	0.879	0.75	0.879	1.5	5.121	∞

Figure 10.16 shows the frequency spectrum of $y(n) = 3nu(n)$.

(iii) Fourier transform of $x(3n) = 3nu(n)$ is:

$$X(\omega) = \sum_{n=0}^{\infty} x(n)\, e^{-j\omega n}$$

$$= x(0) + x(1)e^{-j\omega} + x(2)e^{-j2\omega} + x(3)e^{-j3\omega} + \cdots$$

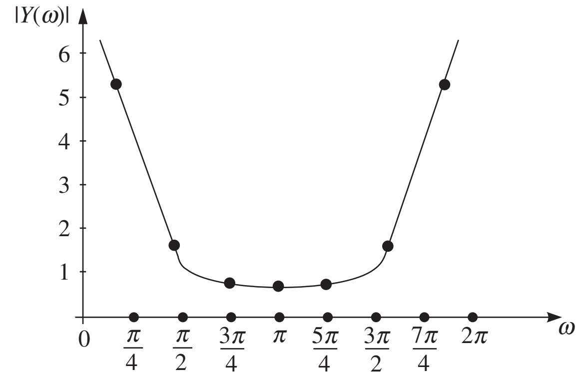


Figure 10.16 Frequency spectrum of $y(n) = 3nu(n)$.

$$= 0 + 3e^{-j\omega} + 6e^{-j2\omega} + 9e^{-j3\omega} + 12e^{-j4\omega}$$

$$= 3e^{-j\omega}[1 + 2e^{-j\omega} + 3e^{-j2\omega} + 4e^{-j3\omega} + \cdots]$$

$$= \frac{3e^{-j\omega}}{[1-e^{-j\omega}]^2} = \frac{3e^{j\omega}}{[e^{j\omega}-1]^2}$$

This is same as spectrum of signal in part (ii), that is, of $x(3n) = 3nu(n)$.

EXAMPLE 10.6 Consider the signal $x(n) = a^n u(n),\ |a| < 1$.

(i) Determine the spectrum of the signal.

(ii) The signal is applied to an interpolator that increases sampling rate by a factor of 2. Determine its output spectrum.

(iii) Show that the spectrum in part (ii) is simply Fourier transform of $x(n/2)$.

Solution: The given signal is $x(n) = a^n u(n),\ |a| < 1$.

(i) Taking Z-transform of $x(n)$, we have

$$X(z) = Z[a^n u(n)] = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n = 1 + az^{-1} + (az^{-1})^2 + (az^{-1})^3 + \cdots$$

$$= [1-az^{-1}]^{-1} = \frac{1}{1-az^{-1}} = \frac{z}{z-a},\ |z| > a$$

The spectrum $X(\omega)$ of the given $x(n)$ is obtained by substituting $z = e^{j\omega}$ in $X(z)$.

$$\therefore \qquad X(\omega) = X(z)\big|_{z=e^{j\omega}} = \frac{e^{j\omega}}{e^{j\omega}-a} = \frac{\cos\omega + j\sin\omega}{(\cos\omega - a) + j\sin\omega}$$

$$\therefore \qquad |X(\omega)| = \frac{\sqrt{\cos^2\omega + \sin^2\omega}}{\sqrt{(\cos\omega - a)^2 + \sin^2\omega}} = \frac{1}{\sqrt{1 - 2a\cos\omega + a^2}}$$

The values of $|X(\omega)|$ for different values of ω is tabulated as follows:

ω	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
$\lvert X(\omega)\rvert$	$\frac{1}{1-a}$	$\frac{1}{\sqrt{1-\sqrt{2}a+a^2}}$	$\frac{1}{\sqrt{1+a^2}}$	$\frac{1}{\sqrt{1+\sqrt{2}a+a^2}}$	$\frac{1}{1+a}$	$\frac{1}{\sqrt{1+\sqrt{2}a+a^2}}$	$\frac{1}{\sqrt{1+a^2}}$	$\frac{1}{\sqrt{1-\sqrt{2}a+a^2}}$	$\frac{1}{1-a}$

The frequency spectrum of the signal $x(n)$ is plotted as shown in Figure 10.17.

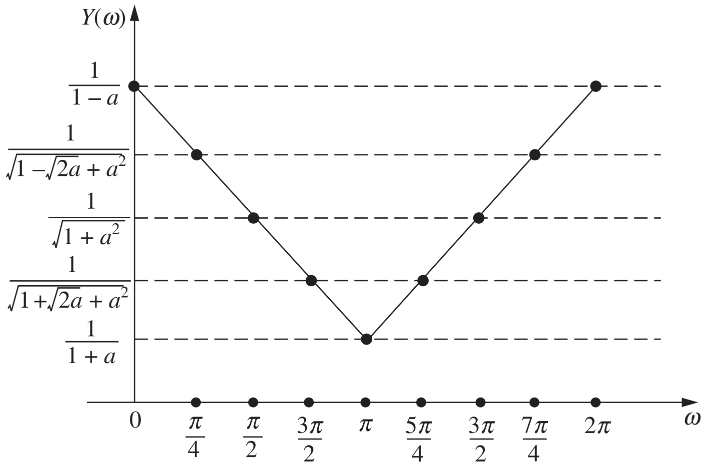


Figure 10.17 Magnitude spectrum of $X(\omega)$.

(ii) The interpolated signal $y(n)$ which is obtained by increasing the sampling rate by a factor of 2 for $a^n u(n)$ can be written as: $y(n) = x\left(\frac{n}{2}\right) = a^{n/2}u(n)$.

Taking Z-transform, we get

$$Z[y(n)] = Y(z) = \sum_{n=0}^{\infty} a^{n/2} z^{-n}$$

$$= 1 + a^{1/2}z^{-1} + az^{-2} + a^{3/2}z^{-3} + a^2 z^{-4} + \cdots$$

$$= 1 + \sqrt{a}z^{-1} + \left(\sqrt{a}z^{-1}\right)^2 + \left(\sqrt{a}z^{-1}\right)^3 + \cdots$$

$$= \left[1 - \sqrt{a}z^{-1}\right]^{-1} = \frac{1}{1 - \sqrt{a}z^{-1}}, \quad \sqrt{a}z^{-1} < 1$$

$$= \frac{z}{z - \sqrt{a}}, \quad |z| > \sqrt{a}$$

The spectrum of signal $y(n)$ can be obtained by substituting $z = e^{j\omega}$ in the above equation.

$$\therefore \qquad Y(\omega) = \frac{e^{j\omega}}{e^{j\omega} - \sqrt{a}} = \frac{\cos\omega + j\sin\omega}{(\cos\omega + j\sin\omega) - \sqrt{a}}$$

$$|Y(\omega)| = \frac{\sqrt{\cos^2\omega + \sin^2\omega}}{\sqrt{(\cos\omega - \sqrt{a})^2 + \sin^2\omega}} = \frac{1}{\sqrt{1 + a - 2\sqrt{a}\cos\omega}}$$

For different values of ω, the value of $|Y(\omega)|$ is tabulated below:

ω	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
$\lvert Y(\omega)\rvert$	$\frac{1}{1-\sqrt{a}}$	$\frac{1}{\sqrt{1-\sqrt{2}a+a}}$	$\frac{1}{\sqrt{1+a}}$	$\frac{1}{\sqrt{1+\sqrt{2}a+a}}$	$\frac{1}{1+\sqrt{a}}$	$\frac{1}{\sqrt{1+\sqrt{2}a+a}}$	$\frac{1}{\sqrt{1+a}}$	$\frac{1}{\sqrt{1-\sqrt{2}a+a}}$	$\frac{1}{1-\sqrt{a}}$

The magnitude spectrum of $y(n) = x(2n)$ is shown in Figure 10.18.

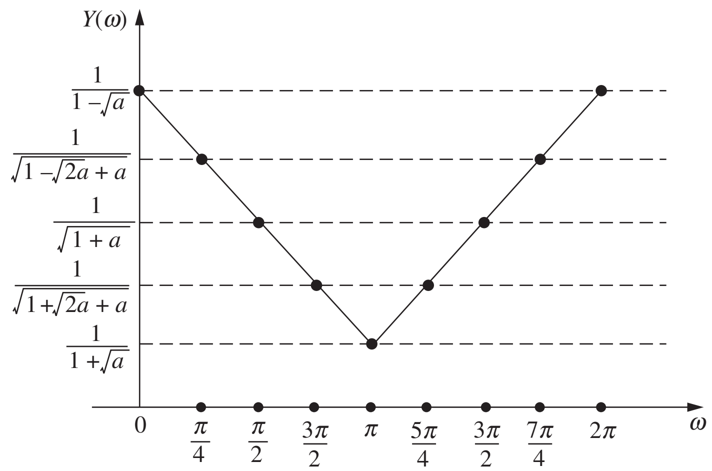


Figure 10.18 Magnitude spectrum of $y(n) = x(n/2)$.

(iii) The Fourier transform for $x(n/2)$ is obtained as:

$$F\left[x\left(\frac{n}{2}\right)\right] = F\left[a^{\frac{n}{2}}\,u(n)\right] = \sum_{n=0}^{\infty} a^{n/2} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} [\sqrt{a}e^{-j\omega}]^n$$

$$= \frac{1}{1-\sqrt{a}e^{-j\omega}}$$

$$= \frac{e^{j\omega}}{e^{j\omega}-\sqrt{a}}$$

This shows that spectrum in part (ii) is simply the Fourier transform of $x(n/2)$.

10.5 SAMPLING RATE CONVERSION

In some applications sampling rate conversion by a non-integer factor may be required. For example transferring data from a compact disc at a rate of 44.1 kHz to a digital audio tape at 48 kHz. Here the sampling rate conversion factor is 48/44.1, which is a non-integer.

A sampling rate conversion by a factor I/D can be achieved by first performing interpolation by factor I and then performing decimation by factor D. Figure 10.19(a) shows the cascade configuration of interpolator and decimator. The anti-imaging filter $H_u(z)$ and the anti-aliasing filter $H_d(z)$ are operated at the sampling rate, hence can be replaced by a simple low-pass filter with transfer function $H(z)$ as shown in Figure 10.19(b), where the low-pass filter has a cutoff frequency of $\omega_c = \min\left[\frac{\pi}{I}, \frac{\pi}{D}\right]$.

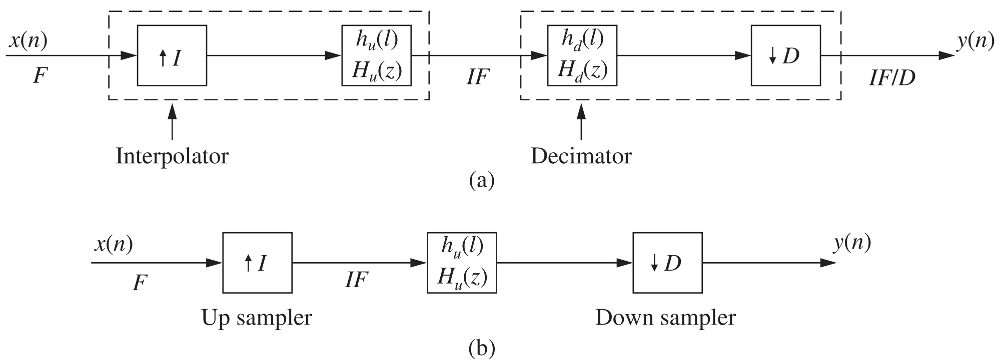


Figure 10.19 Cascading of sample rate converters.

Time domain and frequency domain relations of sampling rate converters

In Figure 10.19(a), $h_u(l)$ is Anti-imaging filter and $h_d(l)$ is Anti-aliasing filter. The overall cutoff frequency of the two cascaded low-pass filters [i.e. $h_u(l)$ and $h_d(l)$] will be the minimum of the two cutoff frequencies.

The frequency response of $h_u(l)$ (anti-imaging filter) is given as:

$$H_d(\omega) = \begin{cases} I, & -\dfrac{\pi}{I} \le \omega \le \dfrac{\pi}{I} \\ 0, & \text{elsewhere} \end{cases}$$

The frequency response of $h_d(l)$ (anti-aliasing filter) is given as:

$$H_d(\omega) = \begin{cases} 1, & -\dfrac{\pi}{D} \le \omega \le \dfrac{\pi}{D} \\ 0, & \text{elsewhere} \end{cases}$$

Time domain relationship

From Figure 10.19(b), the output of the low-pass filter is given as:

$$w(l) = \sum_{k=-\infty}^{\infty} h(l-k)\, v(k)$$

$$= \sum_{k=-\infty}^{\infty} h(l-kI)\, x(k)$$

In Figure 10.19, $y(d)$ is the output of the down sampler and is given by

$$y(d) = w(dD)$$

$$= \sum_{k=-\infty}^{\infty} h(dD-kI)\, x(k)$$

Therefore, the time domain relationship between the input and output of a sampling rate converter is:

$$y(n) = \sum_{k=-\infty}^{\infty} h(nD-kI)\, x(k)$$

Frequency domain relationship

From Figure 10.19(b), $v(k)$ = output of up sampler with frequency ω_v.

Therefore, the output of the up sampler with frequency ω_v is expressed as:

$$V(\omega_v) = X(\omega_v I)$$

The output of the up sampler is passed through a LPF and hence we obtain $w(l)$ with frequency ω_v. Therefore, the output of the low-pass filter with frequency ω_v is given as

$$W(\omega_v) = H(\omega_v)X(\omega_v)$$

$$\therefore \qquad W(\omega_v) = \begin{cases} IX(\omega_v I), & |\omega| \le \min\left(\dfrac{\pi}{D}, \dfrac{\pi}{I}\right) \\ 0, & \text{elsewhere} \end{cases}$$

The spectrum of the output sequence is given by

$$Y(\omega_y) = \frac{1}{D} X\left(\frac{\omega_y}{D}\right)$$

$$\therefore \qquad Y(\omega_y) = \frac{1}{D} W\left(\frac{\omega_y}{D}\right)$$

We know that, $$\omega_v = \frac{\omega_y}{D}$$

$$\therefore \qquad Y(\omega_y) = \frac{1}{D} W(\omega_v)$$

Substituting $W(\omega_v) = IX(\omega_v I)$, we get

$$Y(\omega_y) = \begin{cases} \dfrac{I}{D} X(\omega_v I), & |\omega_v| \le \min\left(\dfrac{\pi}{D}, \dfrac{\pi}{I}\right) \\ 0, & \text{elsewhere} \end{cases}$$

$$Y(\omega_y) = \begin{cases} \dfrac{I}{D} X\left(\dfrac{I\omega_y}{D}\right), & \left|\dfrac{\omega_y}{D}\right| \le \min\left(\dfrac{\pi}{D}, \dfrac{\pi}{I}\right) \\ 0, & \text{elsewhere} \end{cases}$$

So the frequency domain relationship between input and output of a sampling rate converter is:

$$Y(\omega_y) = \begin{cases} \dfrac{I}{D} X\left(\dfrac{I\omega_y}{D}\right), & |\omega_y| \le \min\left(\pi, \dfrac{\pi D}{I}\right) \\ 0, & \text{elsewhere} \end{cases}$$

Figure 10.20 shows the sampling rate conversion by a factor of 5/3. Figure 10.20(a) shows the actual signal $x(n)$. The sampling rate is increased by 5, by inserting 4 zero valued samples between successive values of $x(n)$ as shown in Figure 10.20(b). The output of

anti-imaging filter is shown in Figure 10.20(c). The filtered data is then reduced for every three samples as shown in Figure 10.20(d).

A cascade of a factor of D down sampler and a factor of I up sampler is interchangeable with no change in the input and output relation if and only if I and D are co-prime.

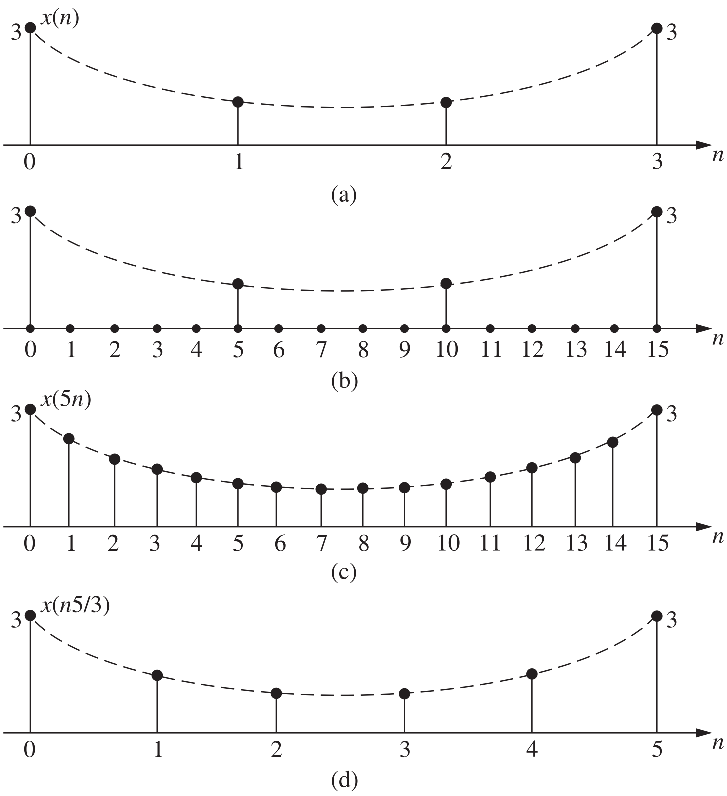


Figure 10.20 Sampling rate conversion by a factor of 5/3.

EXAMPLE 10.7 Considering an example

$$x(n) = \{1, 3, 2, 5, 4, -1, -2, 6, -3, 7, 8, 9, ...\}$$

show that a cascade of D down sampler and I up sampler is interchangeable only when D and I are co-prime.

Solution: Given $x(n) = \{1, 3, 2, 5, 4, -1, -2, 6, -3, 7, 8, 9, ...\}$

(i) Let $D = 2$ and $I = 3$. Here D and I are co-prime.

For the cascading shown in Figure 10.21, we have

$$x_d(n) = \{1, 2, 4, -2, -3, 8, \ldots\}$$

$$y_1(n) = \{1, 0, 0, 2, 0, 0, 4, 0, 0, -2, 0, 0, -3, 0, 0, 8, \ldots\}$$

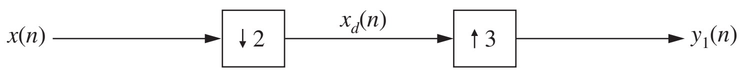


Figure 10.21 Cascading of $D = 2$ and $I = 3$.

Interchanging the cascading as shown in Figure 10.22, we have

$$x_u(n) = \{1, 0, 0, 3, 0, 0, 2, 0, 0, 5, 0, 0, 4, 0, 0, -1, 0, 0, -2, 0, 0, 6, 0, 0, -3, 0, 0, 7, 0, 0, 8, \ldots)$$

$$y_2(n) = \{1, 0, 0, 2, 0, 0, 4, 0, 0, -2, 0, 0, -3, 0, 0, 8, \ldots\}$$

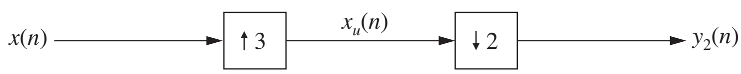


Figure 10.22 Cascading of $I = 3$ and $D = 2$.

Now $y_1(n) = y_2(n)$. This shows that the cascade of an I up sampler and a D down sampler are interchangeable when I and D are co-prime.

(ii) Let $D = 2$ and $I = 4$. Here D and I are not co-prime.
For the cascading shown in Figure 10.23, we have

$$x_d(n) = \{1, 2, 4, -2, -3, 8, \ldots\}$$

$$y_3(n) = \{1, 0, 0, 0, 2, 0, 0, 0, 4, 0, 0, 0, -2, 0, 0, 0, -3, 0, 0, 0, -8, \ldots\}$$

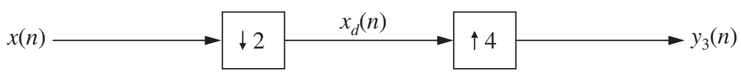


Figure 10.23 Cascading of $D = 2$ and $I = 4$.

Interchanging the cascading as shown in Figure 10.24, we have

$$x_u(n) = \{1, 0, 0, 0, 3, 0, 0, 0, 2, 0, 0, 0, 5, 0, 0, 0, 4, 0, 0, 0, -1, \ldots\}$$

$$y_4(n) = \{1, 0, 3, 0, 2, 0, 5, 0, 4, 0, -1, \ldots\}$$

Now, $y_3(n) \neq y_4(n)$.

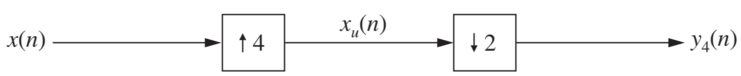


Figure 10.24 Cascading of $I = 4$ and $D = 2$.

This shows that the cascading of up sampler and down sampler is not interchangeable when D and I are not co-prime, i.e., when D and I have a common factor.

EXAMPLE 10.8 Show that the transpose of a factor of D decimator is a factor of D interpolator if the transpose of a factor of D down sampler is a factor of D up sampler.

Solution: The transpose of a digital filter is obtained by reversing all paths, interchanging the input and output nodes, replacing the pick off node with an adder and vice versa. The factor of D decimator is shown in Figure 10.25.

Figure 10.25 Factor of D decimator.

Interchanging the input and output nodes and reversing the paths results in Figure 10.26.

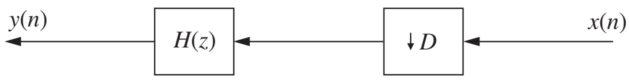


Figure 10.26 Transpose of decimator.

If the transpose of a factor of D down sampler is a factor of D up sampler, we have Figure 10.27.

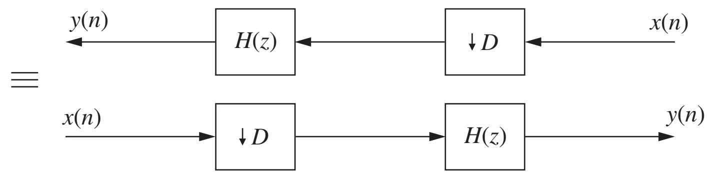


Figure 10.27 Transpose of down sampler.

Hence the transpose of a factor of D decimator is a factor of D interpolator.

10.6 IDENTITIES

1. The scaling of discrete-time signals and their addition at the nodes are independent of the sampling rate. It is illustrated in Figure 10.28.

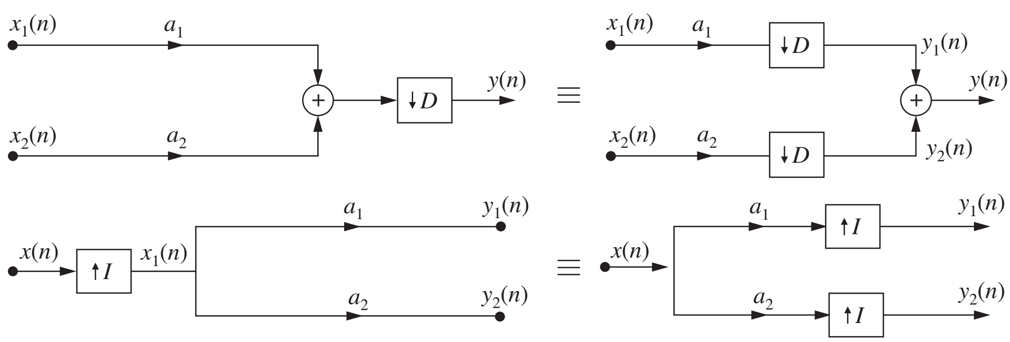


Figure 10.28 Identity 1.

2. A delay of D sample periods before a down sampler is the same as a delay of one sample period after the down sampler. It is illustrated in Figure 10.29.

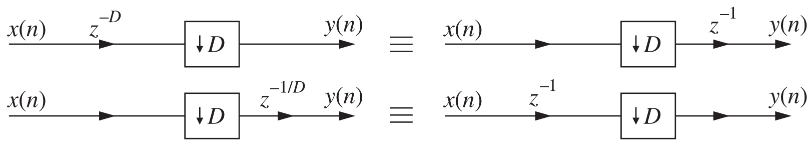


Figure 10.29 Identity 2.

3. A delay of one sample period before up sampling leads to a delay of I sample periods after the up sampling. It is illustrated in Figure 10.30.

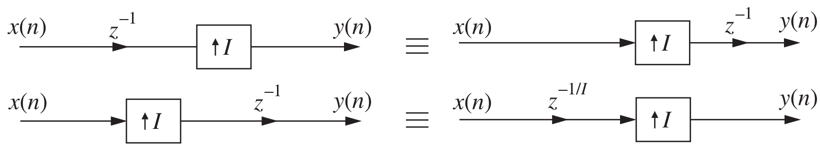


Figure 10.30 Identity 3.

4. Two down samplers with down sampling factors of D_1 and D_2 in cascade can be replaced by a single down sampler with a down sampling factor $D = D_1D_2$. It is illustrated in Figure 10.31.

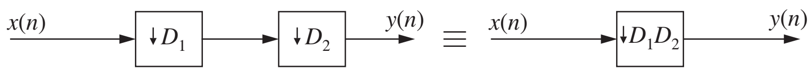


Figure 10.31 Identity 4.

5. Two up samplers with up-sampling factors of I_1 and I_2 in cascade can be replaced by a single up sampler with up-sampling factor $I = I_1I_2$. It is illustrated in Figure 10.32.

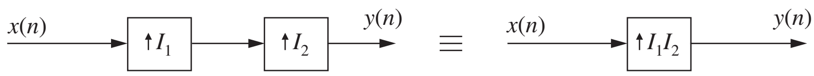


Figure 10.32 Identity 5.

6. An up sampler with a sampling factor I followed by a down sampler with the same sampling factor $D = I$ results in no change in input signal. It is illustrated in Figure 10.33.

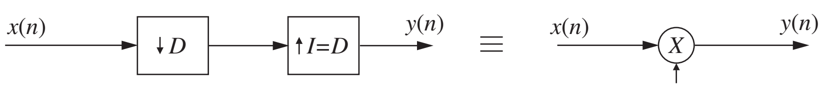


Figure 10.33 Identity 6.

7. A cascade of a down sampler with down sampling factor D followed by an up sampler with up sampling factor $I = D$ results in an output which is same as the input at the new sampling instants. It is illustrated in Figure 10.34.

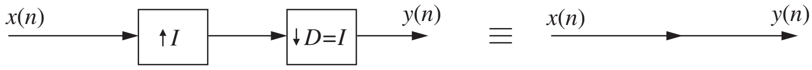


Figure 10.34 Identity 7.

8. and 9.

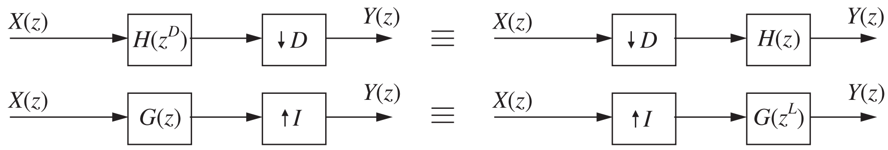


Figure 10.35 Identities 8 and 9.

10.7 POLYPHASE DECOMPOSITION

Polyphase decomposition results in reduction of computation complexity in FIR filter realization. The Z-transform of a filter with impulse response $h(n)$ is given by

$$H(z) = h(0) + z^{-1}h(1) + z^{-2}h(2) + \cdots$$

Rearranging the above equation, we get

$$H(z) = h(0) + z^{-2}h(2) + z^{-4}h(4) + \cdots + z^{-1}[h(1) + z^{-2}h(3) + z^{-4}h(5) + \cdots]$$

Type I Polyphase Decomposition

$$H(z) = P_0(z^2) + z^{-1}P_1(z^2)$$

where $$P_0(z^2) = h(0) + z^{-2}h(2) + z^{-4}h(4) + \cdots$$

and $$P_1(z^2) = h(1) + z^{-2}h(3) + z^{-4}h(5) + \cdots$$

are polyphase components for a factor of 2.

Type II Polyphase Decomposition

$$H(z) = z^{-1}R_0(z^2) + R_1(z^2)$$

where $R_0(z^2)$ and $R_1(z^2)$ are polyphase components for a factor of 2. The above equations represent two branch polyphase decomposition of $H(z)$. In general, a D-branch type I polyphase decomposition is given by

$$H(z) = \sum_{k=0}^{D-1} z^{-k}P_k(z^D)$$

The type II polyphase decomposition is given by

$$H(z) = \sum_{i=0}^{D-1} z^{(-D-1-I)}\, R_i(z^D)$$

The type I and type II polyphase decomposition representation is shown in Figure 10.36.

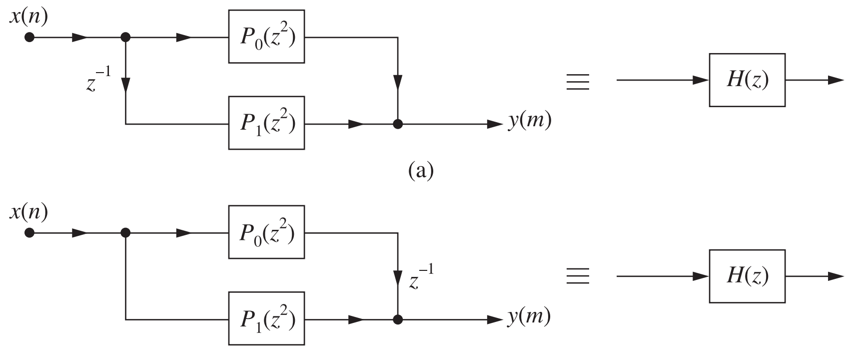


Figure 10.36 The Type I and Type II polyphase decomposition representation.

EXAMPLE 10.9 Obtain the polyphase decompositions of the IIR digital system having following transfer function:

$$H(z) = \frac{1-4z^{-1}}{1+6z^{-1}}$$

Solution: Given

$$H(z) = \frac{1-4z^{-1}}{1+6z^{-1}} = P_0(z^2) + z^{-1}P_1(z^2)$$

where $P_0(z^2)$ and $P_1(z^2)$ are polyphase components of the IIR digital system $H(z)$.

$$H(z) = \frac{1-4z^{-1}}{1+6z^{-1}} = \left(\frac{1-4z^{-1}}{1+6z^{-1}}\right)\left(\frac{1-6z^{-1}}{1-6z^{-1}}\right)$$

$$= \frac{1-10z^{-1}+24z^{-2}}{1-36z^{-2}}$$

$$= \left(\frac{1+24z^{-2}}{1-36z^{-2}}\right) + z^{-1}\left(\frac{-10}{1-36z^{-2}}\right)$$

Comparing the above two expressions for $H(z)$, we get polyphase components of $H(z)$.

$$P_0(z^2) = \frac{1 + 24z^{-2}}{1 - 36z^{-2}}$$

$$P_1(z^2) = \frac{-10}{1 - 36z^2}$$

EXAMPLE 10.10 Develop a two-band polyphase decomposition for the following transfer function.

$$H(z) = \frac{1 + z^{-1} + 3z^{-2}}{1 + 0.6z^{-1} + 0.4z^{-2}}$$

Solution:

$$H(z) = P_0(z^2) + z^{-1}P_1(z^2)$$

$$H(-z) = P_0(z^2) - z^{-1}P_1(z^2)$$

Therefore,

$$P_0(z^2) = \frac{1}{2}\{H(z) + H(-z)\}$$

and

$$z^{-1}P_1(z^2) = \frac{1}{2}\{H(z) - H(-z)\}$$

$$P_0(z^2) = \frac{1}{2}\left\{\frac{1 + z^{-1} + 3z^{-2}}{1 + 0.6z^{-1} + 0.4z^{-2}} + \frac{1 - z^{-1} + 3z^{-2}}{1 - 0.6z^{-1} + 0.4z^{-2}}\right\}$$

$$= \frac{2 + 5.6z^{-2} + 2.4z^{-4}}{1 + 0.44z^{-2} + 0.16z^{-4}}$$

$$z^{-1}P_1(z^2) = \frac{1}{2}\left\{\frac{1 + z^{-1} + 3z^{-2}}{1 + 0.6z^{-1} + 0.4z^{-2}} - \frac{1 - z^{-1} + 3z^{-2}}{1 - 0.6z^{-1} + 0.4z^{-2}}\right\}$$

$$= \frac{z^{-1}(0.8 - 2.8z^{-2})}{1 + 0.44z^{-2} + 0.16z^{-4}}$$

$\therefore$

$$H(z) = \frac{2 + 5.6z^{-2} + 2.4z^{-4}}{1 + 0.44z^{-2} + 0.16z^{-4}} + \frac{z^{-1}(0.8 - 2.8z^{-2})}{1 + 0.44z^{-2} + 0.16z^{-4}}$$

EXAMPLE 10.11 Develop a two-band decomposition of the following transfer function:

$$H(z) = \frac{A + Bz^{-1}}{1 + d_1z^{-1}}, \ |d_1| < 1$$

Solution: Given
$$H(z)=\frac{A+Bz^{-1}}{1+d_1z^{-1}}=\left(\frac{A+Bz^{-1}}{1+d_1z^{-1}}\right)\left(\frac{1-d_1z^{-1}}{1-d_1z^{-1}}\right)$$
$$=\frac{(A-Bd_1z^{-2})}{1-d_1z^{-2}}+\frac{z^{-1}(B-Ad_1)}{1-d_1z^{-2}}$$

Since $H(z) = P_0(z^2) + z^{-1}P_1(z^2)$, we get polyphase components of $H(z)$ as:
$$P_0(z^2)=\frac{A-Bd_1z^{-2}}{1-d_1z^{-2}}$$
$$P_1(z^2)=\frac{B-Ad_1}{1-d_1z^{-2}}$$

10.8 GENERAL POLYPHASE FRAMEWORK FOR DECIMATORS AND INTERPOLATORS

Consider the decimator having an anti-aliasing filter with impulse response $h(n)$ shown in Figure 10.37.

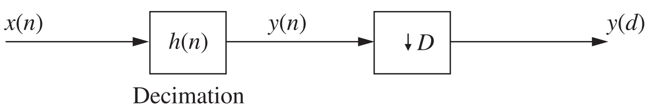


Figure 10.37 Decimation by a factor D.

The Z-transform of the filter is:
$$H(z)=\sum_{n=0}^{\infty}h(n)z^{-n}$$
$$=h(0)+h(1)z^{-1}+h(2)z^{-2}+h(3)z^{-3}+\cdots$$

This $H(z)$ can be partitioned into D sub-signals where D represents decimation factor. Hence
$$H(z)=h(0)+h(D)z^{-D}+h(2D)z^{-2D}+\cdots$$
$$+z^{-1}[h(1)+h(D+1)z^{-D}+h(2D+1)z^{-2D}+\cdots]$$
$$+z^{-(D-1)}[h(D-1)+h(2D-1)z^{-D}+\cdots]$$

The previous equation for $H(z)$ can be written as:

$$H(z) = \sum_{k=0}^{D-1} \sum_{d=0}^{\infty} h(dD + k) z^{-(dD+k)}$$

$$= \sum_{k=0}^{D-1} z^{-k} \sum_{d=0}^{\infty} h(dD + k) z^{-dD}$$

$$= P_0(z^D) + z^{-1}P_1(z^D) + z^{-2}P_2(z^D) + \cdots + z^{-(D-1)}P_{D-1}(z^D)$$

where $P_0(z^D)$, $P_1(z^D)$, ... are polyphase components given by

$$P_0(z^D) = \langle H(z)|_{k=0}$$

$$P_1(z^D) = \langle H(z)|_{k=1}$$

$$P_2(z^D) = \langle H(z)|_{k=2} \quad \ldots.$$

In matrix form,

$$H(z) = [1, z^{-1}, z^{-2}, ..., z^{-(D-1)}] \begin{bmatrix} P_0(z^D) \\ P_1(z^D) \\ \vdots \\ P_1(z^D) \end{bmatrix}$$

With these polyphase components, the anti-aliasing filter and its equivalent using decimator identity can be represented as shown in Figure 10.38. Similarly, with the polyphase components $R_0(z^I), R_1(z^I), ..., R_{I-1}(z^I)$, the equivalent of the interpolation filter of Figure 10.39 using interpolation identity are shown in Figure 10.40.

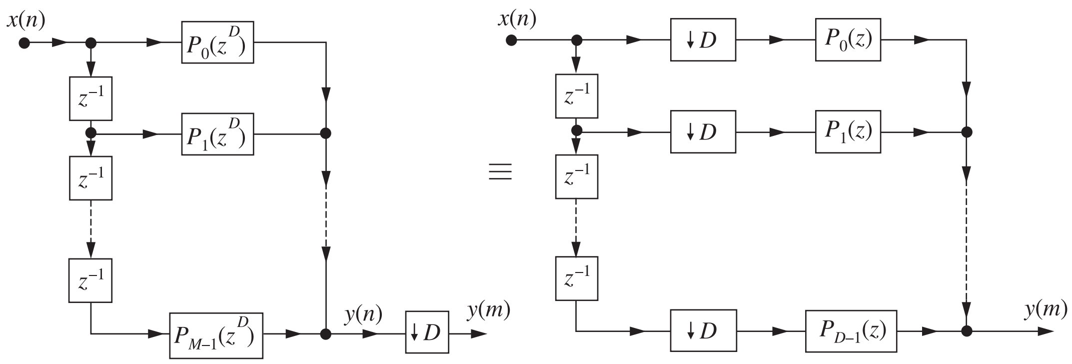


Figure 10.38 Polyphase representation of decimators.

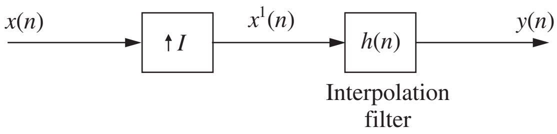


Figure 10.39 Interpolation by a factor *I*.

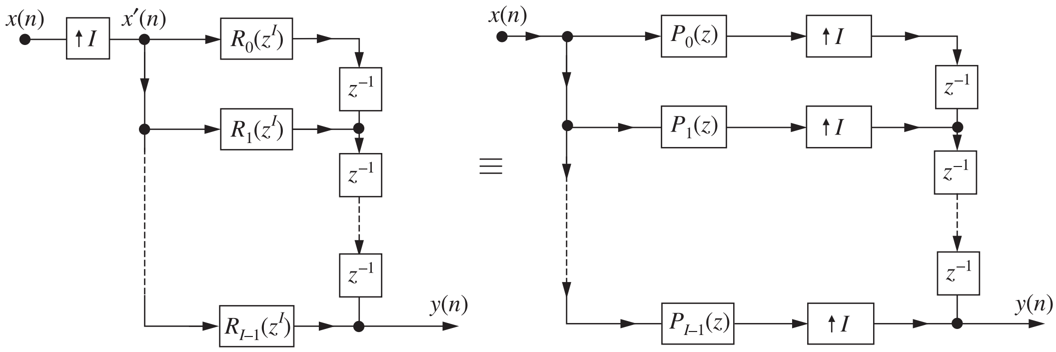


Figure 10.40 Polyphase representation of interpolators.

10.9 MULTISTAGE DECIMATORS AND INTERPOLATORS

In practical applications, mostly sampling rate conversion by a rational factor I/D is required. Figure 10.41 represents the general structure of the system where this conversion is used.

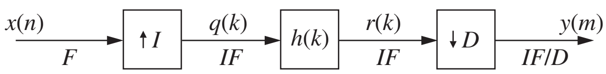


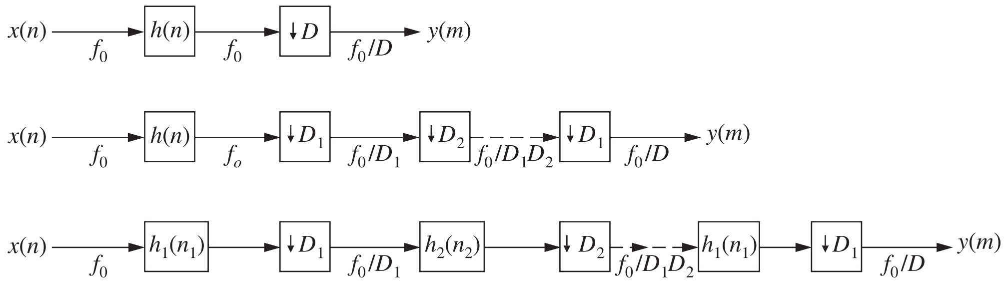
Figure 10.41 Sampling rate conversion by a rational factor *I*/*D*.

If the decimation factor D and/or interpolation factor I are much larger than unity, the implementation of sampling rate conversion in a single stage is computational inefficient. Therefore, for performing sampling rate conversion for either $D >> 1$ and/or $I >> 1$, we go in for multistage implementation.

Consider a system for decimating a signal by an integer factor D. Let the input signal sampling frequency be f_x, then the decimated signal frequency will be $f_y = f_x/D$. If $D >> 1$, then we express D as a product of positive integers as

$$D = \prod_{i=1}^{N} D_i$$

Each decimator D_i is implemented and cascaded to get N stages of filtering and decimators as shown in Figure 10.42.



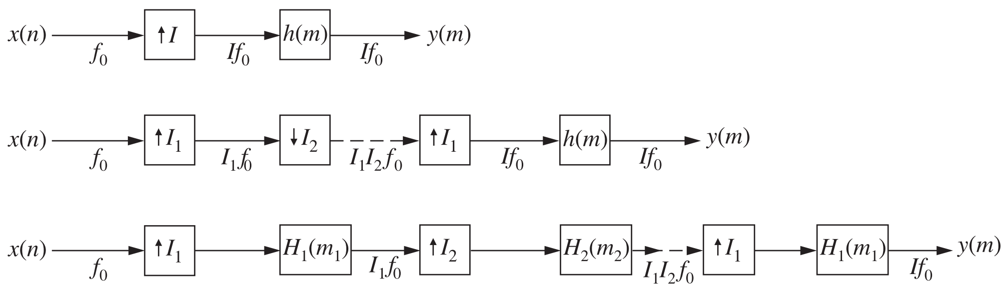
$$D = D_1 D_2 \ldots D_I = \prod_{I=1}^{I} D_i$$

Figure 10.42 Multistage decimator.

Similarly, if the interpolation factor $I >> 1$, then express I as a product of positive integers as:

$$I = \prod_{i=1}^{N} I_i$$

Then each interpolator I_i is implemented and cascaded to get N stages of implementation and filtering as shown in Figure 10.43.



$$I = I_1 I_2 \ldots I_I = \prod_{I=1}^{I} I_i$$

Figure 10.43 Multistage interpolation.

If the sampling rate alteration system is designed as a cascade system, the computational efficiency is improved significantly. The reasons for using multistage structures are as follows:

1. Multistage system requires less computation.
2. Storage space required is less.
3. Filter design problem is simple.
4. Finite word length effects are less.

The demerits of the systems are that proper control structure is required in implementing the system and proper values of I should be chosen.

EXAMPLE 10.12 For the multi-rate system shown in Figure 10.44, develop an expression for the output $y(n)$ as a function of the input $x(n)$.

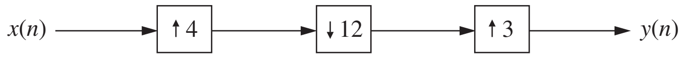


Figure 10.44 Example 10.12.

Solution: In Figure 10.45 the down sampler with D = 12 is split into 2 down samplers with D_1 = 3 and D_2 = 4. The up sampler with I = 4 and the down sampler with D = 3 are interchanged and finally the up sampler with I = 4 and down sampler with D = 4 cancel, and we will be left with a down sampler with D = 3 and an up sampler with I = 3.

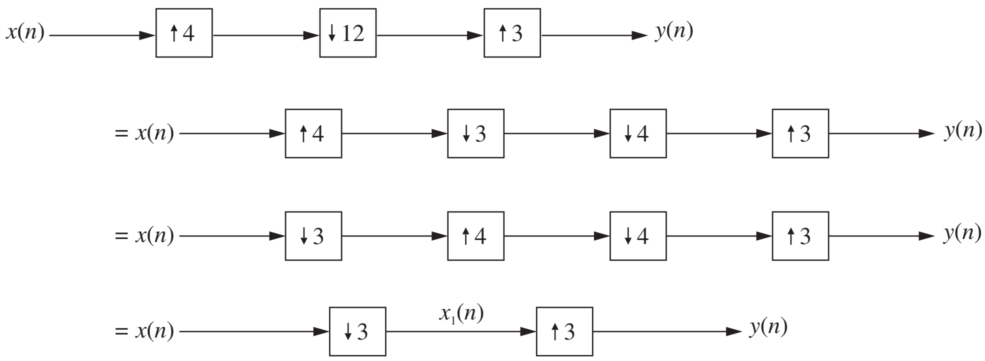


FIGURE 10.45 Simplification of Figure 10.45.

From Figure 10.45, we have

$$x_1(n) = x(3n)$$

and
$$y(n) = \begin{cases} x_1\left(\dfrac{n}{3}\right), & \text{for } n = 3k \\ 0, & \text{otherwise} \end{cases}$$

i.e.
$$y(n) = \begin{cases} x(n), & \text{for } n = 3k \\ 0, & \text{otherwise} \end{cases}$$

EXAMPLE 10.13 A multi-rate system is shown in Figure 10.46. Find the relation between $x(n)$ and $y(n)$.

Solution: The given system is shown in Figure 10.47.

In Figure 10.47 after down sampling $x(n)$, we get

$$v(n) = x(2n)$$

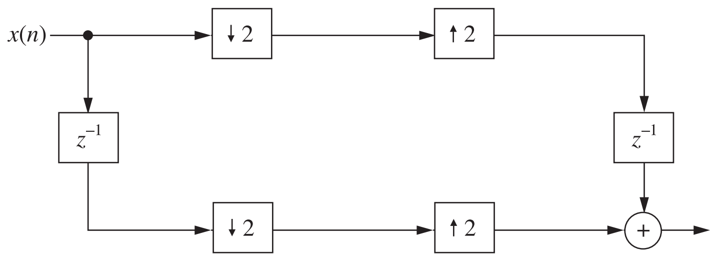


Figure 10.46 Example 10.13.

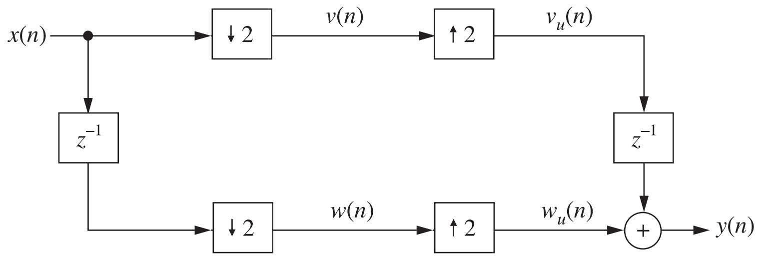


Figure 10.47 Example 10.13.

If we up sample $v(n)$ by 2, we get

$$v_u(n) = v\left(\frac{n}{2}\right) = \begin{cases} x(n), & \text{for } n = 0, \pm 2, \pm 4, \ldots \\ 0, & \text{otherwise} \end{cases}$$

If we delay $x(n)$ and down sample, we get

$$w(n) = x(2n - 1)$$

If we up sample $w(n)$, we get

$$w_u(n) = w\left(\frac{n}{2}\right) = \begin{cases} x(n-1), & \text{for } n = 0, \pm 2, \pm 4, \ldots \\ 0, & \text{otherwise} \end{cases}$$

$$y(n) = v_u(n-1) + w_u(n)$$

$$v_u(n) = \{x(0), 0, x(2), 0, x(3), \ldots\}$$

$$v_u(n-1) = \{0, x(0), 0, x(2), 0, x(3), \ldots\}$$

$$w_u(n) = \{x(-1), 0, x(1), 0, x(3), 0, \ldots\}$$

$$w_u(n) + v_u(n-1) = \{x(-1), x(0), x(1), x(2), x(3), \ldots\}$$

$$\therefore \quad y(n) = x(n-1)$$

10.10 EFFICIENT TRANSVERSAL STRUCTURE FOR DECIMATOR

We discussed earlier that a decimator consists of an anti-aliasing filter followed by a down sampler as shown in Figure 10.48.

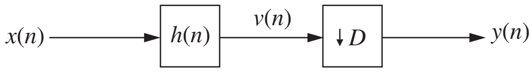


Figure 10.48 A decimator.

Let us assume that the anti-aliasing filter is an FIR filter with N coefficients. The output $v(n)$ of an FIR filter is the convolution of input $x(n)$ and impulse response $h(n)$ and is given by

$$\begin{aligned} v(n) &= x(n) * h(n) \\ &= \sum_{k=0}^{\infty} h(k)\, x(n-k) \end{aligned}$$

The output $v(n)$ is then down sampled to yield

$$\begin{aligned} y(n) &= v(nD) \\ &= \sum_{k=0}^{N-1} h(k)\, x(nD-k) \end{aligned}$$

FIR filters are normally realized with linear phase. Hence impulse response is symmetric given by

$$h(k) = h(N-1-k)$$

With this property, the number of multiplications can be reduced by a factor of two. If N is even,

$$y(n) = \sum_{k=0}^{\frac{N}{2}-1} h(k)\,[x(n-k) + x(N-1-k)]$$

The FIR filter can be realized using direct form structure as shown in Figure 10.49. The direct form realization, shown in Figure 10.49, is very inefficient as it involves the calculation of even the interim values of $v(n)$ which are not used later on.

To avoid unnecessary calculation of the values of $v(m)$, $m \neq nD$ an efficient transversal structure shown in Figure 10.50 is used. Here the multiplications and additions are performed at reduced sampling rate.

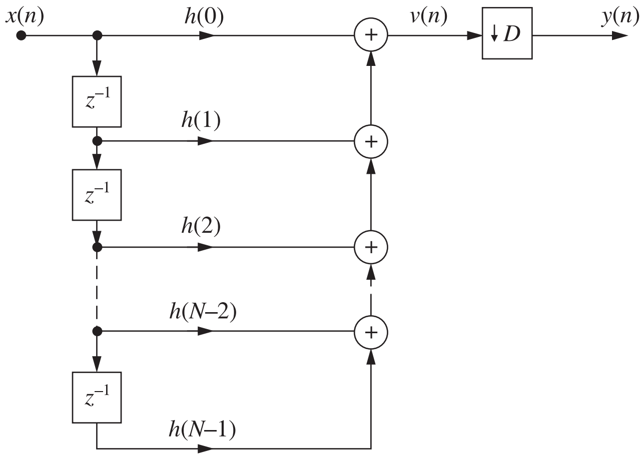


Figure 10.49 Direct form realization of a decimator.

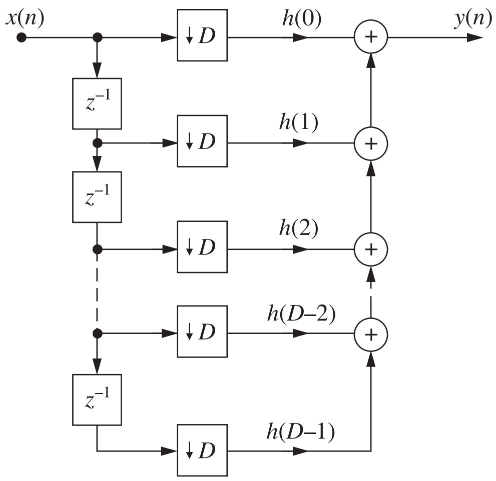


Figure 10.50 Efficient realization for decimator.

10.11 EFFICIENT TRANSVERSAL STRUCTURE FOR INTERPOLATOR

Earlier we discussed that an interpolator consists of an up sampler I followed by anti-imaging filter $h(n)$ (a low-pass filter) as shown in Figure 10.51.

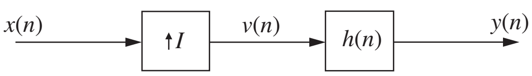


Figure 10.51 An interpolator.

The transposed direct form structure using an FIR filter is shown in Figure 10.52. The up sampling produces an interim signal $v(n)$. The output signal $y(n)$ is obtained by convolving $v(n)$ with the impulse response $h(n)$. If the anti-imaging filter is an FIR low-pass filter with N coefficients, then output

$$y(n) = \sum_{k=0}^{N-1} v(n-k)\, h(k)$$

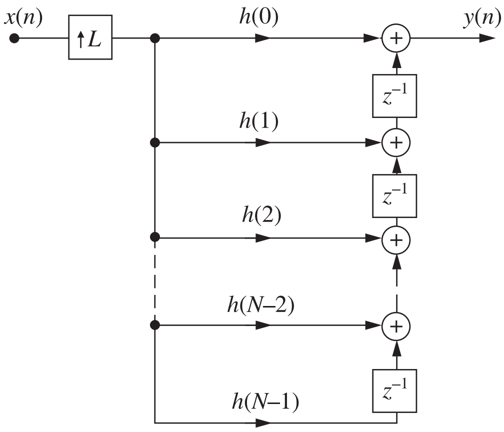


Figure 10.52 Transposed direct form realization for interpolator.

In the process of obtaining $y(n)$ for different values of n, unnecessary calculations are carried out due to zeros inserted because of up sampling. So an efficient transversal structure which avoids these unnecessary computations, is shown in Figure 10.53.

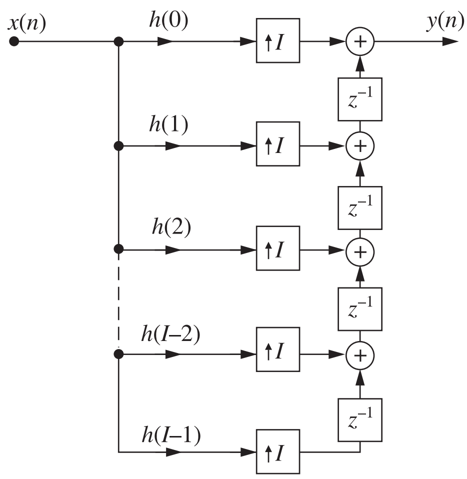


Figure 10.53 Efficient realization of Interpolator.

Duality

From Figures 10.50 and 10.53, we can observe that the structure for interpolator can be obtained by transposing the structure of the decimator. That is the transpose of a decimator is an interpolator, and vice versa. This duality relationship between an interpolator and a decimator is shown in Figure 10.54.

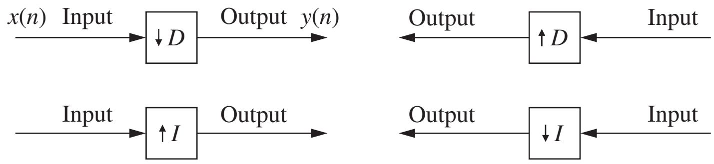


Figure 10.54 Duality between interpolator and decimator.

10.12 IIR STRUCTURES FOR DECIMATORS

The IIR filter is represented by the difference equation

$$y(n) = \sum_{k=1}^{M} a_k y(n-k) + \sum_{k=0}^{N-1} b_k x(n-k)$$

Applying Z-transform,

$$Y(z) = \sum_{k=1}^{M} a_k z^{-k} Y(z) + \sum_{k=0}^{N-1} b_k z^{-k} X(z)$$

i.e.

$$Y(z)\left[1 - \sum_{k=1}^{M} a_k z^{-k}\right] = \sum_{k=0}^{N-1} b_k z^{-k} X(z)$$

So the system function for the above difference equation is given by

$$H(z) = \frac{\sum_{k=0}^{N-1} b_k z^{-k}}{1 - \sum_{k=1}^{M} a_k z^{-k}} = \frac{N(z)}{D(z)}$$

Let $M = N - 1$ so that the numerator and denominator orders are same. Figure 10.55 shows the direct form of the IIR structure for a D to 1 decimator.

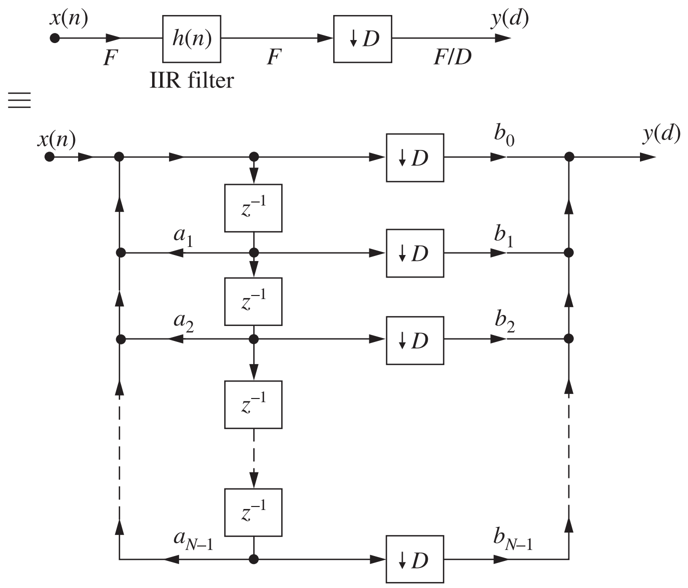


Figure 10.55 The direct form of the IIR structure for a D to 1 decimator.

10.13 FILTER DESIGN FOR FIR DECIMATORS AND INTERPOLATORS

The FIR filter design has already been discussed in Chapter 9. There are various methods for FIR filter design like window method, optimal equiripple linear phase method, half band design, etc. Let us consider the equiripple FIR filter design.

The design equations for calculating the stop band and pass band frequency are discussed below.

Let the highest frequency of the decimated signal or the total bandwidth of the interpolated signal be $\omega_c \le \pi$, then the pass band frequency is given by

$$\omega_p \le \begin{cases} \omega_c / I, & 1 \text{ to } I \text{ interpolator} \\ \omega_c / D, & D \text{ to } 1 \text{ decimator} \\ \min(\omega_c/I, \omega_c/D), & \text{conversion by } I/D \end{cases}$$

The stop band frequency is given by

$$\omega_s = \begin{cases} \pi / I, & 1 \text{ to } I \text{ interpolator} \\ \pi / D, & D \text{ to } 1 \text{ decimator} \\ \min(\pi/I, \pi/D), & \text{conversion by } I/D \end{cases}$$

The assumption is that there is no aliasing in the decimator or imaging in the interpolator. If aliasing is allowed in the decimator or interpolator, then the stop band frequency is given by

$$\omega_s = \begin{cases} (2\pi - \omega_c)/I, & 1 \text{ to } I \text{ interpolator} \\ (2\pi - \omega_c)/D, & D \text{ to } 1 \text{ decimator} \\ \min[(2\pi - \omega_c)/\mathrm{I}, (2\pi - \omega_c)/\mathrm{D})], & \text{conversion by } I/D \end{cases}$$

10.14 FILTER DESIGN FOR IIR INTERPOLATORS AND DECIMATORS

The IIR filter design has already been discussed in Chapter 8. The ideal characteristic for the IIR prototype filter $h(n)$ for a D to 1 decimator, assuming that no constraints are set for the phase is given by

$$H(\omega) = \begin{cases} e^{j\varphi}(\omega'), & |\omega'| \le \pi/D \\ 0, & \text{otherwise} \end{cases}$$

For an interpolator the ideal characteristic becomes,

$$H(\omega) = \begin{cases} Ie^{j\varphi}(\omega'), & |\omega'| \le \pi/I \\ 0, & \text{otherwise} \end{cases}$$

The system function for an IIR filter is given by

$$H(z) = \frac{\sum_{k=0}^{N-1} b_k z^{-k}}{1 - \sum_{k=1}^{M} a_k z^{-k}} = \frac{Y(z)}{X(z)}$$

Represent the decimator polynomial as a polynomial of order R in z^D, where D is the decimation factor. Replace

$$\sum_{k=1}^{M} a_k z^{-k} \text{ by } \sum_{k=1}^{R} c_k z^{-rD}$$

For IIR filter designs, the following approximations are used:

1. The Butterworth approximation
2. The Bessel approximation
3. The Chebyshev approximation
4. The Elliptic approximation

EXAMPLE 10.14 Design one-stage and two-stage interpolators to meet the following specifications. $I = 20$.

(a) Pass band : $0 \le F \le 90$
(b) Transition band : $90 \le F \le 100$

(c) Input sampling rate : 10,000 Hz
(d) Ripple : $\delta_p = 10^{-2}$, $\delta_s = 10^{-3}$

Solution: Given that

Interpolation factor $I = 20$
Pass band frequency $F_p = 90$ Hz
Stop band frequency $F_s = 100$ Hz
Input sampling rate $F_{\text{in}} = 10$ kHz
Pass band ripple $\delta_p = 10^{-2}$
Stop band ripple $\delta_s = 10^{-3}$

Single-stage interpolator

To design an interpolator, initially design a decimator with the same specifications and then transpose the design to get an interpolator structure. Therefore

$$\therefore \qquad D = 20$$

Then, the order of the filter is given by

$$N_1 = \frac{-10\log(\delta_p \delta_s) - 13}{14.6\,\Delta f} + 1$$

where
$$\Delta f = \frac{F_s - F_p}{F_{\text{in}}} = \frac{100-90}{10\text{ k}} = \frac{10}{10\text{ k}} = 0.001$$

$$N_1 = \frac{-10\log(10^{-2}\times 10^{-3}) - 13}{(14.6)\,(0.001)} + 1 = \frac{50-13}{0.0146} + 1 = 2535.25$$

Rounding the order to the next higher integer value, we get

$$N_1 = 2536$$

and the output sampling frequency is given by

$$F_{\text{o}} = \frac{F_{\text{in}}}{D} = \frac{10\text{ k}}{20} = 500\,\text{Hz}$$

Then, the single-stage implementation of decimation by a factor $D = 20$ is shown in Figure 10.56.

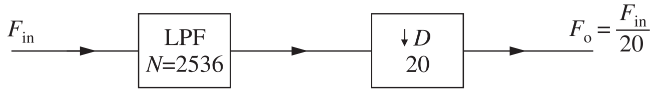


Figure 10.56 Single-stage decimator.

and the transpose of Figure 10.56 gives the single-stage interpolator circuit of $I = 20$ as shown in Figure 10.57.

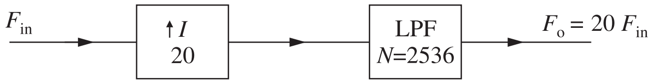


Figure 10.57 Single-stage interpolator.

Two-stage interpolator

Consider two decimation factors as:

$$D_1 = 2,\ D_2 = 10$$

and the respective output sampling frequencies are:

$$F_1 = \frac{10 \text{ kHz}}{2} = 5 \text{ kHz}$$

$$F_2 = \frac{5 \text{ kHz}}{10} = 500 \text{ Hz}$$

First-stage

Pass band : $0 \le F \le 90$ Hz
Transition band : $90 \text{ Hz} \le F \le F_1 - F_{sc}$
: $90 \text{ Hz} \le F \le 4.91$ kHz

$$\Delta f_1 = \frac{4.91 \text{ kHz} - 90}{10 \text{ kHz}} = 0.482$$

Here
$$\delta'_p = \frac{\delta_p}{2} = \frac{10^{-2}}{2} = 0.005$$

Then, the order of the filter is given by

$$N_1 = \frac{-10 \log (\delta'_p \delta_s) - 13}{14.6 \times 0.482} + 1$$

$$= \frac{40.01}{14.6 \times 0.482} + 1$$

$$= 6.685 \approx 7$$

Second-stage

Pass band : $0 \le F \le 90$ Hz
Transition band : $90 \text{ Hz} \le F \le 400$ Hz

$$\Delta f_2 = \frac{400 - 90}{10\text{ k}} = 0.031$$

Here, $$\delta_s' = \frac{\delta_s}{2} = \frac{10^{-3}}{2} = 0.5 \times 10^{-3}$$

Then, the order of the filter is:

$$N_2 = \frac{-10 \log (\delta_p \delta_s') - 13}{14.6\, \Delta f_2} + 1$$

$\therefore$ $$N_2 = \frac{-10 \log (10^{-2} \times 0.5 \times 10^{-3}) - 13}{14.6 \times 0.031} + 1$$

$$= \frac{40.01}{14.6 \times 0.031} + 1 = 89.4 = 90$$

The second stage implementation of decimator by factors $D_1 = 2$ and $D_2 = 10$ in illustrated by Figure 10.58.

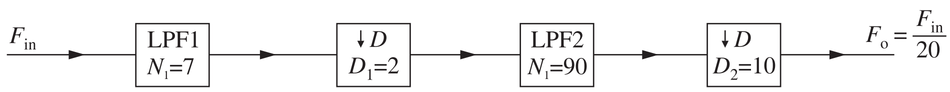


Figure 10.58 Two-stage decimator.

and the transpose of Figure 10.58 gives two-stage interpolator with $I_1 = 10$ and $I_2 = 2$ as shown in Figure 10.59.

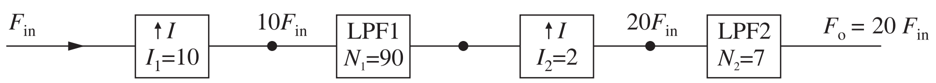


Figure 10.59 Two-stage interpolator.

EXAMPLE 10.15 Implement a two-stage decimator for the following specifications:

Sampling rate of the input signal = 20,000 Hz
Decimation factor $D = 100$
Pass band = 0 to 40 Hz
Transition band = 40 to 50 Hz
Pass band ripple = 0.01
Stop band ripple = 0.002

Solution: The given specifications are as follows:

Pass band edge $F_p = 40$ Hz
Stop band edge $F_s = 50$ Hz

Pass band ripple $\delta_p = 0.01$
Stop band ripple $\delta_s = 0.002$
Sampling rate of the input signal $F_{\text{in}} = 20{,}000$ Hz
Decimation factor $D = 100$.

Single-stage implementation

The single-stage implementation of the filter is shown in Figure 10.60.

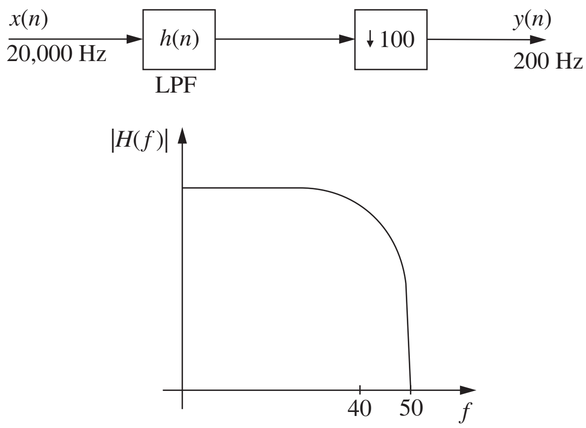


Figure 10.60 Single-stage network.

For an equiripple linear phase FIR filter, the length N is given by

$$N = \frac{-10\log_{10}(\delta_p\delta_s) - 13}{14.6\,\Delta f}$$

where $\Delta f = \dfrac{F_s - F_p}{F_{\text{in}}} = \dfrac{50-40}{20{,}000}$ is the normalized transition band width. Therefore,

$$N = \frac{-10\log_{10}[(0.01)(0.002)] - 13}{14.6\left(\dfrac{50-40}{20{,}000}\right)}$$

$$= 4656$$

In the single-stage implementation, the number of multiplications per second is

$$R_{M,H} = \frac{NF_{\text{in}}}{D} = 4656\,\frac{20{,}000}{100} = 931{,}200$$

Two-stage implementation

The two-stage network for a decimator is shown in Figure 10.61.

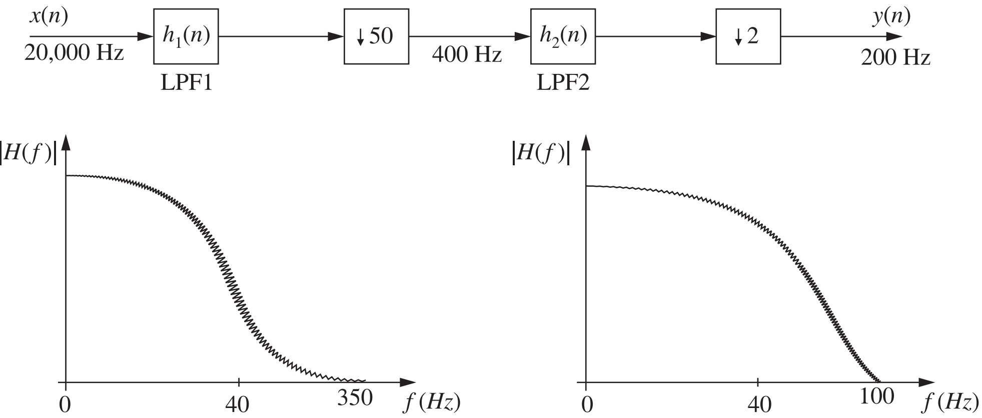


Figure 10.61 Two-stage network.

$H(z)$ can be implemented as a cascade realization in the form of $G(z^{50})F(z)$. The steps in the two-stage realization of the decimator structure is shown in Figure 10.62 and the magnitude response is shown in Figure 10.63.

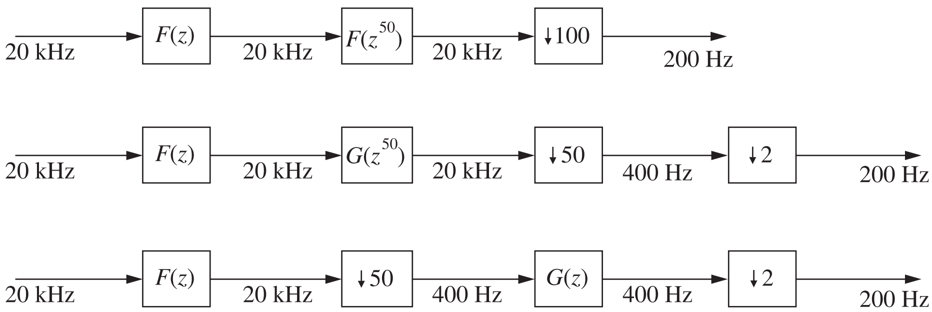


Figure 10.62 Two-stage realization of the decimator structure.

For the cascade realization, the overall ripple is the sum of the pass band ripples of $F(z)$ and $G(z^{50})$. To maintain the stop band ripple atleast as good as $F(z)$ or $G(z^{50})$, δ_s for both can be 0.002. The specifications for the interpolated FIR filter is given by

For $G(z)$, $\delta_p = 0.005, \quad \delta_s = 0.002$

$$\Delta f = \frac{500}{20{,}000}$$

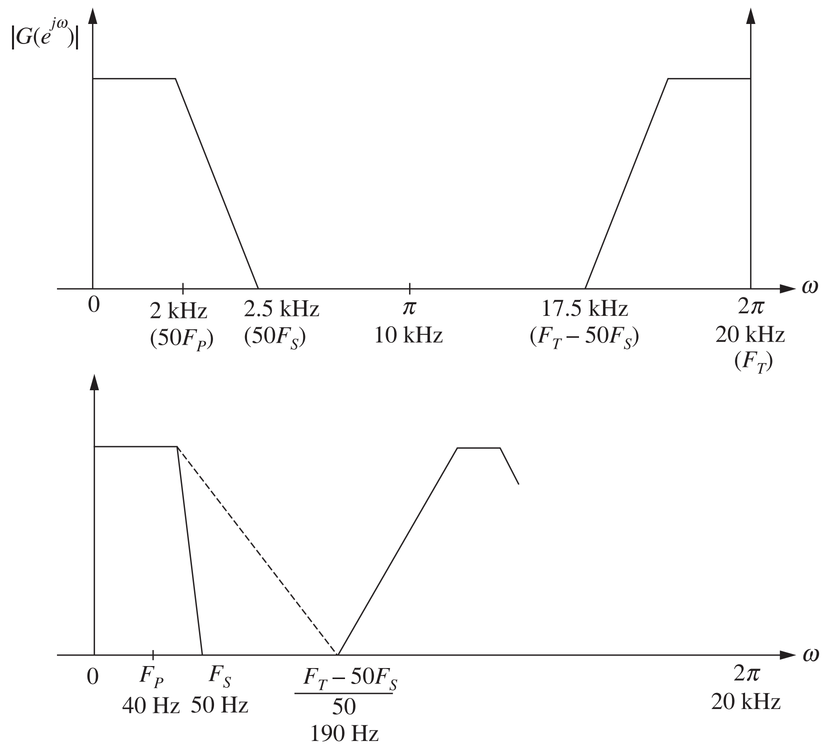


Figure 10.63 Magnitude response for a two-stage decimator.

For $F(z)$, $\delta_p = 0.005, \quad \delta_s = 0.002$

$$\Delta f = \frac{150}{20{,}000}$$

The filter lengths are calculated as follows:

For $G(z)$,

$$N = \frac{-10\log_{10}[(0.005)\,(0.002)] - 13}{14.6\left(\dfrac{2.5k - 2k}{20k}\right)} = 101$$

For $F(z)$,

$$N = \frac{-10\log_{10}[(0.005)\,(0.002)] - 13}{14.6\left(\dfrac{190 - 40}{20k}\right)} = 337$$

The length of the overall filter in cascade is:

$$= 337 + (50 \times 101) + 2 = 5389$$

The filter length in cascade realization has increased, but the number of multiplications per second can be reduced.

$$R_{M,G} = 101 \times \frac{400}{2} = 20{,}200$$

$$R_{M,F} = 337 \times \frac{20{,}000}{50} = 1{,}34{,}800$$

Total number of multiplications per second is given by

$$\begin{aligned} R_{M,G} + R_{M,F} &= 20{,}200 + 1{,}34{,}800 \\ &= 1{,}55{,}000 \end{aligned}$$

EXAMPLE 10.16 Compare the single-stage, two-stage, three-stage and multistage realization of the decimator with the following specification:

Sampling rate of a signal has to be reduced from 10 kHz to 500 Hz. The decimation filter $H(z)$ has the pass band edge (F_p) to be 150 Hz, stop band edge (F_s) to be 180 Hz, pass band ripple (δ_p) to be 0.002 and stop band ripple (δ_s) to be 0.001.

Solution:

Single-stage realization. The length N of an equiripple linear phase FIR filter is given by

$$N = \frac{-10\log_{10}(\delta_p\delta_s) - 13}{14.6\Delta f}$$

where $\Delta f = \dfrac{F_s - F_p}{F_{\text{in}}}$ is the normalized transition bandwidth

$$\therefore \qquad N = \frac{-10\log_{10}\{(0.002)(0.001)\} - 13}{14.6\left(\dfrac{180-150}{10{,}000}\right)}$$

$$= 1004.33 \approx 1004$$

Sampling rate is to be reduced from 10 kHz to 500 Hz. Therefore, decimation factor $D = 20$. For the single-stage implementation of the decimator with a factor of 20, the number of multiplications per second is given by

$$R_{M,H} = \frac{NF_T}{D} = 1004 \times \frac{10{,}000}{20} = 502{,}000$$

Two-stage realization. $H(z)$ can be implemented as a cascade realization in the form of $G(z^{10})$ $H(z)$. The steps in the two-stage realization of the decimator structure are shown in Figure 10.64 and the magnitude response is shown in Figure 10.65.

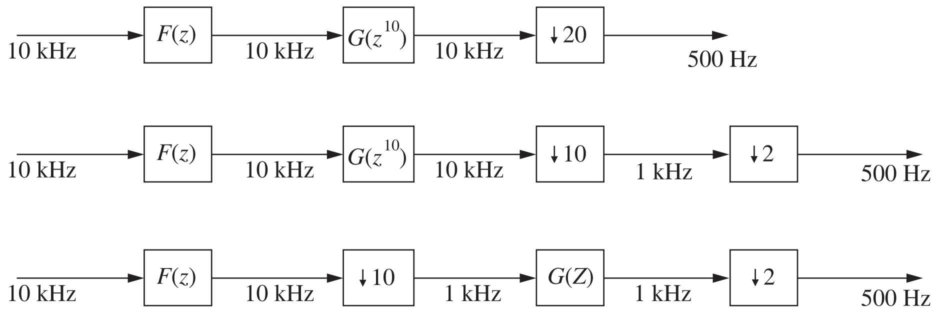


Figure 10.64 Two-stage realization of the decimator structure.

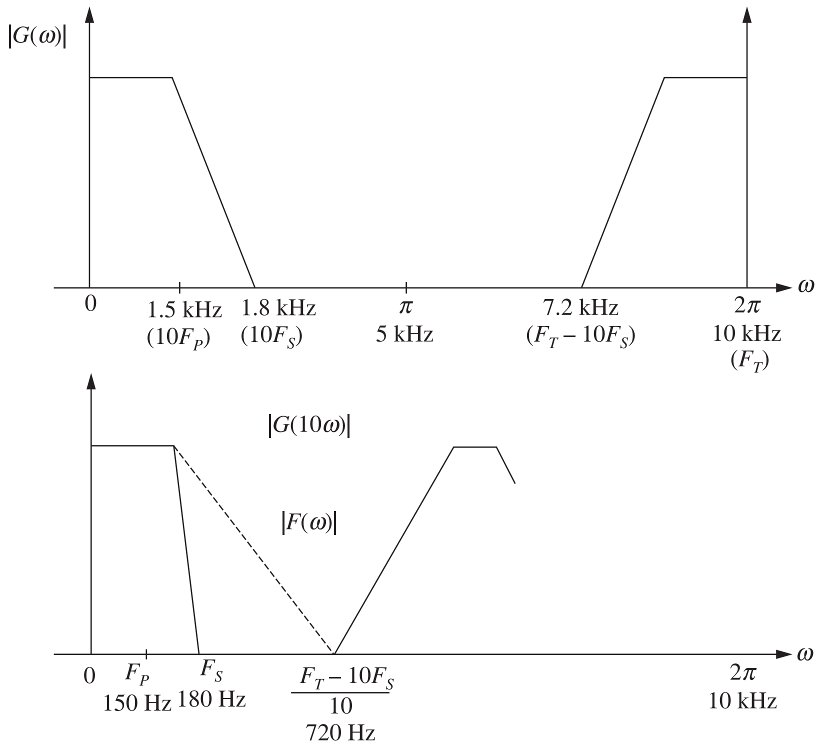


Figure 10.65 Magnitude response of two-stage decimator.

For the cascade realization, the overall ripple is the sum of the pass band ripples of $F(z)$ and $G(z^{10})$. To maintain the stop band ripple atleast as good as $F(z)$ or $G(z^{10})$, δ_s for both can be 0.001. The specifications for the interpolated FIR filters (IFIR) is given by

For $G(z)$,

$$\delta_p = 0.001$$

$$\delta_s = 0.001$$

$$\Delta f = \frac{1.8 \text{ kHz} - 1.5 \text{ kHz}}{F_{\text{in}}} = \frac{300}{10,000}$$

For $F(z)$,

$$\delta_p = 0.001$$

$$\delta_s = 0.001$$

$$\Delta f = \frac{720-150}{10{,}000} = \frac{570}{10{,}000}$$

The filter lengths are calculated as follows:

For $G(z)$,

$$N = \frac{-10 \log [(0.001)\,(0.001)] - 13}{14.6\left(\dfrac{1.8\,\text{k} - 1.5\,\text{k}}{10\,\text{k}}\right)} = 107$$

For $F(z)$,

$$N = \frac{-20 \log [(0.001)\,(0.001)] - 13}{14.6\left(\dfrac{720-150}{10\,\text{k}}\right)} = 56$$

The length of the overall filter in cascade is:

$$56 + (10 \times 107) + 2 = 1128$$

The filter length in cascade realization has increased, but the number of multiplications per second can be reduced.

$$R_{M,G} = 107\ \frac{1000}{2} = 53{,}500$$

$$R_{M,F} = 56\ \frac{10{,}000}{10} = 56{,}000$$

Total number of multiplications per second is:

$$R_{M,G} + R_{M,F} = 53{,}500 + 56{,}000 = 109{,}500$$

Three-stage realization. The decimation filter $F(z)$ can be realized in the cascade form $R(z)\ \delta(z^5)$. The specifications are given as follows:

For $\delta(z)$,

$$\delta_P = 0.0005,\ \delta_s = 0.001$$

$$\Delta f = \frac{570}{10{,}000} \times 5 = 0.285$$

$$N = \frac{-10 \log_{10}[(0.005)\,(0.001)] - 13}{0.285} = 12$$

For $R(z)$,

$$\delta_P = 0.0005,\ \delta_s = 0.001$$

$$\Delta f = \frac{1130}{10{,}000} = 0.113$$

$$N = \frac{-10 \log_{10}[(0.005)(0.001)] - 13}{0.113} = 30$$

The three-stage realization is shown in Figure 10.66.

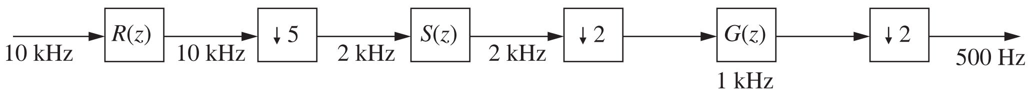


Figure 10.66 Three-stage decimation.

The number of multiplications per second is given by

$$R_{M,S} = 12 \times \frac{2000}{2} = 12{,}000$$

$$R_{M,R} = 30 \times \frac{1000}{5} = 60{,}000$$

The overall number of multiplications per second for a three-stage realization is given by

$$R_{M,G} + R_{M,S} + R_{M,R} = 53{,}500 + 12{,}000 + 60{,}000 = 1{,}25{,}500$$

The number of multiplications per second for a three-stage realization is more than that of a two-stage realization. Hence higher than two-stage realization may not lead to an efficient realization.

10.15 APPLICATIONS OF MULTI-RATE DIGITAL SIGNAL PROCESSING

Here we consider two applications of multi-rate digital signal processing.

1. **Implementation of a narrow band low-pass filter.** A narrow band low-pass filter is characterized by a narrow pass band and a narrow transition band. It requires a very large number of coefficients. Due to high value of N, it is susceptive to finite word length effects. In addition, the number of computations and memory locations required are very high. To overcome these problems multi-rate approach is used in implementing a narrow band low-pass filter. Figure 10.67 shows the cascading stage of a decimator and interpolator. The filters $h_1(n)$ and $h_2(n)$ in the decimator and interpolator are low-pass filters. The input signal is first passed through a low-pass filter. The sampling frequency F of the input sequence $x(n)$ is first reduced by a factor D and then raised by the same factor D and then again low-pass filtering is performed.

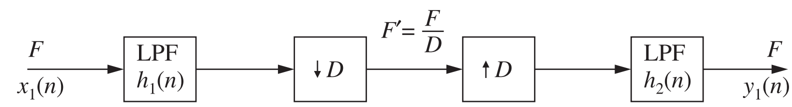


Figure 10.67 A narrow band pass filter.

To meet the desired specifications of a narrow band LPF, the filters $h_1(n)$ and $h_2(n)$ should be identical with the same pass band ripple $\delta_p/2$ and the same stop band ripple δ_s.

2. **Filter banks.** Filter banks are usually classified into two types: (i) Analysis filter bank and (ii) Synthesis filter bank

Analysis filter bank

The D-channel analysis filter bank is shown in Figure 10.68. It consists of D sub-filters. All the sub-filters are equally spaced in frequency and each have the same bandwidth. The spectrum of the input signal lies in the range $0 \le \omega \le \pi$. The filter bank splits the signal into a number of sub-bands each having a bandwidth π/D. The filter $H_0(z)$ is a low-pass filter, $H_1(z)$ to $H_{D-2}(z)$ are band pass and $H_{D-1}(z)$ is high-pass. As the spectrum of the signal is band limited to π/D, the sampling rate can be reduced by a factor D. The down sampling moves all the pass band signals to a base band $0 \le \omega \le \pi/D$.

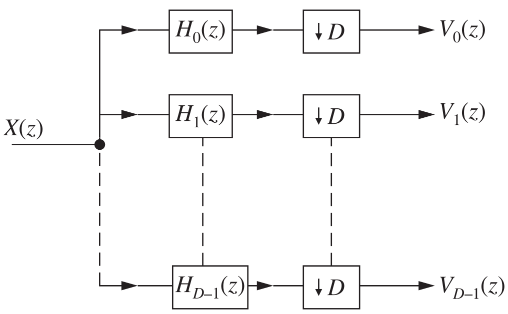


Figure 10.68 Analysis filter bank.

Synthesis filter bank

The D-channel synthesis filter bank shown in Figure 10.69 is dual of the analysis filter bank. In this case, each $V_d(z)$ is fed to an up sampler. The up-sampling process produces the signal $V_d(z^D)$. These signals are applied to filters $G_d(z)$ and finally added to get the output signal $\hat{X}(z)$. The filters $G_0(z)$ to $G_{D-1}(z)$ have the same characteristics as the analysis filters $H_0(z)$ to $H_{D-1}(z)$.

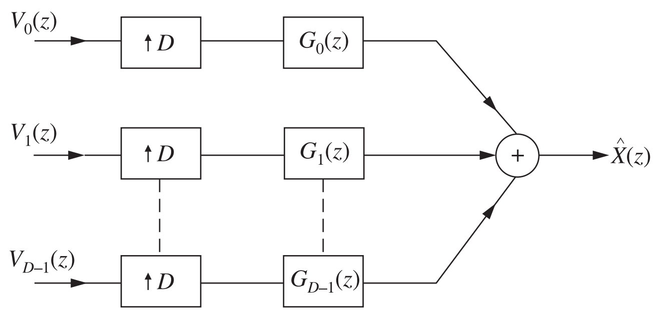


Figure 10.69 Synthesis filter bank.

Sub-band coding filter bank

By combining the analysis filter bank of Figure 10.68 and the synthesis filter bank of Figure 10.69, we can obtain a D-channel sub-band coding filter bank shown in Figure 10.70.

The analysis filter bank splits the broad band input signal $x(n)$ into D non-overlapping frequency band signals $X_0(z)$, $X_1(z)$, ..., $X_{D-1}(z)$ of equal bandwidth. These outputs are coded and transmitted. The synthesis filter bank is used to reconstruct output signal $\hat{X}(z)$ which should approximate the original signal. Sub-band coding is very much used in speech signal processing.

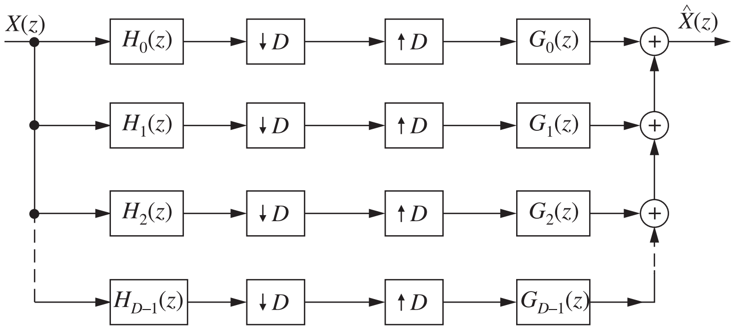


Figure 10.70 Sub-band coding filter bank.

SHORT QUESTIONS WITH ANSWERS

1. What are single-rate systems?

Ans. The systems that use single sampling rate from A/D convertor to D/A convertor are known as single-rate systems.

2. What are multi-rate systems?

Ans. The discrete-time systems that process data at more than one sampling rate are known as multi-rate systems.

3. Where is multi-rate digital signal processing required?

Ans. Multi-rate digital signal processing is required in digital systems where more than one sampling rate is required. For example in digital audio, the different sampling rates used are 32 kHz for broadcasting, 44.1 kHz for compact disc and 48 kHz for audio tape.

4. What are the advantages of multi-rate signal processing?

Ans. The various advantages of multi-rate signal processing are as follows:

(i) Computational requirements are less.
(ii) Storage for filter coefficients is less.
(iii) Finite arithmetic effects are less.
(iv) Filter order required in multi-rate applications is low.
(v) Sensitivity to filter coefficient lengths is less.

5. Name the areas in which multi-rate signal processing is used.

Ans. The various areas in which multi-rate signal processing is used are as follows:

(i) Speech and audio processing systems
(ii) Antenna systems
(iii) Communication systems
(iv) Radar systems

6. Where is multi-rate signal processing used?

Ans. In digital transmission systems like teletype, facsimile, low bit rate speech where data has to be handled at different rates multi-rate signal processing is used. Multirate signal processing also finds applications in

(i) Design of phase shifters
(ii) Interphasing of digital systems with different sampling rates
(iii) Implementation of narrow band low-pass filters (NB-LPF)
(iv) Implementation of digital filter banks
(v) Sub-band coding or speech signals
(vi) Quadratic mirror filters
(vii) Transmultiplexers
(viii) Oversampling A/D and D/A converters

7. State sampling theorem.

Ans. Sampling theorem states that a band limited signal $x(t)$ having a finite energy, which has no spectral components higher than f_h hertz can completely be described and reconstructed from its samples taken at a rate of $\geq 2f_h$ samples per second.

The sampling rate of $2f_h$ samples per second is called the Nyquist rate and its reciprocal $1/2f_h$ is called the Nyquist period.

8. What are the two basic operations in multi-rate signal processing?

Ans. The two basic operations in multi-rate signal processing are decimation and interpolation. Decimation reduces the sampling rate, whereas interpolation increases the sampling rate.

9. How can different sampling rates be obtained?

Ans. Different sampling rates can be obtained using an up sampler and a down-sampler. The basic operations in multi-rate processing to achieve this are decimation and interpolation. Decimation is used for reducing the sampling rate and interpolation is used for increasing the sampling rate.

10. Define down sampling.

Ans. Down sampling a sequence $x(n)$ by a factor D is the process of picking every Dth sample of $x(n)$ and discarding the rest.

11. Define up sampling.

Ans. Up sampling a sequence $x(n)$ by a factor I is the process of inserting $I - 1$ zeros between two consecutive samples.

12. What is decimation?

Ans. Decimation of the sampling rate is the process of reducing the sampling rate by an integer factor D. It is also called down sampling by factor D.

13. What is interpolation?
Ans. Interpolation is the process of replacing the zero valued samples inserted by up sampler with approximated values using some type of filtering process, i.e. interpolation is the complete process of up sampling and filtering to remove image spectra.

14. What for decimation is used in multi-rate digital signal processing system?
Ans. Decimation is used for reducing the sampling rate in a multi-rate digital signal processing system.

15. What for interpolation is used in multi-rate digital signal processing system?
Ans. Interpolation is used for increasing the sampling rate in a multi-rate digital signal processing system.

16. How is decimation achieved?
Ans. Decimation is achieved by picking up every Dth sample of the sequence and discarding the rest. Before down sampling, the signal is linearly filtered to avoid aliasing.

17. How is interpolation achieved?
Ans. Interpolation is achieved by inserting $I - 1$ zeros between successive values of the input signal $x(n)$. After up sampling, the interpolated signal is linearly filtered to eliminate the unwanted images of $X(\omega)$.

18. What is sampling rate conversion?
Ans. Sampling rate conversion is the process of converting the sequence $x(n)$, which is got from sampling the continuous time signal with a period of T, to another sequence $y(k)$ obtained from sampling $x(t)$ with a period of T^1.

19. How can a sampling rate conversion by a factor of I/D achieved?
Ans. A sampling rate conversion by a factor I/D can be achieved by first performing interpolation by a factor I and then performing decimation by a factor D.

20. If $x(n) = \{1, -1, 3, 4, 0, 2, 5, 1, 6, 9, ...\}$, what is $y(n) = x(2n)$, $y(n) = x(3n)$?
Ans. $y(n) = x(2n) = \{1, 3, 0, 5, 6, ...\}$
and $y(n) = x(3n) = \{1, 4, 5, 9, ...\}$

21. If $x(n) = \{1, 2, 3, 7, 4, -1, 5, ...\}$, find $y(n) = x(n/2)$, $y(n) = x(n/3)$.

Ans. $$y(n) = x\left(\frac{n}{2}\right) = \{1, 0, 2, 0, 3, 0, 7, 0, 4, 0, -1, 0, 5, 0, ...\}$$

$$y(n) = x\left(\frac{n}{3}\right) = \{1, 0, 0, 2, 0, 0, 3, 0, 0, 7, 0, 0, 4, 0, 0, -1, 0, 0, 5, ...\}$$

22. If the spectrum of a sequence $x(n)$ is $X(\omega)$, then what is the spectrum of a signal down sampled by a factor 2?
Ans. If the spectrum of a sequence $x(n)$ is $X(\omega)$, then the spectrum $Y(\omega)$ of the signal down sampled by a factor 2 is $Y(\omega) = \frac{1}{2}\left[X\left(\frac{\omega}{2}\right) + X\left(\frac{\omega}{2} - \pi\right)\right]$.

23. If the Z-transform of a sequence $x(n)$ is $X(z)$, then what is the Z-transform of the sequence down sampled by a factor of D?
Ans. If the Z-transform of a sequence $x(n)$ is $X(z)$, then the Z-transform of the sequence down sampled by a factor of D is:

$$Y(z) = \frac{1}{D}\sum_{k=0}^{D-1} X(z^{1/D}e^{-j2\pi k/D})$$

24. If the Z-transform of a sequence $x(n)$ is $X(z)$, then what is the Z-transform of a sequence up sampled by a factor I?
Ans. If the Z-transform of a sequence $x(n)$ is $X(z)$, then the Z-transform $Y(z)$ of a sequence up sampled by a factor I is $Y(z) = X(z^I)$.

25. What do you mean by aliasing?
Ans. The overlapping of the spectra at the output of the down sampler due to the lack of band limiting of the signal fed to the down sampler is called aliasing.

26. What do you mean by imaging?
Ans. The phenomenon of getting image spectra in the output of up sampler in addition to the scaled input spectra is called imaging.

27. What do you mean by image spectra?
Ans. Insertion of $I - 1$ zeros between successive values of input signal $x(n)$ results in a signal whose spectrum $X(e^{j\omega I})$ is an I-fold periodic repetition of the input signal spectrum $X(e^{j\omega})$. These additional spectra are called image spectra.

28. What do you mean by an anti-imaging filter?
Ans. The low-pass filter which is used after the up sampler to remove the image spectra is called the anti-imaging filter.

29. What is the need for anti-aliasing filter prior to down sampling?
Ans. The spectra obtained after down sampling a signal by a factor D is the sum of all the uniformly shifted and stretched version of the original spectrum scaled by a factor $1/D$. If the original spectrum is not band limited to π/D, then down sampling will cause aliasing. In order to avoid aliasing, the signal $x(n)$ is to be band limited to $\pm\pi/D$. This can be done by filtering the signal $x(n)$ with a low-pass filter with a cutoff frequency of π/m. This filter is known as anti-aliasing filter.

30. What is the need for anti-imaging filter after up sampling a signal?
Ans. The frequency spectrum of up sampled signal with a factor I contains $I - 1$ additional images of the input spectrum. Since we are not interested in image spectra, a low-pass filter with a cutoff frequency $\omega_c = \pi/D$ can be used after up sampler to remove these images. This filter is known as anti-imaging filter.

31. When can a cascade of a factor of D down sampler and a factor of I up sampler interchangeable with no change in the input and output relation?
Ans. A cascade of a factor of D down sampler and a factor of I up sampler is interchangeable with no change in the input and output relation if and only if I and D are co-prime.

32. When do you go in for multistage implementation for sampling rate conversion?
Ans. For performing sampling rate conversion we go in for multistage implementation when either $D >> 1$ and/or $I >> 1$.

33. How is a narrow band low-pass filter characterized?
Ans. A narrow band low-pass filter is characterized by a narrow pass band and a narrow transition band.

34. Why multi-rate approach is needed for designing a narrow band low-pass filter?
Ans. A narrow band low-pass filter is characterized by narrow pass band and a narrow transition band. It requires a very large number of coefficients. Due to high value of N it is susceptible to finite word length effects. In addition the number of computations and memory locations required are very high. To overcome these problems multi-rate approach is needed.

35. How many types of filter banks are there? What are they?
Ans. There are two types of filter banks. They are analysis filter bank and synthesis filter bank.

36. What is the relation between D-channel synthesis filter bank and D-channel analysis filter bank?
Ans. The relation between D-channel synthesis filter bank and D-channel analysis filter bank is that they are dual of each other.

REVIEW QUESTIONS

1. Describe the interpolation process with a factor of I.

2. Obtain the necessary expression for interpolation process.

3. Describe the decimation process with a factor of D.

4. Obtain the necessary expression for decimation process.

5. Discuss the applications of multi-rate digital signal processing.

6. Explain multi-rate digital signal processing.

7. Describe the decimation process with a factor of D. Obtain the necessary expression. Sketch frequency response. Also discuss aliasing effect.

8. With the help of block diagram explain the sampling rate conversion by a rational factor I/D. Obtain necessary expression.

9. Explain the effect of aliasing in decimation with the frequency spectrum and discuss how the aliasing can be eliminated.

FILL IN THE BLANKS

1. In _______ systems, single sampling rate is used.

2. The systems that process data at more than one sampling rate are called _______ systems.

3. The two basic operations in multi-rate signal processing are _______ and _______.
4. _______ reduces the sampling rate, whereas _______ increases the sampling rate.
5. Down sampling by a factor D means taking every _______ value of the signal.
6. _______ sampling reduces the sampling rate whereas _______ sampling increases the sampling rate.
7. The complete process of _______ and then _______ is referred to as decimation.
8. The filter used to band limit the signal prior to down sampling is called as _______ filter.
9. The sampling rate of a discrete-time signal can be _______ by a factor I by placing $I - 1$ equally spaced zeros between each pair of samples.
10. The up sampler and down sampler are time- _______ systems.
11. The spectrum of up sampler is given be $Y(\omega)$ = _______.
12. The phenomenon of getting image spectra in the output of an up sampler in addition to the scaled input spectra is called _______.
13. The additional spectra introduced at the output of an up sampler is called _______.
14. Interpolation is the complete process of _______ and _______ to remove image spectra.
15. The low-pass filter which is used after the up sampler to remove the image spectra is called the _______ filter.
16. The scaling of discrete-time signals and their addition at the nodes are independent of the _______.
17. A delay of D sample periods before a down sampler is the same as a delay of _______ after the down sampler.
18. A delay of one sample period before up sampling leads to a delay of I _______ after up sampling.
19. A sampling rate conversion by a factor I/D can be achieved by _______ a factor of I interpolater and a factory of D decimator.
20. A cascade of a factor of D down sampler and a factor of I up sampler is interchangeable with no change in the input and output relation if and only if I and D are _______.
21. The _______ of a decimator is an interpolator and vice versa.
22. For performing sampling rate conversion for either $D >> 1$ and/or $I >> 1$ we go in for _______ implementation.
23. Filter banks may be _______ filter banks or _______ filter banks.
24. The D-channel synthesis filter bank is the _______ of D-channel analysis filter bank.
25. In digital audio, the different sampling rates used are _______ kHz for broadcasting, _______ kHz for compact disc and _______ kHz for audio tape.

26. While designing multi-rate systems, effects of _______ for decimation and _______ for interpolation should be avoided.

27. The decimator is also knows as _______ sampler, _______ sampler or _______ sampler.

28. The sampling rate of $2f_h$ samples per second where f_h is the highest frequency component in the signal is called the _______.

29. The reciprocal of the Nyquist rate is called the _______.

30. In sampling rate conversion, first _______ is to be performed and then _______ is to be performed.

OBJECTIVE TYPE QUESTIONS

1. Decimation results in
(a) decrease in sampling rate
(b) increase in sampling rate
(c) no change in sampling rate
(d) random change in sampling rate

2. Interpolation results in
(a) decrease in sampling rate
(b) increase in sampling rate
(c) no change in sampling rate
(d) random change in sampling rate

3. The down-sampled signal is obtained by multiplying the sequence $x(n)$ with
(a) impulse function
(b) unit step function
(c) train of impulses
(d) unit sample function

4. Anti-aliasing filter is to be kept
(a) before down sampler
(b) after down sampler
(c) before up sampler
(d) after up sampler

5. Anti-imaging filter is to be kept
(a) before down sampler
(b) after down sampler
(c) before up sampler
(d) after up sampler

6. Up sampler and down sampler are
(a) time-varying systems
(b) time-invariant systems
(c) unpredictable systems
(c) may be time-varying or time-invariant

7. Up sampling by a factor of I introduces
(a) I zeros between samples
(b) $I - 1$ zeros between samples
(c) no zeros
(d) $I/2$ zeros between samples

8. Down sampling by a factor of D skips
(a) D samples
(b) $D - 1$ samples
(c) no samples
(d) $D/2$ samples

9. Down sampling by a factor of D introduces how many additional images?
(a) D images
(b) $D - 1$ images
(c) no images
(d) $D/2$ images

10. Up sampling by a factor of I introduces how many additional images?
(a) I images (b) $I-1$ images
(c) no images (d) $I/2$ images

11. A delay of D sample periods before a down sampler is the same as a delay of how many sample periods after the down sampler.
(a) D (b) 1 (c) $D/2$ (d) $D-1$

12. A delay of one sample period before up sampling leads to a delay of how many sample periods after up sampling.
(a) I (b) $I-1$ (c) $I/2$ (d) 1

13. Cascading a factor of I interpolator and a factor of D decimator results in a sampling rate conversion by a factor of
(a) I/D (b) ID (c) D/I (d) $1/ID$

14. A cascade of a factor of D down sampler and a factor of I up-sampler is interchangeable with no change in the input and output relation if
(a) D and I are integers (b) D and I are co-prime
(c) D and I are rational (d) D and I are finite

15. If $x(n) = \{1, 2, 3, 4, 5, 6, 7, ...\}$, then $x\left(\dfrac{n}{2}\right) =$
(a) $\{1, 0, 2, 0, 3, 0, 4, 0, 5, 0, 6, 0, ...\}$ (b) $\left\{\dfrac{1}{2}, \dfrac{2}{2}, \dfrac{3}{2}, \dfrac{4}{2}, \dfrac{5}{2}, \dfrac{6}{2}, \dfrac{7}{2}, \ldots\right\}$
(c) $\{1, 3, 5, 7, ...\}$ (d) $\{2, 4, 6, 8, 10, ...\}$

16. If $x(n) = \{1, 2, 3, 4, 5, 6, 7, ...\}$, then $x(2n) =$
(a) $\{2, 4, 6, 8, 10, ...\}$ (b) $\{1, 0, 2, 0, 3, 0, 4, 0, 5, 0, 6, 0, ...\}$
(c) $\{1, 3, 5, 7, ...\}$ (d) $\{1, 0, 0, 2, 0, 0, 3, 0, 0, 4, 0, 0, 5, 0, 0 ...\}$

PROBLEMS

1. Find the decimated and interpolated version of the following signal; $x(n) = (3, 6, 8, 9, -2, -1\}$ with the factors 2, 3, 4.

2. Consider a signal $x(n) = u(n)$.
(i) Obtain a signal with a decimation factor '2'
(ii) Obtain a signal with a interpolation factor '2'.

3. Consider a signal $x(n) = \sin \pi n u(n)$.
(a) Obtain a signal with a decimation factor '4'
(b) Obtain a signal with an interpolation factor '4'

4. Consider a ramp sequence and sketch its interpolated and decimated versions with a factor of '2'.

5. Obtain the polyphase decompositions of the IIR digital system having the following transfer functions:

(a) $H(z) = \dfrac{1 - 2z^{-1}}{1 + 3z^{-1}}$ (b) $H(z) = \dfrac{1 + z^{-1} + 2z^{-2}}{1 + 0.8z^{-1} + 0.6z^{-2}}$

6. Obtain the expression for the output $y(n)$ in terms of input $x(n)$ for the multirate system shown in Figure 10.71.

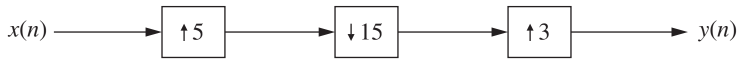


Figure 10.71

7. Consider the signal $x(n) = nu(n)$

(a) Determine the spectrum of the signal
(b) The signal is applied to a decimator that reduces sampling rate by a factor '4'. Determine its output spectrum.
(c) Show that the spectrum in part (b) is simply Fourier transform of $x(3n)$.

8. Consider the signal $x(n) = a^n u(n)$, $|a| < 1$.
(a) Determine the spectrum of the signal
(b) The signal is applied to an interpolator that increases sampling rate by a factor of '3'. Determine its output spectrum.
(c) Show that the spectrum in part (b) is simply Fourier transform of $x(n/3)$.

9. Design a two-stage decimator for the following specifications:

$D = 100$, pass band; $0 \le F \le 50$, transition band; $50 \le F \le 55$

Input sampling; 10,000 Hz; Ripple; $\delta_1 = 10^{-1}$, $\delta_2 = 10^{-3}$.

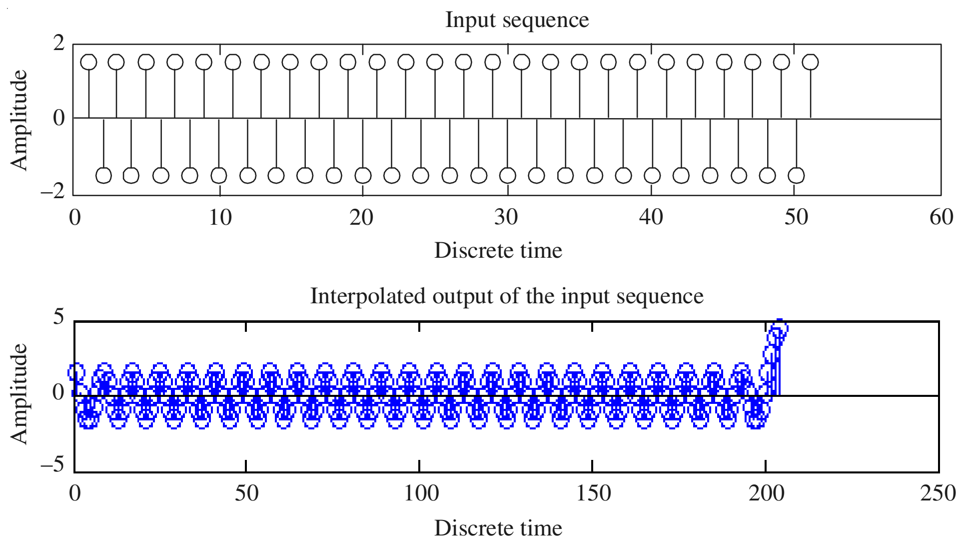
MATLAB PROGRAMS

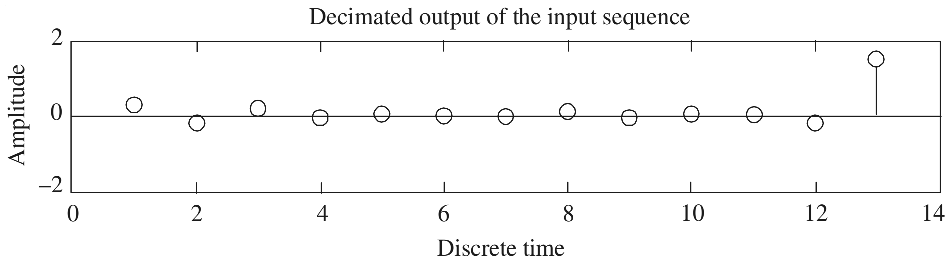
Program 10.1

% Interpolation and Decimation operations on the given signal

```
clc; clear all; close all;
t=0:0.01:0.5;
x=1.5*cos(2*pi*50*t);
y=interp(x,4);
y1=decimate(x,4);
subplot(3,1,1),stem(x);
xlabel('Discrete time')
ylabel('Amplitude')
title('Input sequence')
subplot(3,1,2),stem(y);
xlabel('Discrete time')
ylabel('Amplitude')
title('Interpolated output of the input sequence')
subplot(3,1,3),stem(y1);
xlabel('Discrete time')
ylabel('Amplitude')
title('Decimated output of the input sequence')
```

Output:



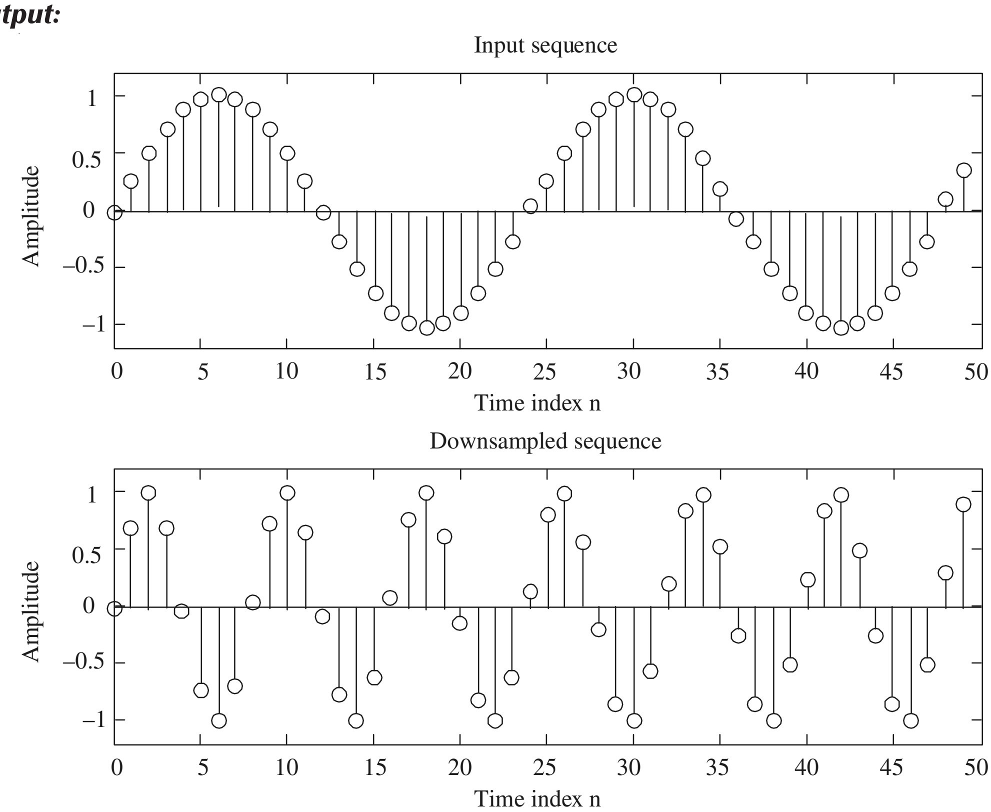


Program 10.2

% Down Sampling by an integer factor

```
clc; clear all; close all;
n = 0: 49;
m = 0: 50*3 – 1;
x = sin(2*pi*0.042*m);
y = x([1 : 3 : length(x)]);
subplot(2,1,1),stem(n, x(1:50));
axis([0 50 –1.2 1.2]);
xlabel('Time index n');
ylabel('Amplitude');
title('Input Sequence');
subplot(2,1,2),stem(n, y);
axis([0 50 –1.2 1.2]);
xlabel('Time index n');
ylabel('Amplitude');
title('Downsampled Sequence');
```

Output:

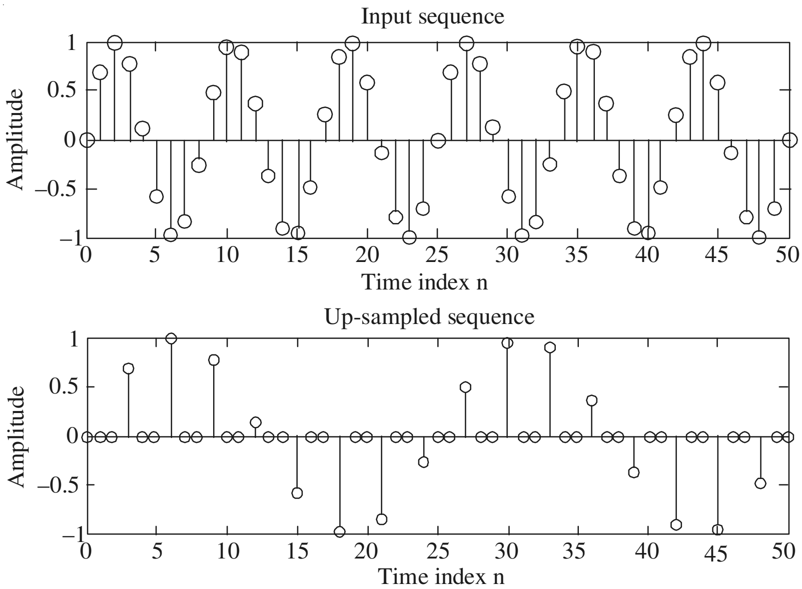


Program 10.3

% Up-Sampling by an integer factor

```
clc; clear all; close all;
n = 0:50;
x = sin(2*pi*0.12*n);
y = zeros(1, 3*length(x));
y([1: 3: length(y)]) = x;
subplot(2,1,1),stem(n,x);
xlabel('Time index n');
ylabel('Amplitude');
title('Input Sequence');
subplot(2,1,2),stem(n,y(1:length(x)));
xlabel('Time index n');
ylabel('Amplitude');
title('Up-sampled sequence ');
```

Output:

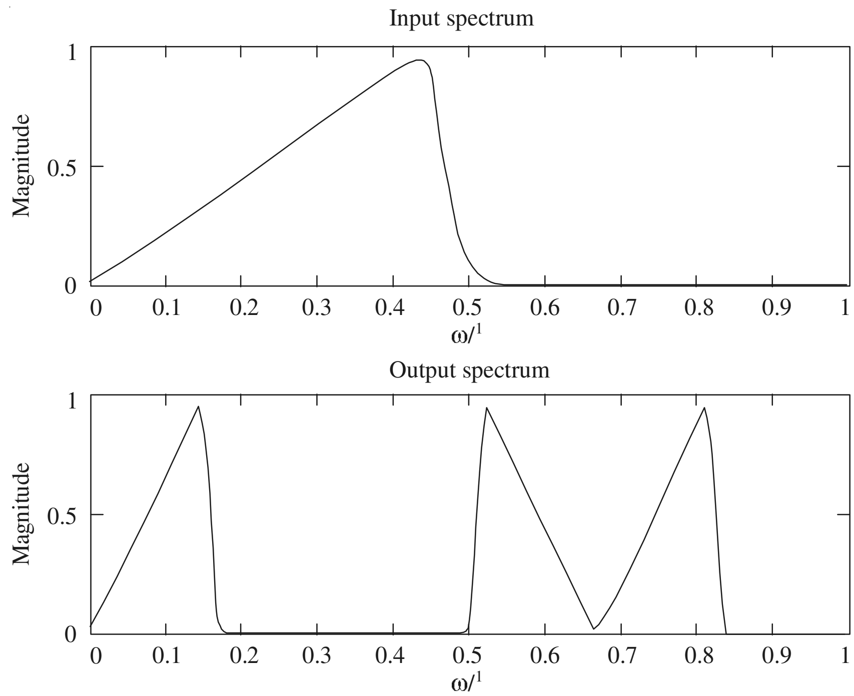


Program 10.4

```
% Effect of Up-sampling in the Frequency Domain
% Use FIR2 to create a band limited input sequence
clc; clear all; close all;
freq = [0 0.45 0.5 1];
mag = [0 1 0 0];
x = fir2(99, freq, mag);
% Evaluate and plot the input spectrum
[Xz, w] = freqz(x, 1, 512);
subplot(2,1,1),plot(w/pi, abs(Xz));
xlabel('\omega/'')
ylabel('Magnitude')
title('Input Spectrum')
% Generate the up-sampled sequence
L = 3 ;%Type in the up-sampling factor
y = zeros(1, L*length(x));
y([1: L: length(y)]) = x;
```

```
% Evaluate and plot the output spectrum
[Yz, w] = freqz(y, 1, 512);
subplot(2,1,2),plot(w/pi, abs(Yz));
xlabel('\omega/'')
ylabel('Magnitude')
title('Output Spectrum')
```

Output:

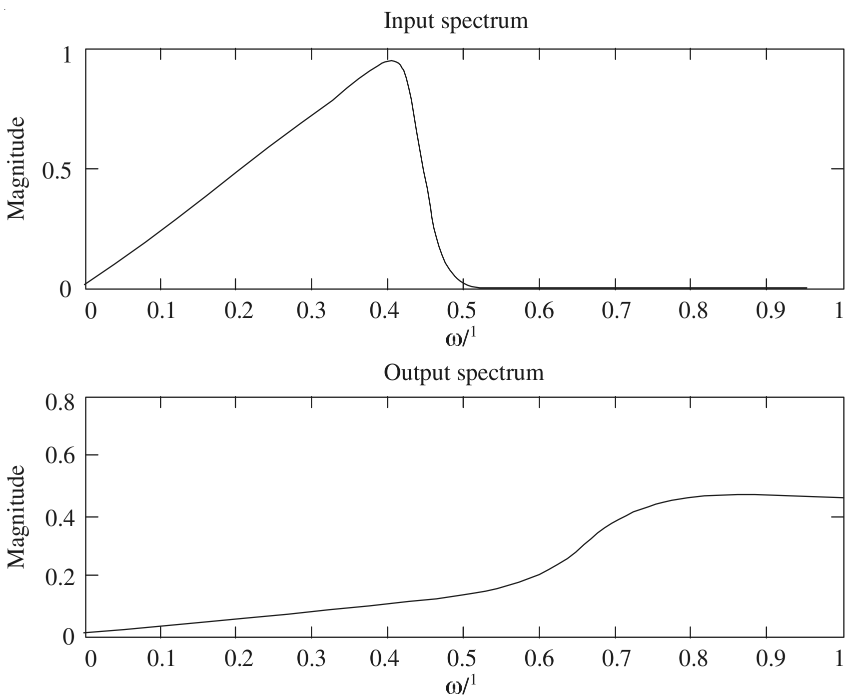


Program 10.5

```
% Effect of Down-sampling in the Frequency Domain
% Use FIR2 to create a bandlimited input sequence
clc; clear all; close all;
freq = [0 0.42 0.48 1];
mag = [0 1 0 0];
x = fir2(101, freq, mag);
% Evaluate and plot the input spectrum
```

```
[Xz, w] = freqz(x, 1, 512);
subplot(2,1,1),plot(w/pi, abs(Xz))
xlabel('\omega/'')
ylabel('Magnitude')
title('Input Spectrum')
% Generate the down-sampled sequence
M = 3 ;%Type in the down-sampling factor
y = x([1: M: length(x)]);
% Evaluate and plot the output spectrum
[Yz, w] = freqz(y, 1, 512);
subplot(2,1,2),plot(w/pi, abs(Yz))
xlabel('\omega/'')
ylabel('Magnitude')
title('Output Spectrum')
```

Output:

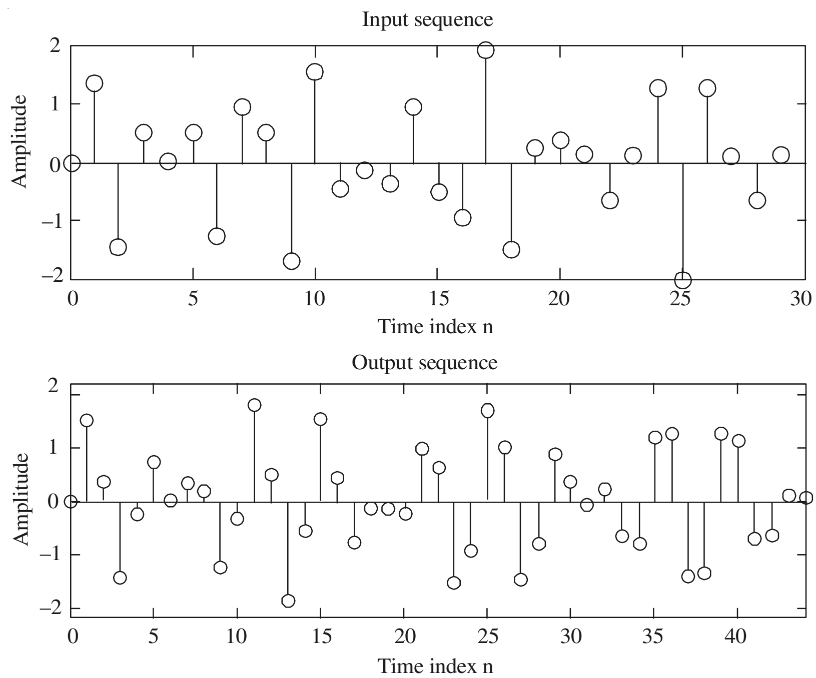


Program 10.6

% Sampling Rate alteration by a ratio of two Integers

```
clc; clear all; close all;
L = 3; %Up-sampling factor
M = 2; %Down-sampling factor
n = 0:29;
x = sin(2*pi*0.43*n) + sin(2*pi*0.31*n);
y = resample(x,L,M);
subplot(2,1,1),stem(n,x(1:30));
xlabel('Time index n');
ylabel('Amplitude');
title('Input Sequence');
m = 0:(30*L/M)–1;
subplot(2,1,2),stem(m,y(1:30*L/M));
axis([0 (30*L/M)–1 –2.2 2.2]);
xlabel('Time index n');
ylabel('Amplitude');
title('Output Sequence');
```

Output:



11

Introduction to DSP Processors

11.1 INTRODUCTION TO PROGRAMMABLE DSPs

A digital signal processor (DSP) is a specialized microprocessor designed specifically for digital signal processing, generally in real time computing. They contain special architecture and instruction set so as to execute computation-intensive DSP algorithms more efficiently. Some advanced microprocessors may have performances close to that of P-DSPs. However, in terms of low power requirements, cost, real time I/O compatibility and availability of high-speed on-chip memories, the P-DSPs have an advantage over the advanced microprocessors or RISC processors. The programmable DSPs (P-DSPs) can be divided into two broad categories. They are (i) General purpose DSPs and (ii) Special purpose DSPs.

1. **General purpose DSPs:** These are basically high speed microprocessors with architecture and instruction sets optimized for DSP operations. They include fixed point processors such as Texas instruments TMS320C5X, TMS320C54X and Motorala DSP563X and floating point processors such as Texas instruments TMS320C4X, TMS320C67XX, and analog devices ADSP21XXX.
2. **Special purpose DSPs:** This type of processors consist of hardware (i) designed for specific DSP algorithms such as FET, (ii) designed for specific applications such as PCM and filtering. Examples for special purpose DSPs are MT93001, PDSP16515A and UPDSP16256.

The factors that influence the selection of DSP for a given application are architectural features, execution speed, and type of arithmetic and word length. Some of the areas of the applications of P-DSPs are: communication systems, audio signal processing, control and data acquisition, biometric information processing, image/video processing, etc.

11.2 ADVANTAGES OF DSP PROCESSORS OVER CONVENTIONAL MICROPROCESSORS

1. The architecture of DSP processors supports fast processing of arrays.
2. The DSP processors require single clock cycle to execute instructions.
3. The DSP processors support parallel execution of instructions.
4. The DSP processors have separate data and program memories.
5. The DSP processors support simultaneous fetching of multiple operands.
6. The DSP processors have three separate computational units. Arithmetic logic unit, multiplier and accumulator, and shifter.
7. The DSP processors consist of powerful interrupt structure and timers.
8. The DSP processors have multiprocessing ability.
9. The DSP processors have on chip program memory and data memory.

11.3 MULTIPLIER AND MULTIPLIER ACCUMULATOR (MAC)

Array multiplication is one of the most common operations required in digital signal processing applications. An example is: convolution and correlation which require array multiplication operation. Implementation of the convolver with single multiplier/adder is shown in Figure 11.1. One of the important requirements of these array multipliers is that they have to process the signals in real time. The array multiplication should be completed before the next sample of the input signal arrives at the input to the array. This requires the multiplication as well as accumulation to be carried out using hardware elements. There are two approaches to solve this problem.

1. A dedicated MAC unit may be implemented in hardware, which integrates multiplier and accumulator in a single hardware unit. Example for this approach is Motorola DSP5600X.
2. Have separate multiplier and accumulator. Example for this approach, is TIDSP320C5X. In this approach, the output of the multiplier is stored into the product register and the content of the product register is added to accumulator register in the central ALU.

In both the approaches, the MAC operation can be completed in one clock cycle. The presence of H/W multipliers and or multiplier accumulator is one of the mandatory requirements of a P-DSP.

In Figure 11.1, the array corresponding to the present and the past $M - 1$ samples of the input is given by

$$x_n = \{x_n x_{n-1} x_{n-2} \ldots x_{n-M+3} x_{n-M+2} x_{n-M+1}\}$$

and the array corresponding to the impulse response of the sequence is given by

$$h = \{h_0 h_1 h_2 \ldots h_{M-3} h_{M-2} h_{M-1}\}$$

The output at the nth sampling instant, y_n is obtained by multiplying the array x_n with the array h. To obtain y_{n+1}, the input signal array x_{n+1} is multiplied with the array h. The vector

x_{n+1} is obtained by shifting the array x_n towards right so that the $(n + 1)$th sample of the input data x_{n+1} becomes the first element and all the elements of x_n are shifted right by 1 position so that the ith element of x_n becomes $(i + 1)$th element of x_{n+i}. The content of the product register is added to the accumulator before the new product is stored. Further, the content of 'dma' is copied to the next location whose address is 'dma + 1'.

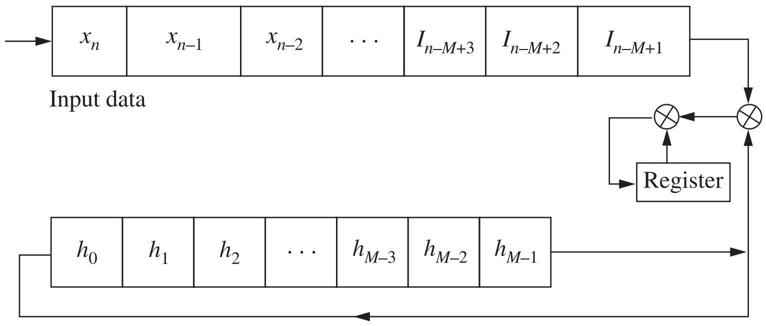


Figure 11.1 Implementation of convolver with single multiplier/adder.

11.4 MODIFIED BUS STRUCTURES AND MEMORY ACCESS SCHEMES IN P-DSPs

The MACD instruction, that is, the MAC operation with data move requires four memory accesses per instruction cycle. The four memory accesses/clock period required for the MACD instructions are as follows:

1. Fetch the MACD instruction from the program memory.
2. Fetch one of the operands from the program memory.
3. Fetch the second operand from the data memory.
4. Write the content of the data memory with address 'dma' into the location with the address 'dma + 1'.

The relatively static impulse response coefficients are stored in the program memory and the samples of the input data are stored in the data memory.

Figure 11.2 shows the Von Neumann architecture. The Von Neumann architecture is most widely used in majority of microprocessors. This architecture consists of 3 buses: The data bus, the address bus and the control bus. In this architecture, the CPU can be either reading an instruction or reading/writing data from/to memory. Both cannot occur at the same time since the instruction and data use the same signal path ways and memory. The execution of each instruction requires four clock cycles because there is a single address bus and there is a single data bus for accessing the program as well as data memory area.

The number of clock cycles required for the memory access can be reduced by using more than one bus for both address and data. Figure 11.3 shows the Harvard architecture. In this, there are two separate buses for the program and data memory. Hence the content of program memory and data memory can be accessed in parallel. The instruction code is fed from the program memory to the control unit and the operand is fed to the processing unit

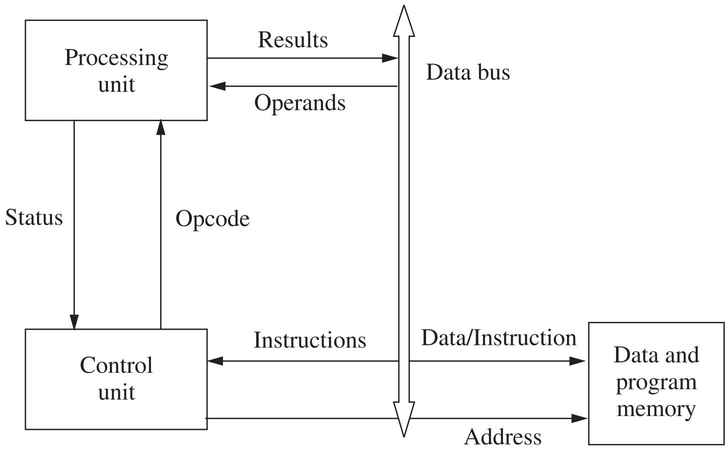


Figure 11.2 Von Neumann architecture.

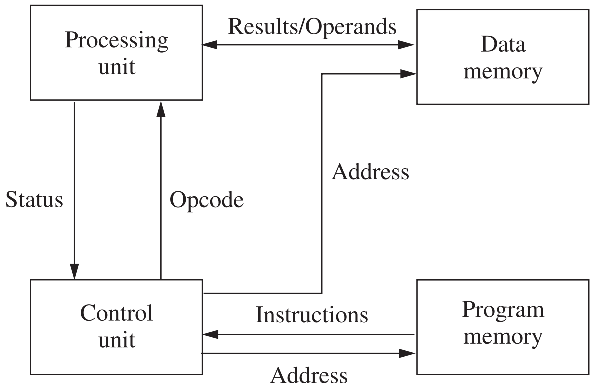


Figure 11.3 Harvard architecture.

from the data memory. Since it has two memories, it is not possible for the CPU to mistakenly write codes into the program memory and, therefore, compute the code while it is executing. However, it is less flexible. It needs two independent memory banks. These two resources are not interchangeable. The processing unit consisting of the registers and processing elements such as MAC units, multiplier, ALU, shifters, etc., are also referred to as data path.

Figure 11.4 shows the modified Harvard architecture. The P-DSPs normally follow this. One set of bus is used to access a memory that has both program and data and another set of bus is used to access data alone. In modified Harvard architecture, data can also be transferred from one memory to another memory. This architecture is used in several P-DSPs.

It may also have multiple bus system for program memory alone or for data memory alone. These multiple bus system increases complexity of CPU, but allow it to access several memory locations simultaneously, thereby increasing the data throughput between memory and CPU.

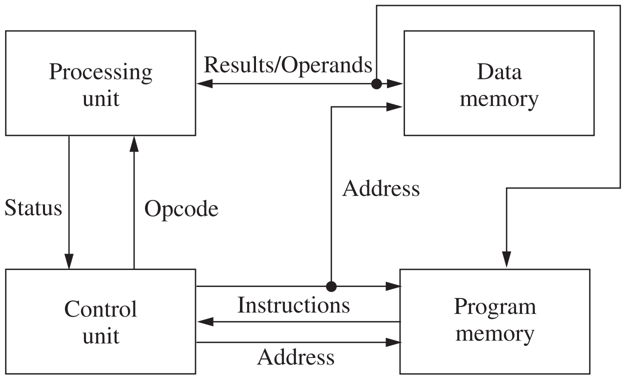


Figure 11.4 Modified Harvard architecture.

By using more number of buses, the number of memory accesses/clock cycle can be increased. Motorola DSP5600X, DSP96002, etc. have three separate buses and so have three memory accesses/clock cycle. TMS320C54X has four address buses and so has four memory accesses per clock cycle.

11.5 MULTIPLE ACCESS MEMORY AND MULTIPORTED MEMORY

The different techniques adopted for increasing the number of memory accesses/instruction cycle are: (i) Multiple access memory and (ii) Multiported memory.

Multiple access memory

The number of memory accesses/clock period can be increased by using a high speed memory that permits more than one memory access/clock period. The memory that permits more than one access/clock period is called multiple access memory. Two memory accesses per clock period can be achieved using the DRAM, the dual access RAM. By using the Harvard architecture, multiple access RAM may be connected to the processing unit of the P-DSP. Four memory accesses/clock period can also be achieved with dual access RAM, when it is connected to a programmable DSP with two independent address and data buses.

Multiported memory

The number of accesses/clock period can also be increased by using multiported memory. The dual port memory shown in Figure 11.5 has two independent data and address buses and hence two memory accesses can be achieved in one clock period. Multiported memories dispense with the need for storing the program and data in two different memory chips in order to permit simultaneous access to both program and data memory.

One of the major limitations of the dual ported memory is the increase in the cost compared to two single port memories of the same total capacity. This is due to the increased number of pins and larger chip area required for the dual port memory. Larger and more expensive package and a larger die size is required for larger number of I/O pins.

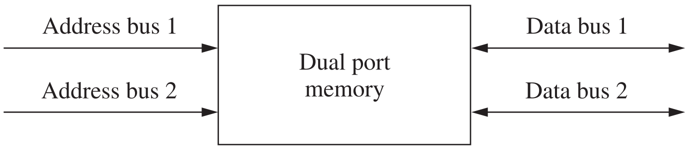


Figure 11.5 Block diagram of a dual ported memory.

Some P-DSPs combine the modified Harvard architecture with the dual ported memories. For example, the Motorola DSP56IXX processors have a single ported program memory and a dual ported data memory. Hence one program memory access and two data memory access can be achieved per clock period.

11.6 VLIW ARCHITECTURE

Very long instruction word (VLIW) architecture is another architecture used for P-DSPs (Example: in TMS320C6X). The VLIW processor consists of architecture that reads a relatively large group of instructions and executes them at the same time. These P-DSPs have a number of processing units (data paths). In other words, they have a number of ALUs, MAC units, shifters etc. The VLIW is accessed from memory and is used to specify the operands and operations to be performed by each of the data paths. The VLIW processing increases the number of instructions that are processed per cycle. Figure 11.6 shows the block diagram of the VLIW architecture. The multiple functional units share a common multiported register file for fetching the operands and storing the results.

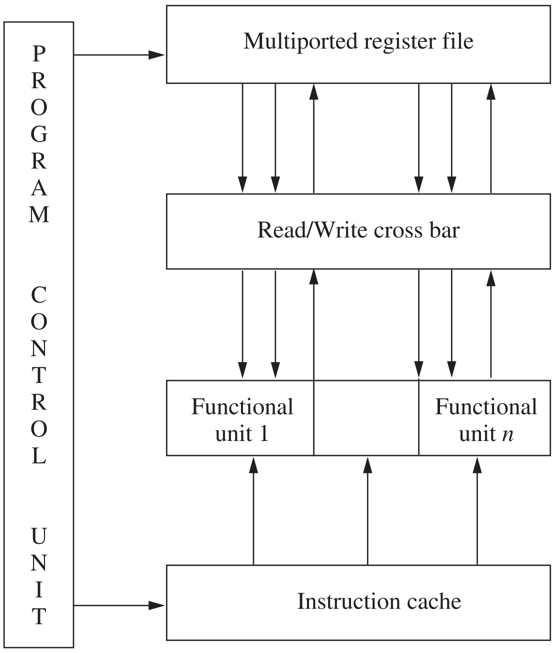


Figure 11.6 VLIW architecture.

The read/write cross bar provides parallel random access by multiple functional units to the multiported register file. Execution of the operations in the functional units is carried out concurrently with the load/store operation of data between a RAM and the register file.

The performance gains that can be achieved with VLIW architecture depends on the degree of parallelism in the algorithm selected for a DSP application and the number of functional units. The throughput will be higher only if the algorithm involves execution of independent operations.

11.7 PIPELINING

Instruction pipelining is a mechanism used for increasing the efficiency of the advanced microprocessors as well as P-DSPs. Pipelining a processor means breaking down its instruction into a series of discrete pipeline stages which can be completed in sequence by specialized hardware.

An instruction cycle starting with the fetching of an instruction and ending with the execution of instruction including the time for storage of the results can be split into a number of microinstructions. Execution of each of the microinstructions is also referred to as one phase of an instruction. For example, an instruction cycle requiring four micro-instructions can be said to be in four phases as follows.

1. *Fetch phase:* In this phase, the instruction is fetched from program memory.
2. *Decode phase:* In this phase, the instruction is decoded.
3. *Memory read phase:* In this phase, the operand required for the execution of the instruction may be read from the data memory.
4. *Execution phase:* In this phase, the execution as well as storage of the results in either one of the register or memory is carried out.

In a modern processor, the above four steps get repeated over and over again till the program is finished executing. Each one of the above microinstructions may be carried out separately by four functional units. If we assume that each of the above phases take equal time for completion, then if there is no pipelining, each of the functional units is busy only for 25% of the time. The functional units can be kept busy almost all the time by using pipelining and processing a number of instructions simultaneously in the CPU. If one clock cycle of the processor corresponds to T, in a period of 12T, only three instructions can be executed in a machine without pipelining, whereas in the same period, nine instructions can be carried out with pipelining. Hence the throughput is increased by a factor of 3 in this case.

The performance and programming simplification is achieved by pipeline operation with the elimination of pipeline interlocks; the control of pipeline is simplified. The bottlenecks in the program fetch, multiply operations and data access are eliminated by increased pipelining.

The number of instructions that are processed simultaneously in the CPU, also referred to as depth of the instruction pipeline, differs in different families of DSPs.

11.8 SPECIAL ADDRESSING MODES IN P-DSPs

In addition to the addressing modes such as direct, indirect and immediate supported by the conventional microprocessors, P-DSPs have special addressing modes that permit single word/instruction format and thereby speed up the execution by making effective use of the instruction pipelining. Further there are also special addressing modes such as cyclic addressing and bit reversed addressing that are specifically tailored for DSP applications. The special addressing modes in P-DSPs are as follows:

1. Short immediate addressing
2. Short direct addressing
3. Memory mapped addressing
4. Indirect addressing
5. Bit reversed addressing
6. Circular addressing

Short immediate addressing

In short immediate addressing mode, the operand is specified as a short constant that forms part of a single word instruction. The length of the short constant depends on the programmable DSP and the instruction type. In TMS320C5XDSPs, an 8 bit constant can be specified as one of the operands in the single word instruction like AND, OR, addition, subtraction, etc.

Short direct addressing

In short direct addressing mode, the lower order address of the operand is specified in the single word instruction. In TITMS320DSPs, the higher order 9 bits of the memory are stored in the data page pointer and only the lower 7 bits are specified as part of the instruction. Using short direct addressing in the Motorola DSP5600X processor, an instruction is specified with a 6 bit address.

Memory-mapped addressing

In this addressing mode, the CPU registers and the I/O registers are accessed as memory locations by storing them in either the starting page or the final page of the memory space. For example, in TMS320C5X, page 0 corresponds to the CPU registers and I/O registers.

In the case of Motorola DSP5600X, the last page of the memory space containing 64 locations is used as the memory map for the CPU and I/O registers.

Indirect addressing

This addressing mode has a number of options in P-DSPs. This mode permits an array of data to be efficiently processed, fetched and stored. The address of the operands can be stored in one of the registers called indirect address registers. In the case of TI processors, the indirect address registers are called auxiliary registers ARs. When the operands fetched by these registers are being executed, these registers can be updated. This is made possible

by having an additional ALU in the CPU core specifically for the indirect address registers of ARs. The increment or decrement of ARs can be performed either in steps of 1 or in steps specified by the content of an offset register. In P-DSPs from Texas instruments, the offset register is known as INDEX register and in the case of analog devices the offset register is known as modifier register. The content of the indirect address registers may also be updated by a constant using bit reversed addressing mode.

Bit reversed addressing

The binary pattern corresponding to a particular decimal number is obtained by writing the natural binary equivalent of the number in the reverse order. Therefore, the least significant bit of the bit reversed number becomes the most significant bit of the natural binary number and vice versa. In this addressing mode, the address is incremented or decremented by the number represented in the bit reversed form.

Circular addressing mode

In real time processing of signals, the input signal is continuously stored in memory. The processed data is stored in another memory space continuously and may be written on to the output device. In this case, the input as well as output program will be simple. However, since the input as well as the output memory space is finite it would be exhausted after processing the input signal for some time, if the data is written into the memory by using linear addressing mode. This problem may be overcome by checking continuously whether the range of either the input or the output memory space is exceeded. In that case, the new data is to be stored starting from the beginning of the particular memory space. Checking this condition is an overhead that can be overcome using the circular addressing mode. In this mode, the memory can be organized as a circular buffer with the beginning memory address and the ending memory address corresponding to this buffer designed by the programmer. In the circular addressing mode, when the address pointer is incremented, the address will be checked with the ending memory address of the circular buffer. If it exceeds that, the address will be made equal to the beginning address of the circular buffer.

11.9 ON-CHIP PERIPHERALS

The P-DSPs have a number of on-chip peripherals that relieve the CPU from routine functions. Further they also help to reduce the chip count on the DSP system based around P-DSP. Some of the on-chip peripherals in P-DSPs and their functions are as follows.

11.9.1 On-chip Timer

Two of the common applications of timers are generation of periodic interrupts to the DSPs and generation of the sampling clocks for A/D converters. The timer can be programmed by the P-DSPs.

11.9.2 Serial Port

The serial port enables the data communication between the P-DSP and an external peripheral such as A/D converter, D/A converter or an RS232C device. These ports normally have input and output buffers so that the P-DSP writes or reads from the serial port in parallel form and the serial port sends and receives data to the peripherals in serial form.

11.9.3 TDM Serial Port

The TDM serial port is a special serial port the P-DSPs have. This port permits a P-DSP to communicate with other devices or P-DSPs by using time division multiplexing (TDM).

11.9.4 Parallel Ports

Parallel ports enable communication between the P-DSP and other devices to be faster compared to the serial communication by using a number of lines in parallel. In addition they have additional lines, which are for strobing or for hand shaking purpose.

11.9.5 Bit I/O Ports

These are additional I/O ports the P-DSPs have that are single bit wide. These port bits may be individually set, reset or read. These bits are normally used for control purposes, but they can also be used for data transfer.

11.9.6 Host Ports

Host port is a special parallel port the P-DSPs have. This enables the P-DSPs to communicate with a microprocessor or a PC, which is called a host. In addition to data communication, the host can generate interrupts and also cause the P-DSP to load a program from ROM to the RAM on reset.

11.9.7 Common Ports

These are parallel ports that are used for interprocess communication between a number of identical P-DSPs in a multiprocessor system.

11.9.8 On-chip A/D and D/A Converters

Some of the P-DSPs targeted towards voice applications such as cellular telephones and tapeless answering machines have A/D and D/A converters inside the P-DSPs.

11.10 P-DSPs WITH RISC AND CISC

P-DSPs may be implemented using either the RISC processor or the CISC processors. The relative advantages of each of these processors is given below.

11.10.1 Advantages of Restricted Instruction Set Computer (RISC) Processors

1. In RISC processor, the control unit uses only around 20% of the chip area because of the reduced number of instructions. Hence the remaining area can be used for incorporating other features.
2. The delayed branch and call instructions are used to improve the speed of the RISC processors.
3. The execution time required for all the instructions of RISC processor is same because all the instructions are of uniform length.
4. The RISC processors have smaller and simpler control units, which have fewer gates.
5. The speed of the RISC processor is high because of smaller control unit and smaller propagation delays.
6. Since a simplified instruction set allows for a pipelined superscalar design, RISC processors often achieve two to four times the performance of CISC processor using comparable semiconductor technology and the same clock rates.
7. Because the instruction set of a RISC processor is simpler, it uses much less chip space than a CISC processor. Extra functional units such as memory management units or floating point arithmetic units can be placed on the same chip.
8. The throughput of the processor can be increased by applying pipelining and parallel processing.
9. Since RISC processors can be designed more quickly, they can take advantage of new technological developments sooner than corresponding CISC design.
10. High level language (HLL) support; The programs can be written in C and C^{++}. It relieves the programmer from learning the instruction set of a P-DSP which in turn increases the throughput of the programmer.

11.10.2 Advantages of Complex Instruction Set Computer (CISC) Processors

1. The CISC processors have a very rich instruction set that even support high level language constructs similar to "if condition true then do", "for" and "while".
2. The CISC processors have instructions specifically required for DSP applications such as MACD, FIRS, etc.
3. The assembly language program of a CISC processor is very short and easy to follow.
4. For RISC architecture compliers are essential. So this becomes costly. For CISC compilers are not required. Hence they are of low cost.
5. New CISC processors are designed to be upward compatible with the older processors. This makes the learning curve steeper.
6. Microprogramming is easier to implement and much less expensive than hardwiring a control unit.

7. Since microprogram instruction sets can be written to match the constructs of high-level languages, the complier need not be very complicated.
8. As each instruction is more capable, fewer instructions could be used to implement a given task. This made more efficient use of the relatively slow main memory.

11.11 ARCHITECTURE OF TMS320C50

The TMS320C5X generation of the Texas instruments TMS320C50 digital signal processor is fabricated with CMOS IC technology. It is a fixed point, 16 bit processor running at 40 MHz. The single instruction execution time is 50 nsec. Its architectural design is based on the combination of advanced Harvard architecture, on-chip peripherals and on-chip memory.

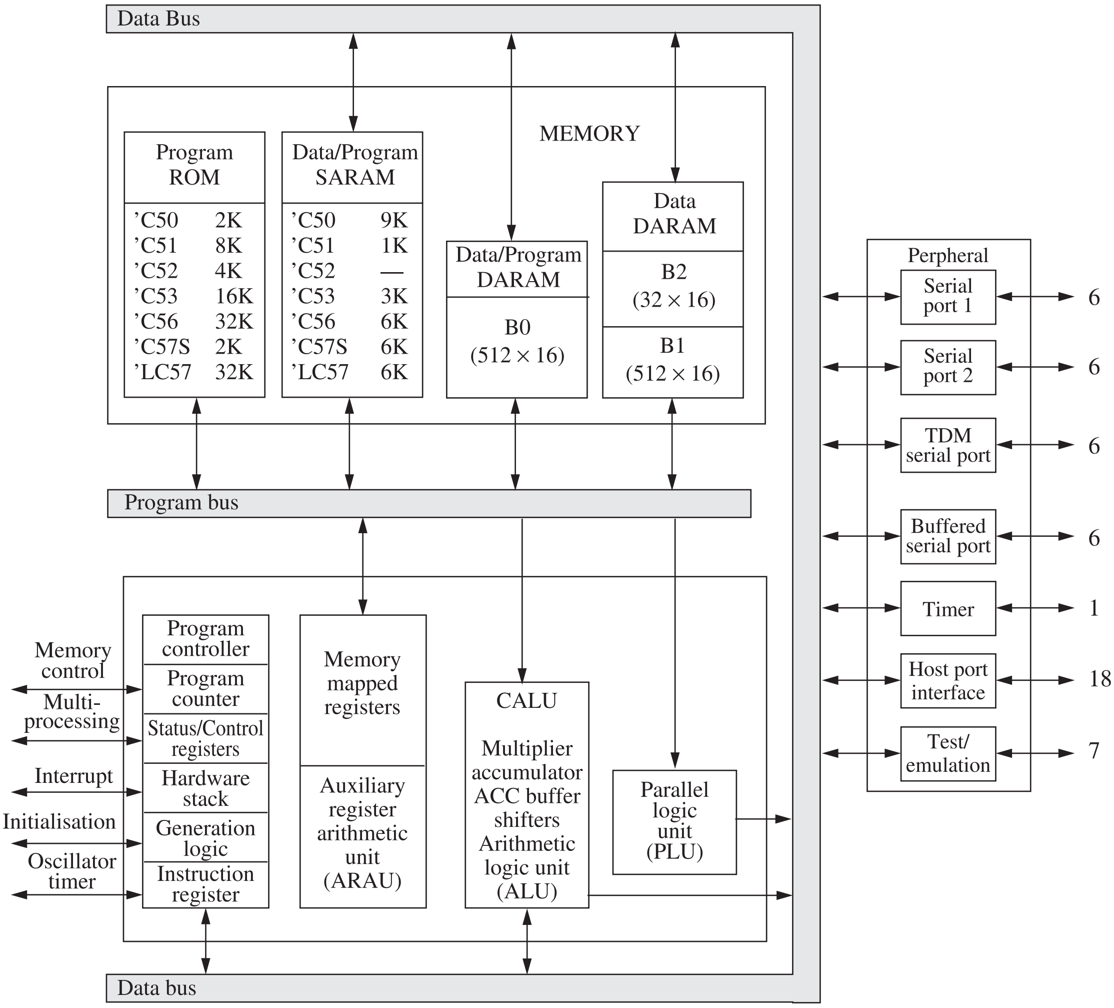


Figure 11.7 Architecture of TMS320C50.

Moreover, the TMS320C50 has a highly specialized instruction set. These features enable the operational flexibility and the device speed, which together with the cost effectiveness make the signal processor as the suitable device for a wide range of applications.

The TMS320C50 has a programmable memory map which can vary for each application. On-chip memory includes 10K words of the RAM and 2K words of the ROM. All C5X DSPs have the same CPU structure. However, they have different on-chip memory configuration and on-chip peripherals.

The functional block diagram of TMS320CX is shown in Figure 11.7. It can be divided into four sub blocks. They are: (1) Bus structure, (2) Central processing unit, (3) On-chip memory and (4) On-chip peripherals.

11.12 BUS STRUCTURE

Separate program and data buses in the advance Harvard architecture of C5X maximize the processing power and provide a high degree of parallelism. Many DSP applications are accomplished using single cycle multiply/accumulate instruction with a data move option. The C5X included the control mechanism to manage interrupts, repeated operations and function calling. The 'C5X' architecture has four buses:

1. Program bus (PB)
2. Program read bus (PRB)
3. Data read bus (DB)
4. Data read address bus (DRB)

The program bus carries the instruction code and immediate operands from program memory to the CPU. The program address bus provides address to program memory space for both read and write.

The data read bus interconnects various elements of the CPU to data memory space. The data read address bus provides the address to access the data memory space.

11.13 CENTRAL PROCESSING UNIT

The CPU consists of the following elements:

1. Central arithmetic logic unit (CALU)
2. Parallel logic unit (PLU)
3. Auxiliary register arithmetic unit (ARAU)
4. Memory mapped registers
5. Program controller

11.13.1 Central Arithmetic Logic Unit (CALU)

The CPU uses the CALU to perform 2's complement arithmetic. It consists of the following:

1. Parallel multiplier (16×16 bit)
2. Accumulator (32 bit)

3. Accumulator buffer (ACCB) (32 bit)
4. Product register (PREG)
5. Shifters
6. Arithmetic logic unit (ALU)

All 32 bit signed/unsigned multiplication operations can be performed in parallel multiplier within one machine cycle. All multiply instructions except the MPYU (multiply unsigned) instruction perform a signed multiply operation in the multiplier. One of the operands to the multiplier is from the 16 bit temporary register O (TREGO) and the second input is from the program bus or data bus. The product register (PREG) holds the product.

The 32 bit ALU along with 16 bit accumulator carries out arithmetic and logic operations executing most of them in one machine cycle. Here the accumulator provides one of the inputs to the ALU, whereas the product register, accumulation buffer, or scaling shifter output provides the second input. The results of operations performed in ALU are stored in accumulator.

The scaling shifter has a 16 bit input connected to the data bus and a 32 bit output connected to the ALU. The scaling shifters produce a left shift of 0 to 16 bits on the input data. A 5 bit register TREGI specifies the number of bits by which the scaling shifter should shift or the shift count is specified by a constant embedded in the instruction word.

11.13.2 Parallel Logic Unit (PLU)

The parallel logic unit (PLU) is another logic unit that executes logic operations on data without affecting the contents of the accumulator. The multiplier bit in a status/control register or any memory location can be directly set, clear, test or toggled by the PLU. After executing the logical operation the PLU writes the result of the operation to the same memory location from which the first operand was fetched.

11.13.3 Auxiliary Register Arithmetic Unit (ARAU)

The C5X consists of a register file containing eight auxiliary registers (ARO-AR7) each of 16 bit length, a 3 bit auxiliary register pointer (ARP) and an unsigned 16 bit ALU. The auxiliary register file is connected to the auxiliary register arithmetic unit.

Auxiliary registers

The eight 16 bit auxiliary registers (ARO-AR7) can be accessed by the CALU and modified by the ARAU or the PLU. The primary function of ARs is to provide 16 bit address for indirect addressing to data space or for temporary data storage. The ARs can also be used as general purpose registers or counters. The contents of the ARs can be stored in the data memory or used as inputs to the CALU.

Index registers (INDX)

The index register (INDX) is a 16 bit register used by the ARAU as a step value to modify the address in the ARs during indirect addressing. The INDX can be added to or subtracted

from the current AR or any AR update cycle. The INDX can be used to increment or decrement the address in steps larger than 1.

Auxiliary register compare register (ARCR)

The ARCR is a 16 bit register used for address bound comparison. It limits blocks of data and supports logical comparisons between the current AR and ARCR in conjunction with the CMPR instruction. The result of this comparison is placed in the TC bit of STI.

Block move address register (BMAR)

The BMAR is a 16 bit register that holds the address value of a source destination space of the block move. It can also hold the address value of an operand in program memory for a multiply accumulate operation.

11.13.4 Memory Mapped Registers

The C5X has 96 registers mapped into page 0 of the data memory space. This memory mapped register space contains various controls and status registers including those for CPU, serial port, timer and software wait generators. Additionally the first sixteen I/O port locations are mapped into this data memory space, allowing them to be accessed either as data memory using single word instruction or as I/O locations with two word instruction.

Instruction registers (IREG)

The 16 bit instruction register (IREG) holds the op code of the instruction being executed.

Interrupt register (IMR, IFR)

The 16 bit interrupt mask register (IMR) individually masks specific interrupts at required time. The 16 bit interrupt flag register indicates the current status of the interrupts.

Status registers

The two 16 bit status registers contain status and control bits for the CPU.

11.13.5 Program Controller

The program controller contains logic circuitry that decodes the operational instructions, manages the CPU pipeline, stores the status of CPU operation and decodes the conditional operations. It consists of the following elements:

(i) Program counter (PC)
(ii) Status and control registers
(iii) Hardware stack

(iv) Program memory addresses generation
(v) Instruction registers

(i) *Program counter (PC)*

It is a 16 bit counter which contains the address of internal or external program memory used to fetch instructions. The PC addresses program memory either on chip or off chip, via the program address bus. Through the PAB an instruction is loaded into the instruction register (IREG). Then the PC is ready to start the next instruction fetch cycle.

(ii) *Status and control registers*

The C5X has four status and control registers. They are circular buffer control register, process mode status register, status registers ST0 and ST1.

(iii) *Hardware stack*

The stack is 16 bit wide and 8 levels deep and is accessible via the push and pop instructions. The stack is used during interrupts and subroutine to save and restore the PC contents.

(iv) *Program memory addresses generation*

It contains the code for application and holds table information and immediate operands. The program memory is accessed only by the program address bus.

(v) *Instruction registers*

The 16 bit instruction registers (IREG) hold the op code of the instruction being executed.

11.14 SOME FLAGS IN THE STATUS REGISTERS

The bit assignment details for flags in status registers ST0 and ST1 are shown in Figure 11.8.

15–13	12	11	10	9	8–0
ARP	OV	OVM	1	INTM	DP

(a)

15–13	12	11	10	9	8–7	6	5	4	3–2	1–0
ARB	CNF	TC	SXM	C	11	HM	1	XF	11	PM

(b)

Figure 11.8 (a) Status Register 0 (ST0) bit assignment, (b) Status Register 1 (ST1) bit assignment.

Significance of the various bits of ST0 and ST1 are as follows:

ARP (Auxiliary register pointer): These bits select the AR to be used in indirect addressing.

OV (Overflow) flag bit: This bit indicates an arithmetic operation overflow in the ALU.

OVM (Overflow mode): This bit enables/disables the accumulator overflow saturation mode in the ALU.

INTM (Interrupt mode): This bit globally masks or enables all interrupts. The INTM bit has no effect on the non-maskable $\overline{RS}$ and $\overline{NMI}$ interrupts.

DP (Data memory page register): These bits specify the address of the current data memory page.

ARB (Auxiliary register buffer): This 3 bit field holds the previous value contained in ARP in ST0.

CNF (On-chip RAM configuration control bit): This 1 bit field enables the on-chip dual access RAM block 0 (DARAM B0) to be addressable in data memory space or programmable memory space.

TC (Test/Control flag bit): This 1 bit flag stores the results of the ALU or PLU test bit operations. The status of the TC bit determines if the conditional branch, call and return instructions are to be executed.

SXM (Sign extension mode bit): This 1 bit field enables/disables sign extension of an arithmetic operation.

C (Carry bit): This 1 bit field indicates whether the CPU stops or continues execution when acknowledging an active $\overline{HOLD}$ signal.

XF (Pin status bit): This 1 bit field determines the level of the external flag (XF) output pin.

PM (Product shift mode bits): This 2 bit field determines the product shifter mode and shift value for PREG output into the ALU.

11.15 ON-CHIP MEMORY

The C5X structure has a total memory address range of 224K words × 16 bits. The memory space is divided into four memory segments.

64K word program memory space: It contains the instruction to be executed.

64K word local data memory space: It stores data used by the instruction.

64K word input/output ports: It interfaces to external memory mapped peripherals.

32K word global data memory space: It can share data with other peripherals within the system.

The large on-chip memory of C5X includes:

1. Program read only memory
2. Data/Program single access RAM (SARAM)
3. Data/Program dual access RAM (DARAM)

11.16 ON-CHIP PERIPHERALS

All C5X DSPs have the same CPU structure; however they have different on-chip peripherals connected to their CPUs. A TMS320C50 digital signal processor contains the following on-chip peripherals.

1. Clock generator
2. Hardware timer
3. Software programmable wait stage generators
4. General purpose I/O pins
5. Parallel I/O ports
6. Serial port interface
7. Buffered serial port
8. TDM serial port
9. Host port interface
10. User-maskable interrupts

11.16.1 Clock Generator

The clock generator consists of an internal oscillator and a phase locked loop (PLL) circuit. The clock generator can be driven internally by a crystal oscillator or driven externally by a clock source. A clock source with a frequency lower than that of the CPU can be used because the PLL circuit can generate an internal CPU clock by multiplying the clock source by a specific factor.

11.16.2 Hardware Timer

The programmable hardware timer with 4 bit prescaler clocks at a rate that is between 1/2 and 1/32 of the machine cycle rate, depending upon the timer's divide down ratio. It acts as a down-counter producing interrupts to CPU at regular intervals. The timer can be stopped, restarted, reset or disabled by specific status bits. Three registers namely the timer counter register (TIM), the timer period register (PRD) and timer control register (TCR) control and operate the timer. The timer counter register gives the current count of the timer. The timer period register defines the period for the timer. The timer control register controls the operations of the timer.

11.16.3 Software-programmable Wait-state Generators

Software-programmable wait-state generators can be interfaced without any external hardware with slower off chip memory and I/O devices. This feature consists of multiple wait-state generating circuits. Each circuit is user-programmable to operate in different wait states for off chip memory accesses.

11.16.4 General Purpose I/O Pins

There are two general purpose I/O pins namely branch control input (BIO) pin and external flag output (XF) pin that are controlled by software. The pin BIO keeps track of the status of external devices and executes conditional branches depending on the requirement. The pin XF communicates with external devices through software. The instruction SETCXF sets this pin to high and CLRCXF resets to 0.

11.16.5 Parallel I/O Ports

There is a total of 64K parallel I/O ports of which 16 are memory mapped in data memory space. All the I/O ports can be addressed through the instructions IN, OUT or through data memory read and write instructions. There is a signal IS which differentiates accessing of memory mapped IO space to that of program and data space. Interfacing with external I/O devices can be done through the I/O ports with minimal off-chip address decoding circuits.

11.16.6 Serial Port Interface

Three different kinds of serial ports are available: a general purpose serial port, a time division multiplexed (TDM) serial port and a buffered serial port (BSP). Each C5X contains atleast one general purpose high-speed synchronous, full duplexed serial port interface. The serial port is capable of operating at upto one fourth the machine cycle rate.

The serial port control (SPC) register, the data receive register (DRR), the data transmit receiver (DXR), the data transmit shift register (XSR) and the data receive shift register (RSR) control and operate the serial port interface.

The serial port control register contains the mode control and status bits of the serial port. The data receive register holds the incoming serial data. The data transmit shift register controls the shifting of the data from the data transmit register to the output pin.

The data receive shift register controls the storing of the data from the input pin to the data receive register.

11.16.7 Buffered Serial Port

The buffered serial port (BSP) is available on the C56 and C57 devices. It is a full duplexed, double-buffered serial port and an auto buffering unit. The BSP provides flexibility on the data stream length. The ABU supports high-speed data transfer and supports internal latencies. Five BSP registers control and operate the BSP.

11.16.8 TDM Serial Port

This is the third type of serial port that is utilized by the devices C50, C51, C53. It is a full duplexed serial port that can be configured by software either for synchronous operation or TDM operation. The TDM serial port is commonly used in multiprocessor applications.

11.16.9 Host Port Interface

The host port is available on the C57S and LC57. It is an 8 bit parallel I/O port that provides an interface to a host processor. Information is exchanged between the DSP and the host processor through on-chip memory that is accessible to both the host processor and the C57.

11.17 USER MASKABLE INTERRUPTS

Four external interrupt lines ($\overline{\text{INT1}}$, $\overline{\text{INT2}}$, $\overline{\text{INT3}}$, $\overline{\text{INT4}}$) and five internal interrupts, a timer interrupt and four serial port interrupts are user maskable.

When an interrupt service routine (ISR) is executed, the contents of the program counter are saved on an 8 level hardware stack and the contents of 11 specific CPU registers are saved in one deep stack. When a return from interrupt instruction is executed, the CPU contents registers are restored.

11.17.1 Various Interrupt Types Supported by TMS320C5X Processor

The TMS320C5X devices have four external, maskable user interrupts ($\overline{\text{INT4}}$ – $\overline{\text{INT1}}$) that external devices can use to interrupt the processor; there is one external nonmaskable interrupt ($\overline{\text{NMI}}$). The internal interrupts are generated by the serial port (RINT and XINT),

TABLE 11.1 Interrupt locations and priorities

Interrupt type	*Location*		*Priority*	*Function*
	Dec	*Hex*		
$\overline{\text{RS}}$	0	0	1 (Highest)	External reset signal
$\overline{\text{INT1}}$	2	2	3	External interrupt line-1
$\overline{\text{INT2}}$	4	4	4	External interrupt line-2
$\overline{\text{INT3}}$	6	6	5	External interrupt line-3
TINT	8	8	6	Internal timer interrupt
RINT	10	A	7	Serial port receive interrupt
XINT	12	C	8	Serial port transmit interrupt
TRNT	14	E	9	TDM port receive interrupt
TXNT	16	10	10	TDM port transmit interrupt
$\overline{\text{INT4}}$	18	12	11	External user interrupt line-4
—	20–33	14–21	N/A	Reserved
TRAP	34	22	N/A	TRAP instruction vector
$\overline{\text{NMI}}$	36	24	2	Non-maskable interrupt
—	38–39	26–27	N/A	Reserved
—	40–63	28–3F	N/A	Software interrupts

the timer (TINT), the TDM port (TRNT and TXNT), and the software interrupt instructions (TRAP, NMI, and INTR). Interrupt priorities are set so that reset ($\overline{RS}$) has the highest priority and $\overline{INT4}$ has the lowest priority. The $\overline{NMI}$ has the second highest priority.

Vector-relative locations and priorities for all external and internal interrupts are shown in Table 11.1. No priority is set for the TRAP instruction, but it is included because it has its own vector location. Each interrupt address has been spaced apart by two locations so that branch instructions can be accommodated in those locations.

11.18 ON-CHIP PERIPHERALS AVAILABLE IN TMS320C3X PROCESSOR

The TMS320C3X generation is the first of Texas Instruments 32 bit floating-point DSP. The C3X devices provide an easy-to-use, high performance architecture and can be used in a wide variety of areas including automotive applications, digital audio, industrial automation and control, data communication, and office equipment that include multifunction peripherals, copiers and laser printers.

The C3X processors peripherals include serial ports, timers and on-chip DMA (Direct Memory Access) controllers. The timers and DMA controllers are discussed below.

Timers

The C3X timers are general-purpose 32 bit timers/event counters. Figure 11.9 shows the block diagram of the timer. The timer operates in two signaling modes, internal or external clocking. With internal clock we can use the timer to signal external devices such as A/D converter, D/A converters etc. When external clock is applied to the timer it can count the external events and interrupts the CPU after a specified number of events. Each timer has an I/O pin that can be configured as an input, output or general purpose I/O pin. The three memory mapped registers, global control register, period register and counter register are used by the timer. These registers can be accessed using the memory map address values.

The 32 bit counter present in the timer increments for the rising edge or falling edge of the input clock. The input clock can be half of the internal clock of C3X or it can be external clock signal on TCLKX pin. The counter register holds the value of the counter and its present value is compared with the content of the period register. When the values are equal, the counter is zeroed. This causes the internal interrupt. The pulse generator present generates two types of external clock signals, either pulse or clock.

DMA controller

The DMA controller is an on-chip peripheral present in C3X devices. It can be used to read or write the 32 bit operands in any location in the memory map. The ‘C30’ and ‘C31’ devices have only one DMA controller, where as ‘C32’ has two DMA controllers. The DMA controller consists of address generators, source and destination registers and transfer counter. Figure 11.10 shows the block diagram of DMA controller. The DMA controller has its own address bus and data bus and this minimize the conflict with CPU. The DMA and

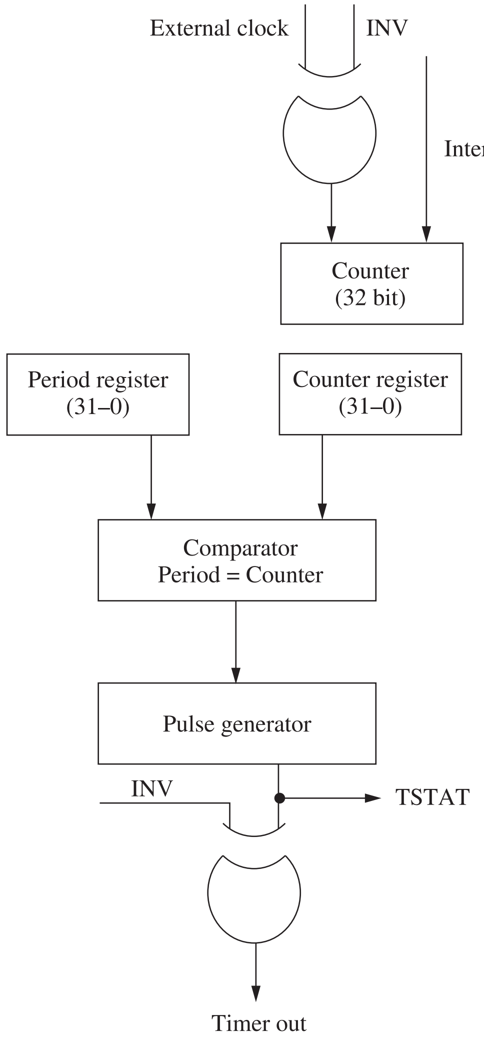


Figure 11.9 Time block diagram.

CPU controller buses can function independently, but when they access the same on chip or the external memory location the priority is provided. As far as C30 and C31 devices are concerned, the highest priority is for the CPU access, but in C32 devices the user can configure the priorities.

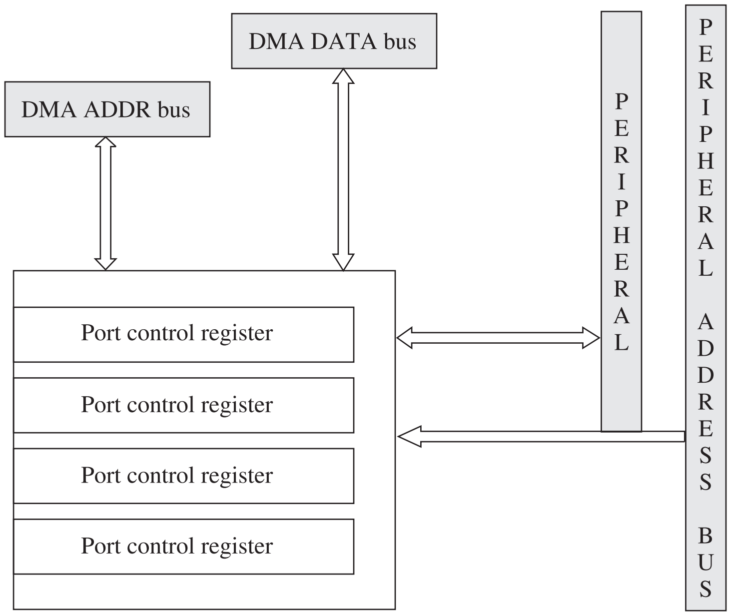


Figure 11.10 Block diagram of DMA controller.

SHORT QUESTIONS WITH ANSWERS

1. Name three leading manufacturers of P-DSPs.
   Ans. Texas instruments (TI), Analog Devices, and Motorola are the three leading manufacturers of P-DSPs.
2. Why P-DSPs are preferred over advanced microprocessors and the RISC processors?
   Ans. Eventhough an advanced microprocessor or RISC processor may use some of the techniques adopted in P-DSPs or may even have instructions that are specifically required for DSP applications and also may have performances close to that of a P-DSP for certain operations, P-DSPs are preferred over advanced microprocessors and the RISC processors because in terms of low power requirement, cost, real-time I/O compatibility and availability of high speed on-chip memories , the P-DSPs have an advantage over them.
3. What are the two categories of DSPs?
   Ans. The DSPs are divided into two categories: General purpose DSPs and Special purpose DSPs.
4. What are general purpose DSPs? Name one.
   Ans. General purpose DSPs are basically high speed microprocessors with architecture and instruction sets optimized for DSP operations. They include fixed point processors as well as floating point processors. Examples are TMS320C5X, TMS320C4X respectively.

5. What are the factors that influence the selection of DSP for a given application?
   Ans. The factors that influence the selection of DSP for a given application are: architectural features, execution speed and type of arithmetic and word length.
6. Why is any operation that involves an off-chip memory is slow compared to that using the on-chip memory in P-DSPs?
   Ans. The P-DSPs use multiple buses only for connecting the on-chip memory to the control unit and data path but for accessing off-chip memory only a single bus is used for accessing both the program memory and the data memory. Hence any operation that involves an off-chip memory is slow compared to that using the on-chip memory in P-DSPs.
7. What is an instruction cycle?
   Ans. An instruction cycle is the time that elapses since an instruction is fetched till the particular instruction completes execution including the time taken for writing the result into a register or memory.
8. How can the number of memory access per clock cycle be increased?
   Ans. The number of memory access per clock period can be increased by using high speed memory or multiport memory.
9. Name few architectures used for P-DSPs.
   Ans. Some architectures used for P-DSPs are: (i) Von Neumann architecture, (ii) Harvard architecture, (iii) Modified Harvard architecture, and (iv) VLIW architecture.
10. Name one P-DSP which uses VLIW architecture.
    Ans. One P-DSP which uses VLIW architecture is TMS320C6X.
11. What is data path?
    Ans. The processing unit consisting of the registers and processing elements such as MAC units, multiplier, ALU, shifter, etc., is referred to as data path.
12. Which architecture does the P-DSPs follow?
    Ans. The P-DSPs follow the modified Harvard architecture.
13. Mention any approach for increasing the efficiency of P-DSPs.
    Ans. One approach for increasing the efficiency of P-DSPs is instruction pipelining.
14. What do you mean by depth of the instruction pipeline?
    Ans. The number of instructions that are processed simultaneously in the CPU are referred to as depth of the instruction pipeline.
15. Which addressing modes are specifically tailored for DSP applications?
    Ans. Cyclic addressing and bit reversed addressing modes are specifically tailored for DSP applications.
16. What are the applications of on-chip timer?
    Ans. Two of the common applications of on-chip timer are generation of periodic interrupts to the P-DSPs and generation of the sampling clocks for the A/D converters. The timers can generate a single pulse or a periodic train of pulses. They can also generate a single square wave or a periodic square wave.

17. What is the use of TDM serial port?
Ans. The TDM serial port permits a P-DSP to communicate with other devices or P-DSPs by using time division multiplexing.

18. What is the use of single bit I/O ports?
Ans. Single bit ports are normally used for control purposes but they can also be used for data transfer. Some of the bits are also used for conditional branching or calls.

19. What do you mean by host port?
Ans. A host port is a special parallel port in P-DSPs that enable them to communicate with a microprocessor or PC.

20. What is the function of the host port?
Ans. In addition to data communication, the host port can generate interrupts and also cause the P-DSP to load a program from ROM to the RAM on reset.

21. What do you mean by comm ports?
Ans. Communication ports are parallel ports that are used for interprocess communication between a number of identical P-DSPs in a multiprocessor system.

22. In RISC processors what percentage of chip area may be used for the control unit and in CISC processors what percentage?
Ans. In RISC processors 20% of the chip area may be used for the control unit and in CISC processors 30 to 40% may be used.

23. How many buses does C5X architecture have? Name them.
Ans. The C5X architecture has four buses and they are: Program bus (PB), Program address bus (PAB), Data read bus (DB), Data read address bus (DAB).

24. What is the function of program bus (PB)?
Ans. The function of program bus is it carries the instruction code and immediate operands from program memory space to the CPU.

25. What is the function of program address bus (PAB)?
Ans. The function of program address bus is it provides addresses to program memory space for both reads and writes.

26. What is the function of data read bus (DB)?
Ans. The function of data read bus is it interconnects various elements of the CPU to data memory space.

27. What is the function of data read address bus?
Ans. The function of data read address bus is it provides the address to access the data memory space.

28. What does CALU consist of?
Ans. The central arithmetic logic unit (CALU) consists of the following elements: Parallel multiplier, arithmetic logic unit (ALU), accumulator (ACC), accumulator buffer (ACCB), product register (PREG), left barrel shifter and right barrel shifter.

29. What does ARAU consist of?
Ans. The auxiliary register ALU (ARAU) consists of auxiliary register, auxiliary register pointer and an unsigned ALU.

30. What is the use of INDX?
Ans. The index register INDX is used by the ARAU as a step value to modify the address in the ARs during indirect addressing.

31. What is the use of ARCR?
Ans. The auxiliary register compare register ARCR is used for address boundary comparision.

32. What is the use of BMAR?
Ans. The block move address register holds an address value to be used with block moves and multiply/accumulate operations.

33. What is the function of PLU?
Ans. The PLU performs Boolean operations or the bit manipulations required of high-speed controllers. It can set, clear, test or toggle bits in a status register control register, or any data memory location. It allows logic operations to be performed on data memory values directly without affecting the control of the ACC and PREG.

34. What is the use of program controller?
Ans. The program controller contains logic circuitry that decodes the instructions, manages the CPU pipeline, stores the status of CPU operations and decodes the conditional operations.

35. What does program controller consist of?
Ans. The program controller consists of the following elements: program counter (PC), status registers, hardware stack, address generation logic, instruction register and interrupt flag register and interrupt mask register.

REVIEW QUESTIONS

1. What are the advantages of DSP processors over conventional processors?

2. Explain the implementation of convolver with single multiplier/adder.

3. Explain the modified bus structures and memory access schemes in digital signal processors.

4. With the help of a block diagram explain the dual port memory.

5. Explain the different techniques adopted for increasing the number of memory accesses/instruction cycle.

6. Explain the VLIW architecture with its block diagram.

7. Give the advantages and disadvantages of VLIW architecture.

8. Explain how a higher throughput is obtained using VLIW architecture. Give an example of DSP that has VLIW architecture.

9. What is instruction pipelining? Briefly explain the pipeline operation.
10. What are the different stages in pipelining and explain.
11. Explain the special addressing modes in P-DSPs.
12. Explain memory mapped addressing modes used in P-DSPs.
13. What are the on-chip peripherals available on P-DSPs and explain their functions.
14. What are the advantages of RISC?
15. What are the advantages of CISC?
16. Draw the block diagram of TMS320C50 digital signal processor and explain the functionality of CALU and PLU.
17. Discuss various interrupt types supported by TMS320C5X processor.
18. Explain the on-chip peripherals interface connected to the TMS320C50 processor.
19. List the status register bits of TMS320C5X and their functions.
20. What are the on-chip peripherals available in TMS320C3X processor? Explain any two peripherals.

FILL IN THE BLANKS

1. The MACD instruction requires _______ memory accesses per instruction cycle.
2. The cost of an IC increases with the number of _______ in the IC.
3. Extension of number of buses outside the chip would _______ the price.
4. Any operation that involves an off-chip memory is _______ compared to that using the on-chip memory in P-DSPs.
5. The P-DSPs use _______ buses only for connecting the on-chip memory to the control unit and data path.
6. The number of memory access per clock period can be increased by using _______ memory or _______ memory.
7. The P-DSPs follow the _______ architecture.
8. Dual port memory is _______ compared to two single port memory of the same capacity.
9. Some architectures used for P-DSPs are: (i) _______ architecture, (ii) _______ architecture, (iii) _______ architecture, and (iv) _______ architecture.
10. The processing unit consisting of the registers and processing elements such as MAC units, multiplier, ALU, shifter, etc., is also referred to as _______.
11. The throughput will be higher only if the algorithm involves execution of _______ operations.
12. By using 8 functional units, the time required for convolution can be reduced by a factor _______ compared to the case where a single functional unit is used.

13. The efficiency of the P-DSPs can be increased by _______ pipelining.

14. P-DSPs may be implemented using either the _______ processor or _______ processor.

15. In RISC processors _______ of the chip area may be used for the control unit and in CISC processors _______ may be used.

16. RISC has a _______ number of instructions compared to CISC.

17. For P-DSP with _______ architecture, compilers are essential.

18. A majority of P-DSPs are _______ based.

19. Execution of each of the micro instructions is referred to as _______ of an instruction.

20. The number of instructions that are processed simultaneously in the CPU are referred to as _______ of the instruction pipeline.

21. _______ addressing and _______ addressing modes are specifically tailored for DSP applications.

22. In TI processors, the indirect address registers are called _______ registers.

23. The TMS320C5X is a _______ bit, _______ point processor.

OBJECTIVE TYPE QUESTIONS

1. The features in which P-DSP is superior to advanced microprocessor is
(a) low cost (b) low power
(c) computational speed (d) real time I/O capability

2. VLIW architecture differs from conventional P-DSP in which of the following aspects?
(a) instruction cache (b) number of functional units
(c) use pipelining

3. The serial port that permits the data from a number of I/O devices to be sent using a single serial port is called
(a) common port (b) Host port
(c) time division multiplexing (d) bit I/O port

4. The addressing mode that is convenient for FFT computation is
(a) indirect addressing (b) circular mode addressing
(c) bit reversed addressing (d) memory mapped addressing

5. Which of the following characteristics are true for a RISC processor?
(a) smaller control unit (b) small instruction set
(c) short program length (d) less traffic between CPU and memory

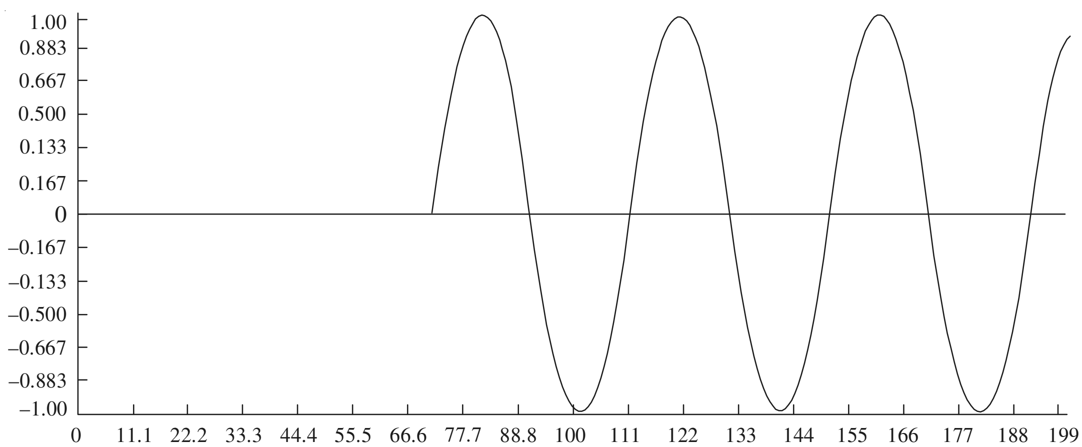
PROGRAMS

Program 11.1

% Sine wave generation

```
#include<stdio.h>
#include<math.h>
#define pi 3.1415625
float a[200];
main()
{int i;
for(i=0;i<200;i++)
a[i]=sin(2*pi*i/200 *5);
}
```

Output:



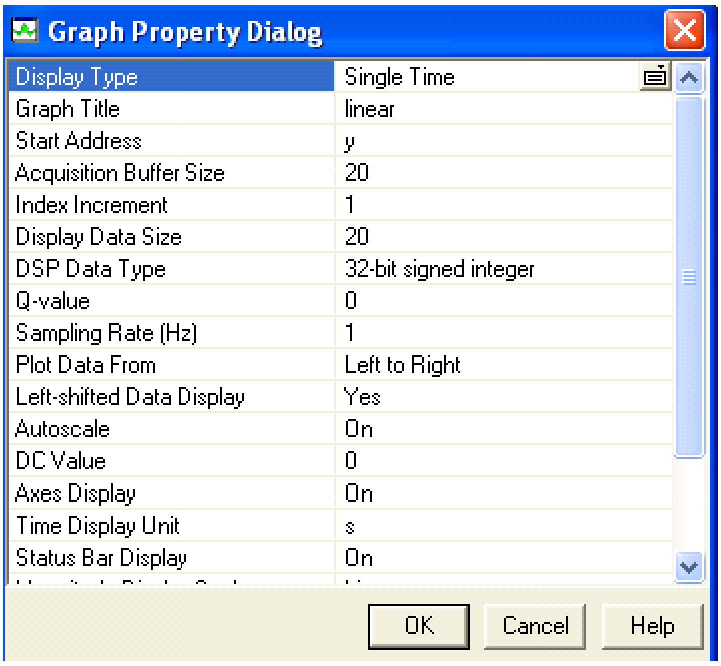
Program 11.2

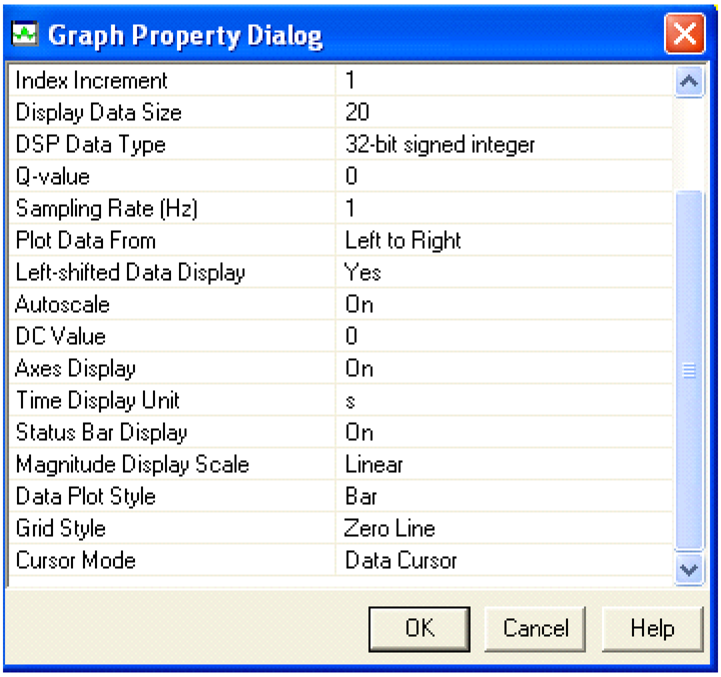
% Linear convolution

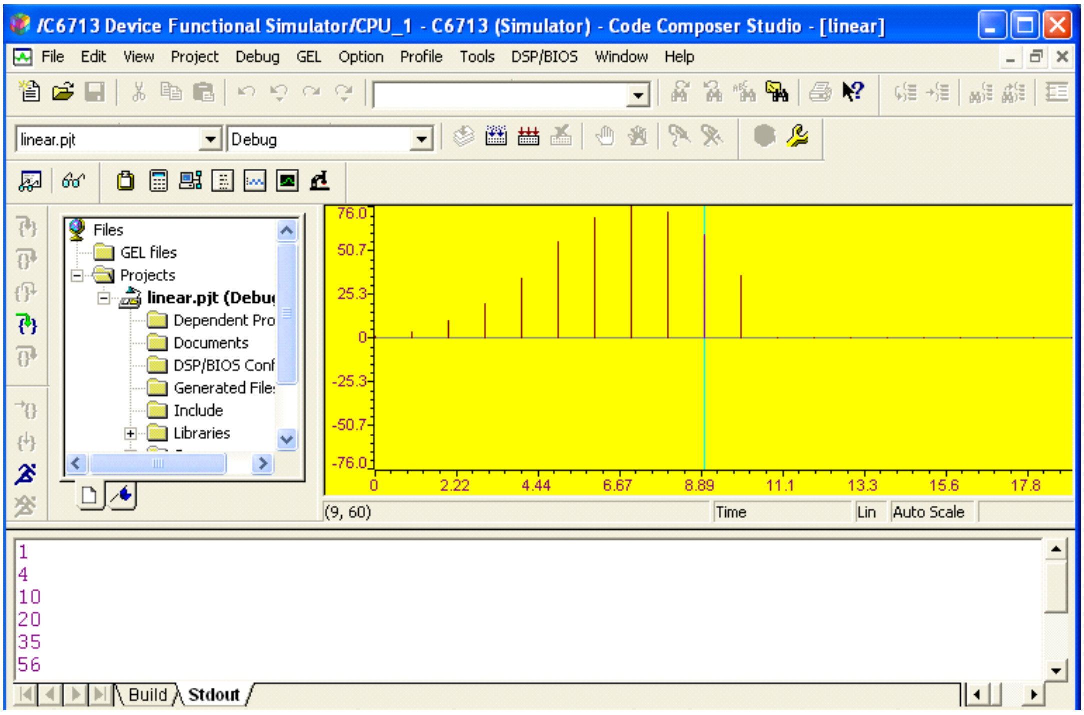
```
#include<stdio.h>
int m=6;
        int n=6;
        int i=0,j;
         int x[15]={1,2,3,4,5,6,0,0,0,0,0,0};
         int h[15]={1,2,3,4,5,6,0,0,0,0,0,0};
        int y[20];
main()
{
for(i=0;i<m+n–1;i++)
{
y[i]=0;
for(j=0;j<=i;j++)
                y[i]+=x[j]*h[i–j];
}
    for(i=0;i<m+n–1;i++)
     printf("%d \n",y[i]);
}
```

Output:

```
 4
10
20
35
56
70
76
73
60
36
```


Graph Property Dialog
Display Type
Single Time
Graph Title
linear
Start Address
y
Acquisition Buffer Size
20
Index Increment
1
Display Data Size
20
DSP Data Type
32-bit signed integer
Q-value
0
Sampling Rate (Hz)
1
Plot Data From
Left to Right
Left-shifted Data Display
Yes
Autoscale
On
DC Value
0
Axes Display
On
Time Display Unit
s
Status Bar Display
On
OK
Cancel
Help


Graph Property Dialog
Index Increment
1
Display Data Size
20
DSP Data Type
32-bit signed integer
Q-value
0
Sampling Rate (Hz)
1
Plot Data From
Left to Right
Left-shifted Data Display
Yes
Autoscale
On
DC Value
0
Axes Display
On
Time Display Unit
s
Status Bar Display
On
Magnitude Display Scale
Linear
Data Plot Style
Bar
Grid Style
Zero Line
Cursor Mode
Data Cursor
OK
Cancel
Help



Program 11.3

% Circular convolution

```
#include<stdio.h>
int m,n,x[30],h[30],y[30],i,j,temp[30],k,x2[30],a[30];
void main()
{
printf("enter the length of the 1st sequence\n");
scanf("%d",&m);
printf("enter the length of the second sequence\n");
scanf("%d",&n);
printf("enter the 1st sequence\n");
for(i=0;i<m;i++)
scanf("%d",&x[i]);
printf("enter the second sequence\n");
for(j=0;j<n;j++)
```

```
scanf("%d",&h[j]);
if(m–n!=0)
{
if(m>n)
{
for(i=n;i<m;i++)
h[i]=0;
n=m;
}
for(i=m;i<n;i++)
x[i]=0;
m=n;
}
y[0]=0;
a[0]=h[0];
for(j=1;j<n;j++)
a[j]=h[n–j];
for(i=0;i<n;i++)
y[0]+=x[i]*a[i];
for(k=1;k<n;k++)
{
y[k]=0;
for(j=1;j<n;j++)
x2[j]=a[j–1];

x2[0]=a[n–1];
for(i=0;i<n;i++)
{
a[i]=x2[i];
    y[k]+=x[i]*x2[i];
    }
    }
    printf("the circular convolution is\n");
for(i=0;i<n;i++)
printf("%d\t",y[i]);
}
```

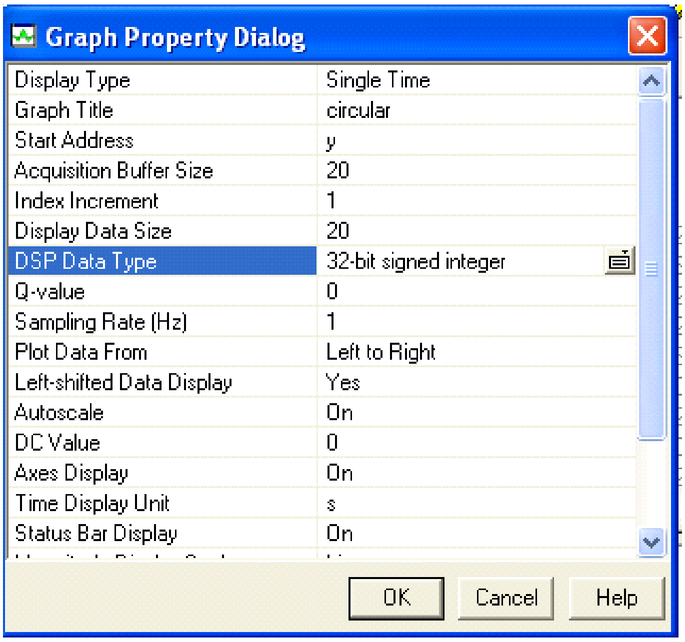
Input:

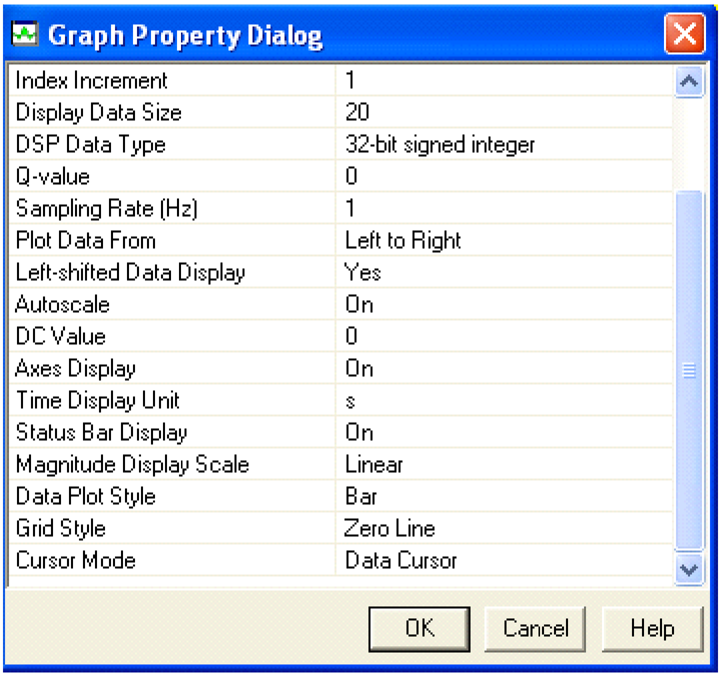
enter the length of the 1st sequence
5
enter the length of the second sequence
5
enter the 1st sequence
1
2
3
4
5
enter the second sequence
1
2
3
4
5

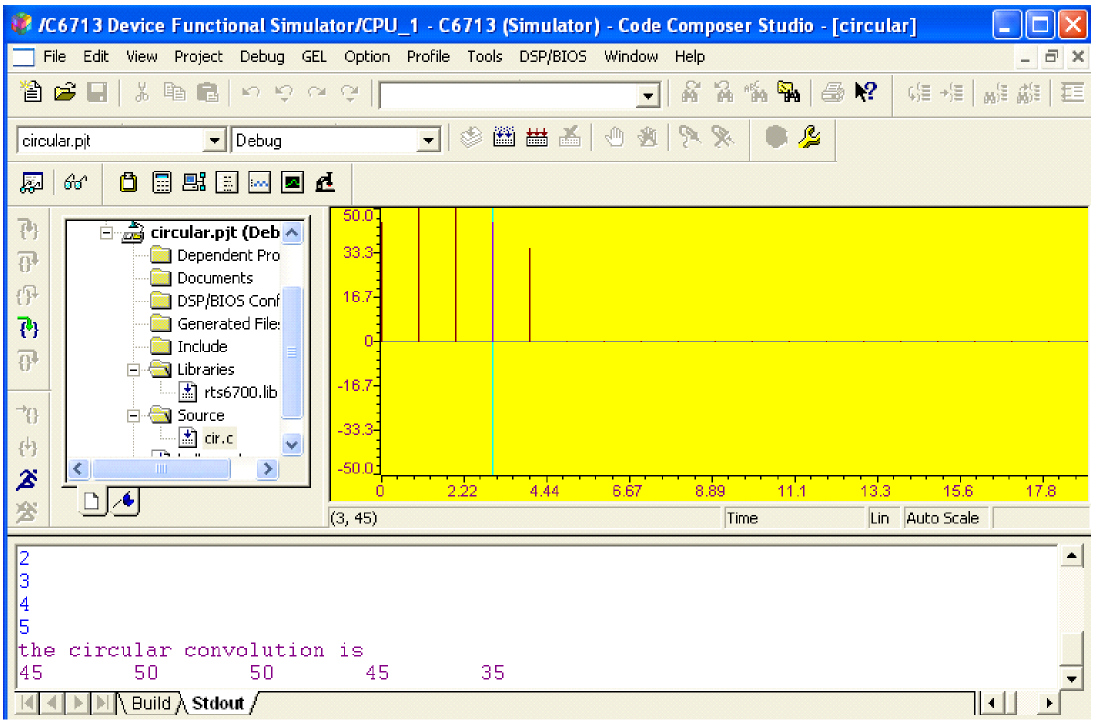
Output:

the circular convolution is

45 50 50 45 35




Graph Property Dialog
Index Increment 1
Display Data Size 20
DSP Data Type 32-bit signed integer
Q-value 0
Sampling Rate (Hz) 1
Plot Data From Left to Right
Left-shifted Data Display Yes
Autoscale On
DC Value 0
Axes Display On
Time Display Unit s
Status Bar Display On
Magnitude Display Scale Linear
Data Plot Style Bar
Grid Style Zero Line
Cursor Mode Data Cursor
OK
Cancel
Help


/C6713 Device Functional Simulator/CPU_1 - C6713 (Simulator) - Code Composer Studio - [circular]
File Edit View Project Debug GEL Option Profile Tools DSP/BIOS Window Help
circular.pjt
Debug
circular.pjt (Deb
Dependent Pro
Documents
DSP/BIOS Conf
Generated File
Include
Libraries
rts6700.lib
Source
cir.c
50.0
33.3
16.7
0
-16.7
-33.3
-50.0
0 2.22 4.44 6.67 8.89 11.1 13.3 15.6 17.8
(3, 45)
Time
Lin
Auto Scale
2
3
4
5
the circular convolution is
45 50 50 45 35
Build
Stdout

Program 11.4

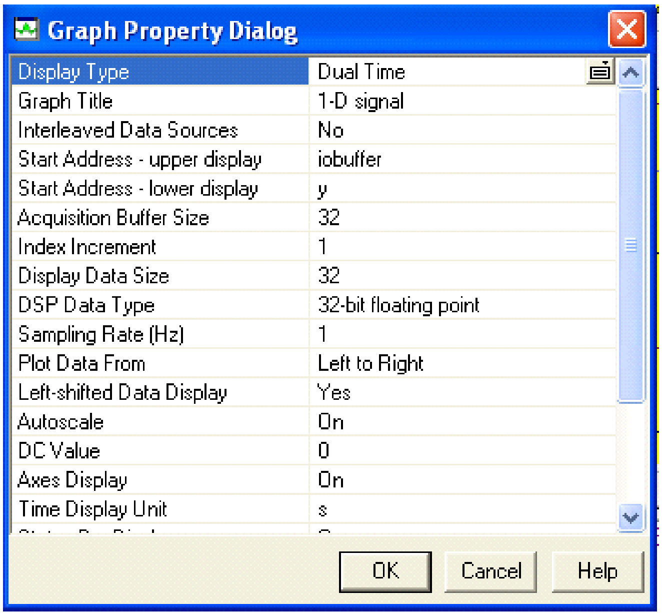
% Fast Fourier Transform

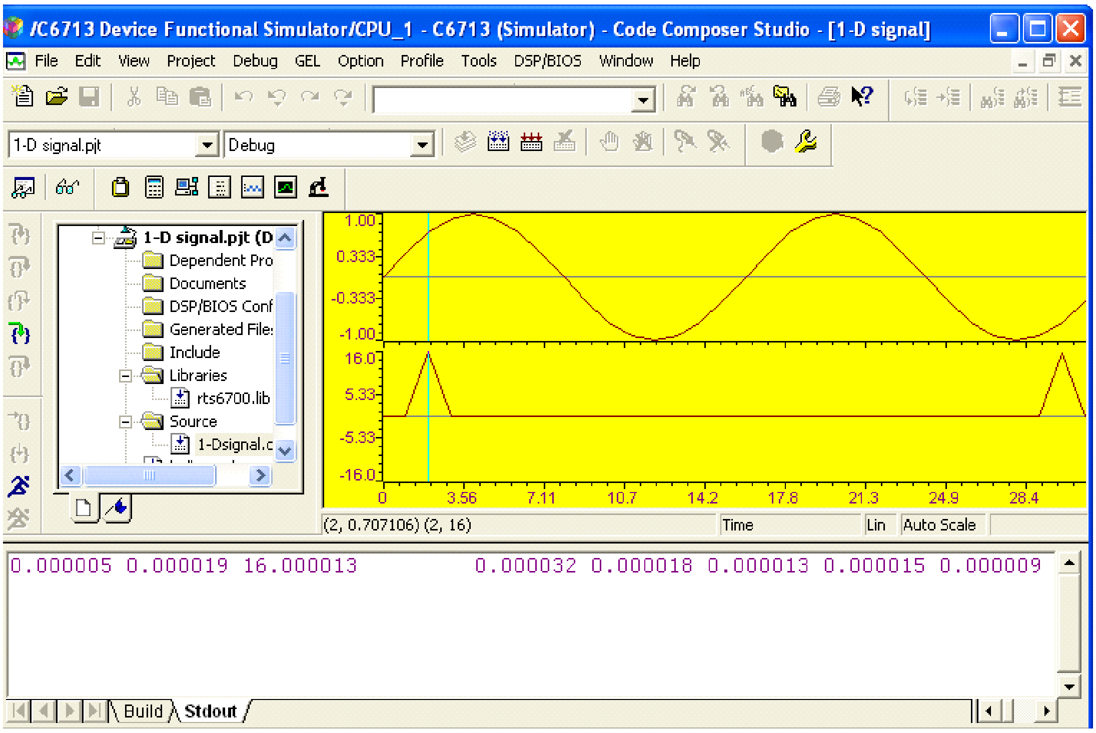
```
#include<stdio.h>
#include<math.h>
#define N 32
#define PI 3.14159
typedef struct
{
float real,imag;
}
complex;
float iobuffer[N];
float y[N];
main()
{
int i;
complex w[N];
complex x[N];
complex temp1,temp2;
int j,k,upper_leg,lower_leg,leg_diff,index,step;
for(i=0;i<N;i++)
{
iobuffer[i]=sin((2*PI*2*i)/32.0);
}
for(i=0;i<N;i++)
{
x[i].real=iobuffer[i];
x[i].imag=0.0;
}
for(i=0;i<N;i++)
{
w[i].real=cos((2*PI*i)/(N*2.0));
w[i].imag=–sin((2*PI*i)/(N*2.0));
}
leg_diff=N/2;
```

```
step=2;
for(i=0;i<5;i++)
{
index=0;
for(j=0;j<leg_diff;j++)
{
for(upper_leg=j;upper_leg<N;upper_leg+=(2*leg_diff))
{
lower_leg=upper_leg+leg_diff;
temp1.real=(x[upper_leg]).real+(x[lower_leg]).real;
temp1.imag=(x[upper_leg]).imag+(x[lower_leg]).imag;
temp2.real=(x[upper_leg]).real-(x[lower_leg]).real;
temp2.imag=(x[upper_leg]).imag-(x[lower_leg]).imag;
(x[lower_leg]).real=temp2.real*(w[index]).real-temp2.imag*(w[index]).imag;
(x[lower_leg]).imag=temp2.real*(w[index]).imag+temp2.imag*(w[index]).real;
(x[upper_leg]).real=temp1.real;
(x[upper_leg]).imag=temp1.imag;
}
index+=step;
}
leg_diff=(leg_diff)/2;
step=step*2;
}
j=0;
for(i=1;i<(N–1);i++)
{
k=N/2;
while(k<=j)
{
j=j–k;
k=k/2;
}

j=j+k;
if(i<j)
```

```
{
temp1.real=(x[j]).real;
temp1.imag=(x[j]).imag;
(x[j]).real=(x[i]).real;
(x[j]).imag=(x[i]).imag;
(x[i]).real=temp1.real;
(x[i]).imag=temp1.imag;
}
}
for(i=0;i<N;i++)
{
y[i]=sqrt((x[i].real*x[i].real)+(x[i].imag*x[i].imag));
}
for(i=0;i<N;i++)
{
printf("%f\t",y[i]);
}
return(0);
}
```

Output:




/C6713 Device Functional Simulator/CPU_1 - C6713 (Simulator) - Code Composer Studio - [1-D signal]
File Edit View Project Debug GEL Option Profile Tools DSP/BIOS Window Help
1-D signal.pjt
Debug
1-D signal.pjt (D
Dependent Pro
Documents
DSP/BIOS Conf
Generated File:
Include
Libraries
rts6700.lib
Source
1-Dsignal.c
1.00
0.333
-0.333
-1.00
16.0
5.33
-5.33
-16.0
0
3.56
7.11
10.7
14.2
17.8
21.3
24.9
28.4
(2, 0.707106) (2, 16)
Time
Lin
Auto Scale
0.000005 0.000019 16.000013 0.000032 0.000018 0.000013 0.000015 0.000009
Build
Stdout

Glossary

Adder A device used to add two or more signals.

Aliasing The phenomenon in which a high frequency component in the frequency spectrum of a signal takes identity of a lower frequency component in the spectrum of the sampled signal.

Amplitude spectrum A plot of amplitude of Fourier coefficients versus frequency.

Anti-aliasing filter The low-pass filter placed before the down sampler to prevent the effect of aliasing by band limiting the input signal.

Anti-imaging filter The low-pass filter placed after the up sampler to remove the images created due to up-sampling.

Aperiodic signal A signal which does not repeat at regular intervals of time.

Approximation of derivatives method A method of converting an analog filter into a digital filter by approximating the differential equation into an equivalent difference equation.

Autocorrelation A measure of similarity or match or relatedness or coherence between a signal and its time delayed version.

Average power The power dissipated by a voltage applied across a 1 Ω resistor (or by a current flowing through a 1 Ω resistor).

BIBO stability criterion A necessary and sufficient condition for a system to be BIBO stable. It states that the impulse response must be absolutely integrable for a system to be stable.

Bilinear transformation It is a conformal mapping that transforms the s-plane into z-plane on one-to-one basis.

Butterfly diagram A signal flow-graph resembling a butterfly using which 2-point DFTs can be computed.

Butterworth filter A filter designed by selecting an error function such that the magnitude is maximally flat in the pass band and monotonically decreasing in the stop band.

Canonical structure A form of realization in which the number of delay elements used is equal to the order of the difference equation.

Cascade form A series interconnection of the sub-transfer functions.

Cascade form realization Realization of complex system as a cascade of subsystems.

Causal signal A signal which does not exist for $n < 0$.

Causal system A system in which the output at any instant depends only on the present and past values of the input and not on future inputs.

Chebyshev Type-I filter A filter designed by selecting the error function such that the magnitude response is equiripple in the pass band and monotonic in the stop band.

Chebyshev Type-II filter A filter designed by selecting the error function such that the magnitude response is monotonic in the pass band and equiripple in the stop band.

Circular convolution Convolution that can be performed only when atleast one of the two sequences is periodic.

Communication ports Parallel ports that are used for interprocess communication between a number of identical P-DSPs in a multi processor system.

Constant multiplier A device used to multiply the signal by a constant.

Continuous-time Fourier transform Fourier transform of continuous-time signals.

Continuous-time signals Signals defined for all instants of time.

Continuous-time system A system which transforms continuous-time input signals into continuous-time output signals.

Convolution A mathematical operation which is used to express the input-output relationship of an LTI system.

Correlation theorem A theorem which states that the cross correlation of two energy signals corresponds to the multiplication of the Fourier transform of one signal by the complex conjugate of the Fourier transform of the second signal.

Correlation An operation between two signals which gives the degree of similarity between those two signals.

Cross correlation A measure of similarity or match or relatedness or coherence between one signal and the time delayed version of another signal.

Decimation The process of decreasing the sampling rate of a signal by an integer factor D by keeping every Dth sample and removing D–1 in between samples.

Decimator The anti-aliasing filter and down sampler together.

Deconvolution The process of finding the input once the output and impulse response are given.

Deterministic signal A signal exhibiting no uncertainty of its magnitude and phase at any given instant of time. It can be represented by a mathematical equation.

DIF algorithm An algorithm used to compute the DFT efficiently by decimating the frequency domain sequence into smaller sequences.

Digital signal processor A large programmable digital computer or microprocessor programmed to perform the desired operations on the signal.

Digital signals Signals which are discrete in time and quantized in amplitude.

Direct form-I structure Direct implementation of the difference equation or transfer function of the IIR system.

Direct form-II structure Realization of IIR filter such that the number of delay elements used is equal to the order of the difference equation.

Discrete Fourier series The Fourier series representation of a periodic discrete time sequence.

Discrete Fourier transform A sequence obtained by sampling one period of the Fourier transform of a discrete sequence at a finite number of frequency points.

Discrete-time Fourier transform A transformation technique which transforms signals from the discrete-time domain to the corresponding frequency domain.

Discrete-time signals Signals defined only at discrete instants of time.

Discrete-time system A system which transforms discrete-time input signals into discrete-time output signals.

DIT algorithm An algorithm used to compute the DFT efficiently by decimating the given time sequence and computing the DFTs of smaller sequences and combining them.

Down sampling Reducing the sampling rate of a discrete-time signal.

Dynamic system A system in which the output is due to past or future inputs also.

Elliptic filter Also called Cauer filter is a filter designed by selecting an error function such that the magnitude response has equiripple in both pass band and stop band.

Energy signal A signal whose total energy is finite and average power is zero.

Even signal A symmetric signal with $x(-n) = x(n)$ for all n.

Even symmetry Also called mirror symmetry which is said to exist if the signal $x(n)$ satisfies the condition $x(n) = x(-n)$.

Fast Fourier transform An algorithm for computing the DFT efficiently.

Filter A frequency selective network.

Finite duration signal A signal which is equal to zero for $n < N_1$ and for $n > N_2$, where N_1 and N_2 are finite and $N_1 < N_2$.

FIR Systems Systems whose impulse response has finite number of samples.

Flipping a sequence Time reversing a sequence.

Forced response The response due to input alone when the initial conditions are zero.

Fourier transform A transformation technique which transforms signals from the continuous-time domain to the corresponding frequency domain.

Frequency response The transfer function in frequency domain.

Frequency spectrum Amplitude spectrum and phase spectrum together.

Frequency warping The distortion in frequency axis introduced when s-plane is mapped into z-plane using bilinear transformation.

Functional representation A way of representing a discrete-time signal where the amplitude of the signal is written against the value of n.

Fundamental period The smallest value of N that satisfies the condition $x(n + N) = x(n)$ for all values of n.

General purpose DSPs High speed microprocessors with architecture and instruction sets optimized for DSP operations.

Gibbs phenomenon The ripples present at the point of discontinuity in signal approximation.

Graphical representation A way of representing a discrete-time signal as a plot with value of $x(n)$ at the sampling instant n indicated.

Host port A special parallel port in P-DSPs that enables them to communicate with a microprocessor or PC.

IIR Systems Systems whose impulse response has infinite number of samples.

Imaging The phenomenon of producing additional spectra by the up-sampler.

Impulse invariant transformation Transformation of an analog filter into a digital filter without modifying the impulse response of the filter.

Impulse response The output of the system for a unit impulse input.

Instruction cycle The time that elapses since an instruction is fetched till the particular instruction completes execution including the time taken for writing the result into a register or memory.

Interpolation The process of increasing the sampling rate by an integer factor I by interpolating I–1 new samples between successive sampling instants.

Interpolator The Anti-imaging filter and up-sampler together.

Inverse discrete-time Fourier transform The process of finding the discrete-time sequence from its frequency response.

Inverse Fourier transform A transformation technique which transforms signals from the frequency domain to the corresponding continuous-time domain.

Inverse Z-transform A transformation technique which transforms signals from the Z-domain to the corresponding discrete-time domain.

Invertible system A system which has a unique relationship between its input and output.

Ladder structure A structure obtained by dividing the numerator by the denominator sequentially and substituting the result in a ladder fashion.

Laplace transform A transformation technique which transforms signals from the continuous-time domain to the corresponding complex frequency domain (s-domain).

Left-sided signal A signal which is equal to zero for $n > N_2$ for some finite N_2.

Linear phase systems Systems for which the phase drops linearly with increase in frequency.

Linear system A system which obeys the principle of superposition and homogeneity.

LPF A frequency selective network which allows transmission of only low frequency signals.

LTI system A system which is both linear and time-invariant.

LTV system A system which is linear but time-variant.

MACD instruction Multiplier accumulator operation with data move.

Memory system Same as dynamic system.

Memoryless system Same as static system.

Mixed radix FFT FFT computed when N is a composite number which has more than one prime factor.

Multiple access memory The memory that permits more than one memory access/clock period.

Multirate systems Discrete-time systems that process date at more than one sampling rate.

Narrow band low-pass filter A low-pass filter characterized by a narrow pass band and narrow transition band.

Natural response Also called the free response is the response due to the initial conditions alone, when the input is zero.

Non-recursive filters Filters which do not employ any kind of feedback connection.

Non-canonical structure A form of realization in which the number of delay elements used is more than the order of the difference equation.

Non-causal signal A signal which exists for $n < 0$ also.

Non-causal system A system in which the output at any time depends on future inputs.

Non-invertible system A system which does not have a unique relationship between its input and output.

Non-linear system A system which does not obey the principle of superposition and homogeneity.

Non-recursive system A system whose output depends only on the present and past inputs and not on past outputs.

Nyquist interval The time interval between any two adjacent samples when sampling rate is Nyquist rate.

Nyquist rate of sampling The theoretical minimum sampling rate at which a signal can be sampled and still be recovered from its samples without any distortion.

Odd signal An antisymmetric signal with $x(-n) = -x(n)$ for all n.

One-dimensional signal A signal which depends on only one independent variable.

Overlap save method A method of sectioned convolution in which the first N–1 values of each block are discarded and the remaining samples are retained.

Overlap-add method A method of sectioned convolution in which the samples of the overlapped regions are added and the samples of the non-overlapped regions are retained as such.

Parallel form realization Realization of a complex system as a parallel connection of subsystems.

Pass band edge frequency The largest frequency upto which pass band is considered.

Periodic signal A signal which repeats itself at regular intervals of time.

Phase factor A factor which is equal to Nth root of unity. Exploiting its symmetry properties the DFT is computed efficiently.

Power signal A signal whose average power is finite and total energy is infinite.

Prewarping The conversion of the specified digital frequencies to analog equivalent frequencies to nullify the effect of warping in IIR filter design using bilinear transformation.

Program bus A bus that carries the instruction code and immediate operands from program memory space to CPU.

Radix The length of the smallest sequences into which the larger sequence is decimated.

Random signal A signal characterized by uncertainty about its occurrence. It cannot be represented by a mathematical equation.

Realization Obtaining a network corresponding to the difference equation or transfer function of the system.

Recursive filters Filters which make use of feedback connection to get the desired filter implementation.

Recursive system A system whose output depends on the present input and any number of past inputs and outputs.

Right-sided signal A signal which is equal to zero for $n < N_1$ for some finite N_1.

ROC of Z-transform The range of values of $|z|$ for which $X(z)$ converges.

Sampling frequency The reciprocal of sampling period which indicates the number of samples per second.

Sampling interval Same as sampling period.

Sampling period The time interval between two successive sampling instants.

Sampling rate conversion The process of converting a sequence with one sampling rate into another sequence with a different sampling rate.

Sampling theorem A condition to be satisfied by the sampling frequency for a band limited signal to be recovered from its samples without distortion.

Sampling The process of converting a continuous-time signal into a discrete-time signal.

Sectioned convolution A method of finding the convolution of longer sequence by sectioning it into smaller sequences, finding their convolutions and combining them.

Sequence representation A way of representing a discrete-time signal where the values of $x(n)$ at the sampling instants are written as a sequence with an arrow pointing towards $x(0)$.

Signal processing A method of extracting information from the signal.

Signal A single-valued function of one or more independent variables which contain some information.

Stable system A system which produces a bounded output for a bounded input.

Standard signals Signals like unit step, unit ramp, unit impulse etc. in terms of which any given signal can be expressed.

Static system A system in which the response is due to present input alone.

Steady state response The response due to the poles of the input function. It remains as $n \to \infty$.

Step response The output of the system for a unit step input.

Stop band edge frequency The smallest frequency beyond which stop band is considered.

Stop band The band of frequencies that is rejected by the filter.

System An entity that acts on an input signal and transforms it into an output signal.

Tabular representation A way of representing a discrete-time signal where the magnitude of the signal at the sampling instant is represented in tabular form.

TDM serial port A serial port that permits a P-DSP to communicate with other devices or P-DSPs by using time division multiplexing.

Time convolution theorem A theorem which states that convolution in time domain is equivalent to multiplication of their spectra in frequency domain.

Time invariant system A system whose input/output characteristics do not change with time.

Time reversed sequence A sequence obtained by wrapping it around the circle in the clockwise direction.

Time variant system A system whose input/output characteristics change with time.

Total response The sum of the natural and forced responses.

Transfer function The ratio of the Fourier transform/Laplace transform/Z-transform of the output to the Fourier transform/Laplace transform/Z-transform of the input of the system when the initial conditions are neglected. It is also the Fourier transform/ Laplace transform/DTFT/Z-transform of the impulse response of the system.

Transient response The response due to the poles of the system function. It vanishes after some time.

Transposed form structure A structure obtained by reversing the direction of all branch transmittances and interchanging the input and output in the direct form structure.

Twiddle factor A complex valued phase factor which is an Nth root of unity using whose symmetry properties DFT is computed efficiently.

Two-sided signal A signal which extends from $-\infty$ to ∞.

Unit delay elemant A device used to delay the signal passing through it by one sampling time.

Unit impulse function A function which exists only at $n = 0$ with a magnitude of unity.

Unit parabolic function A function whose magnitude is zero for $n < 0$ and is $n^2/2$ for $n \geq 0$.

Unit ramp function A function whose magnitude is zero for $n \leq 0$, and rises linearly with a slope of unity for $n > 0$.

Unit step function A function whose magnitude is zero for $n < 0$, suddenly jumps to 1 level at $n = 0$ and remains constant at that value for $n > 0$.

Unstable system A system which produces an unbounded output for a bounded input.

Up-sampling Increasing the sampling rate of a discrete-time signal.

Window A finite weighing sequence with which the infinite impulse response is multiplied to obtain a finite impulse response.

Zero padding Appending zeros to a sequence in order to increase the size or length of the sequence.

Z-transform A transformation technique which transforms signals from the discrete-time domain to the corresponding Z-domain.

Answers

CHAPTER 1

Answers to Fill in the Blanks

1. one-dimensional	**2.** signal modelling	**3.** discrete, continuous
4. digital	**5.** time reversal	**6.** deterministic
7. random	**8.** discrete	**9.** $x(n)$
10. $-x(n)$	**11.** finite, zero	**12.** finite, infinity
13. $n > 0$	**14.** past and future	**15.** past, future
16. memoryless	**17.** memory	**18.** present and past
19. future	**20.** non-anticipative	**21.** anticipative
22. non-causal	**23.** linear	**24.** nonlinear
25. linearity, time-invariance	**26.** input/output characteristics	
27. input/output characteristics	**28.** bounded	**29.** absolutely summable
30. invertible	**31.** non-invertible	

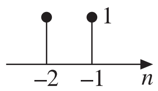
Answers to Objective Type Questions

1. (c)	**2.** (c)	**3.** (a)	**4.** (b)
5. (c)	**6.** (d)	**7.** (c)	**8.** (b)
9. (a)	**10.** (c)	**11.** (d)	**12.** (c)
13. (c)	**14.** (c)	**15.** (d)	**16.** (d)
17. (a)	**18.** (d)	**19.** (d)	

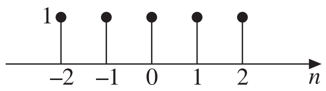
Answers to Problems

1. (a) e^4 (b) 9 (c) 1 (d) 0

2.



(a)



(b)

3. (a) periodic, $N = 50$ (b) non-periodic (c) periodic, $N = 6$ (d) non-periodic

4. (a) Energy signal, $E = 9/8$, $P = 0$ (b) Power signal, $E = \infty$, $P = 1$
(c) Energy signal, $E = 6$, $P = 0$

5. (a) non-causal (b) non-casual (c) non-causal

6. (a) $x_e(n) = \{2, 3.5, 1, 3.5, 2\}$, $x_o(n) = \{-4, 0.5, 0, -0.5, 4\}$

(b) $x_e(n) = \{1.5, 1.5, 4, 1.5, 1.5\}$, $x_o(n) = \{1.5, -3.5, 0, 3.5, -1.5\}$

(c) $x_e(n) = \{2.5, 1.5, 2, 0.5, 2, 0.5, 2, 1.5, 2.5\}$

$x_o(n) = \{-2.5, -1.5, -2, -0.5, 0, 0.5, 2, 1.5, 2.5\}$

7. even

8. (a) static (b) dynamic (c) dynamic

9. (a) causal (b) non-causal (c) non-causal

10. (a) nonlinear (b) nonlinear (c) linear

11. (a) time-variant (b) time-invariant (c) time-invariant

12. (a) stable (b) unstable (c) stable

13. (a) dynamic, linear, non-causal, time-invariant
(b) static, nonlinear, causal, time-invariant
(c) dynamic, linear, non-causal, time-invariant

CHAPTER 2

Answers to Fill in the Blanks

1. FIR

2. zero state

3. multiplying

4. correlation

5. impulse response

6. $y(n) = x(n) * h(n)$

7. $y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$

8. $y(n) = \sum_{k=0}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{n} h(k)x(n-k)$

9. $y(n) = \sum_{k=-\infty}^{n} x(k)h(n-k) = \sum_{k=0}^{\infty} h(k)x(n-k)$

10. $y(n) = \sum_{k=0}^{n} x(k)h(n-k) = \sum_{k=0}^{n} h(k)x(n-k)$

11. $x(n) * h(n) = h(n) * x(n)$

12. $[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$

13. $x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$

14. If $x(n) * h(n) = y(n)$, then $x(n-k) * h(n-m) = y(n-k-m)$

15. $x(n) * \delta(n) = x(n)$

16. running sum

17. $N_1 + N_2 - 1$

18. $n_1 + n_2 - 1$

19. $n_3 + n_4 - 1$

20. leading

21. trailing

22. deconvolution

23. periodic

24. N

25. correlation

26. correlation

27. $R_{xy}(n) = x(n) * h(-n) = \sum_{k=-\infty}^{\infty} x(k)\,h(k-n)$

28. $n = 0$

Answers to Objective Type Questions

1. (a)	**2.** (b)	**3.** (c)	**4.** (a)
5. (b)	**6.** (c)	**7.** (d)	**8.** (a)
9. (a)	**10.** (b)	**11.** (a)	**12.** (b)
13. (a)	**14.** (b)	**15.** (a)	**16.** (a)
17. (b)	**18.** (a)	**19.** (b)	**20.** (a)

Answers to Problems

1. $\left(\frac{1}{2}\right)^n \left[\frac{4^{n+1}-1}{3}\right]$

2. $\left(\frac{1}{5}\right)^n \left[\frac{1-5^{n+1}}{1-5}\right]$

3. (a) $y(n) = \{4, 2, 3, 3, 3, -1, 1\}$

(b) $y(n) = \left\{\begin{matrix} 4, 2, 2, 13, 1, 5, 5 \\ \uparrow \end{matrix}\right\}$

(c) $y(n) = \left\{\begin{matrix} 2, 3, 5, 10, 3, 9 \\ \uparrow \end{matrix}\right\}$

(d) $y(n) = \left\{\begin{matrix} 2, 4, 7, 2, 3 \\ \uparrow \end{matrix}\right\}$

(e) $y(n) = \{2, 0, 4, 0, 7, 0, 2, 0, 3\}$

(f) $y(n) = \{2, 4, 7, 2, 3, 0, 0, 0, 0\}$

4. (a) $x(n) = \{5, 15, 8\}$

(b) $x(n) = \{7, 6, 4\}$

5. (a) $\{12, 10, 14\}$

(b) $\{14, 12, 18\}$

6. $x(n) = \{4, 1, 3\}$

7. $h(n) = \{2, -1, 3\}$

8. (a) $\{0, 3, 3, 0\}$

(b) $\{-18, 14, -18, -2\}$

(c) $\{100, 120, 120, 100, 60\}$

9. (a) $x(n)*h(n)=\{2,0,-5,-10,2,11,12\}$, $x(n)\oplus h(n)=\{4,11,7,-10\}$

$\sum y(n)=12$ in both cases.

(b) $x(n)*h(n)=\{11,11,14\}$, $x(n)\oplus h(n)=\{3,8,14,8,3\}$

$\sum y(n)=36$ in both cases.

10. $y(n)=\{3,0,6,2,0,4,0\}$

11. $y(n)=\{2,3,-4,-5,-4,-1,9,6\}$

12. (a) one to $x(n)$ and four to $h(n)$ (b) $y(n)=\{2,7,3,-5,-13,6\}$

(c) same as (b) (d) $y(n)=\{2,7,3,-5,-13,6,0,0,0,0,0\}$

13. (a) $y(n)=\{18,20,11,-7,11,-5\}$ (b) $y(n)=\{9,2,-6,-4,9\}$

14. $R_{xh}(n)=\{4,8,9,17,9,9,4\}$

15. $R_{xh}(n)=\{2,6,11,17,13,7,4\}$ $R_{hx}(n)=\{4,7,13,17,11,6,2\}$

16. $R_{xx}(n)=\{-8,14,-13,30,-13,14,-8\}$

17. (a) $R_{xyp}(n)=\{11,-4,11,-9\}$ and $R_{yxp}(n)=\{5,8,-7,-3\}$

(b) $R_{xyp}(n)=\{-2,-6,5,3\}$ and $R_{yxp}(n)=\{5,-6,-2,3\}$

18. (a) $R_{xxp}(n)=\{-25,20,-25,30\}$ (b) $R_{xxp}(n)=\{-7,-7,14\}$

CHAPTER 3

Answers to Fill in the Blanks

1. difference, algebraic **2.** discrete-time, continuous-time
3. ROC of $X(z)$ **4.** $x(n)r^{-n}$ **5.** $r=1$ **6.** intersection
7. exterior **8.** interior **9.** poles, infinity **10.** $\underset{z\to\infty}{\mathrm{Lt}}\ X(z)$
11. $\underset{z\to 1}{\mathrm{Lt}}\ (z-1)X(z)$ **12.** two **13.** $z=e^{j\omega}$
14. inside the unit circle **15.** unit circle **16.** forced response
17. free response **18.** total response **19.** impulse response

Answers to Objective Type Questions

1. (c) **2.** (c) **3.** (d) **4.** (c)
5. (a) **6.** (d) **7.** (a) **8.** (c)
9. (a) **10.** (a)

Answers to Problems

1. (a) $X_1(z) = 2 + z^{-1} + 3z^{-2} - 4z^{-3} + z^{-4} + 2z^{-5}$

(b) $X_2(z) = z^5 + 3z^4 - 2z^3 + 2z + 4$

(c) $X_3(z) = 2z^3 + 4z^2 + z + z^{-1} + 3z^{-2} + 5z^{-3}$

2. (a) $X_1(z) = \dfrac{z}{(z-1)^2}$ (b) $X_2(z) = \dfrac{z(z+1)}{(z-1)^3}$

(c) $X_3(z) = \dfrac{2z}{(z-2)^2}$ (d) $X_4(z) = \dfrac{z[z-2\cos 3]}{z^2 - 4z\cos 3 + 4}$

(e) $X_5(z) = \dfrac{(1/3\sqrt{2})\,z}{[z-(1/3)e^{j(\pi/4)}][z-(1/3)e^{-j(\pi/4)}]}$

(f) $X_6(z) = \dfrac{z^{10}-1}{z^9(z-1)}$ (g) $X_7(z) = \dfrac{15z}{5z-4} - \dfrac{9z}{9z-4}$

(h) $X_8(z) = \dfrac{1}{1-3z}$ (i) $X_9(z) = \dfrac{z}{z-(1/2)}\left(1 - \dfrac{z^{-8}}{2^7}\right)$

(j) $X_{10}(z) = \dfrac{3}{1-(1/2)z}$ (k) $X_{11}(z) = z^4\dfrac{(9z^2 - 5.5z + 1)}{[z^3 - (1/3)z^2]^3}$

3. (a) $x(n) = \{1, 1.5, 1.75, 1.875, 1.9375, \ldots\}$ (b) $x(n) = \{\ldots, 62, 30, 14, 6, 2, 0, \underset{\uparrow}{0}\}$

4. (a) $x(n) = \left[\dfrac{3}{4}(-1)^n - \dfrac{1}{2}n(-1)^n + \dfrac{1}{4}\right]u(n)$ (b) $x(n) = \dfrac{9}{2}\left(-\dfrac{1}{2}\right)^n u(n) - \dfrac{7}{2}\left(-\dfrac{1}{3}\right)^n u(n)$

(c) $x(n) = \left(-\dfrac{1}{3}\right)^n u(n)$

(d) $x(n) = \delta(n-1) + \left(-\dfrac{1}{2}\right)^n u(-n-1) - 2(-1)^n u(-n-1)$

(e) $x(n) = \left[4\left(\dfrac{1}{2}\right)^n u(n) - 3\left(-\dfrac{1}{4}\right)^n u(n)\right]$

(f) $x(n) = (-2)^n u(n) + (3)^n u(n) + \dfrac{2}{3}(n+1)(3)^{n+1} u(n+1)$

(g) $x(n) = \dfrac{9}{2}u(n) + \dfrac{3}{2}\left(\dfrac{1}{3}\right)^n u(n) - 9(n+1)\left(\dfrac{1}{3}\right)^{n+1} u(n+1)$

5. (a) (i) $x(n)=\left(\frac{1}{2}\right)^n u(n)-\left(\frac{1}{4}\right)^n u(n)$; ROC; $|z|>\frac{1}{2}$

(ii) $x(n)=\left(\frac{1}{4}\right)^n u(-n-1)-\left(\frac{1}{2}\right)^n u(-n-1)$; ROC; $|z|<\frac{1}{4}$

(iii) $x(n)=-\left(\frac{1}{2}\right)^n u(-n-1)-\left(\frac{1}{4}\right)^n u(n)$; ROC; $\frac{1}{4}<|z|<\frac{1}{2}$

(b) (i) $x(n)=2(2)^n u(n)-\left(\frac{1}{2}\right)^n u(n)$; ROC; $|z|>2$

(ii) $x(n)=-2(2)^n u(-n-1)+\left(\frac{1}{2}\right)^n u(-n-1)$; ROC; $|z|<\frac{1}{2}$

(iii) $x(n)=-2(2)^n u(-n-1)-\left(\frac{1}{2}\right)^n u(n)$; ROC; $\frac{1}{2}<|z|<2$

(c) (i) $x(n)=2\delta(n)-9\left(\frac{1}{2}\right)^n u(n)+8u(n)$; ROC; $|z|>1$

(ii) $x(n)=2\delta(n)+9\left(\frac{1}{2}\right)^n u(-n-1)-8u(-n-1)$; ROC; $|z|<\frac{1}{2}$

(iii) $x(n)=2\delta(n)-9\left(\frac{1}{2}\right)^n u(n)-8u(-n-1)$; ROC; $\frac{1}{2}<|z|<1$

(d) (i) $x(n)=(-2)^{n-3}u(n-3)+(3)^{n-3}u(n-3)+\frac{2}{3}(n-2)(3)^{n-2}u(n)$; ROC; $|z|>3$

(ii) $x(n)=-(-2)^{n-3}u(-n+2)-(3)^{n-3}u(-n+2)-\frac{2}{3}(n-2)(3)^{n-2}u(-n+1)$; ROC; $|z|<2$

(iii) $x(n)=(-2)^{n-3}u(n-3)-(3)^{n-3}u(-n+2)-\frac{2}{3}(n-2)(3)^{n-2}u(-n+1)$; ROC; $2<|z|<3$

6. (a) $nu(n)$ (b) $\frac{n(n-1)}{2}u(n)$

7. $x(n)=\sum_{k=0}^{\infty}\delta(n-3k)$

8. $h(n)=\left[3\left(\frac{1}{4}\right)^n-2\left(\frac{1}{3}\right)^n\right]u(n)$, $H(z)=\frac{1-(1/2)z^{-1}}{[1-(1/3)z^{-1}][1-(1/4)z^{-1}]}$

9. $x(n)=-\frac{1}{2}\delta(n)+\frac{1}{6}u(n)+\frac{4}{3}\left(-\frac{1}{2}\right)^n u(n)$

10. (a) Not both causal and stable; (b) Both causal and stable.

11. $y_n(n) = \frac{1}{12}\left(\frac{1}{2}\right)^n u(n) + \frac{1}{24}\left(-\frac{1}{4}\right)^n u(n)$

12. $y_f(n) = \frac{2}{5}\left(\frac{1}{3}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(n) + \frac{8}{5}(2)^n u(n)$

13. $y(n) = \frac{1}{4}u(n) - \frac{9}{4}(-3)^n u(n)$

14. $y(n) = \left[-\frac{79}{36}\left(\frac{1}{3}\right)^n - \frac{77}{72}\left(-\frac{1}{3}\right)^n + \frac{27}{8}\right]u(n)$

15. $y(n) = \frac{2}{3}u(n) + \left(\frac{1}{2}\right)^n u(n) + \frac{1}{3}\left(\frac{1}{4}\right)^n u(n);\ n \ge 0$

16. $y(n) = \frac{1}{1-\alpha}[1-\alpha^{n+2}]\,u(n)$

17. $h(n) = -\left(\frac{1}{4}\right)^n u(n) + 2\left(\frac{1}{2}\right)^n u(n) \qquad s(n) = \frac{1}{3}\left(\frac{1}{4}\right)^n u(n) - 2\left(\frac{1}{2}\right)^n u(n) + \frac{8}{3}u(n)$

18. $y(n) = \frac{25}{3}u(n) + \frac{8}{3}\left(\frac{1}{4}\right)^n u(n) - \frac{19}{2}\left(\frac{1}{2}\right)^n u(n)$

19. $y(n) = -\frac{1}{3}(-2)^n u(n) + \frac{1}{3}u(n) + nu(n)$

20. $y(n) = \frac{25}{9}\left(\frac{1}{2}\right)^n u(n) + \frac{2}{9}\left(\frac{1}{8}\right)^n u(n) + \frac{8}{3}n\left(\frac{1}{2}\right)^n u(n)$

21. $H(z) = \frac{z + (1/2)}{z + (1/4)}$, ROC $|z| > \frac{1}{4}$, $y(n) = \left(-\frac{1}{4}\right)^n u(n) + \frac{1}{2}\left(-\frac{1}{4}\right)^{n-1} u(n-1)$

$$\left|H(e^{j\omega})\right| = \left[\frac{1.25 + \cos\omega}{1.065 + 0.5\cos\omega}\right]^{1/2} \qquad \underline{/H(e^{j\omega})} = \tan^{-1}\frac{\sin\omega}{0.5 + \cos\omega} - \tan^{-1}\frac{\sin\omega}{0.25 + \cos\omega}$$

22. $y(n) = \frac{1}{6}u(n) - \frac{17}{2}(3)^n u(n) + \frac{31}{3}(4)^n u(n)$

CHAPTER 4

Answers to Fill in the Blanks

1. non-recursive **2.** recursive
3. adder, constant multiplier, unit delay element
4. computational complexity
5. memory requirements
6. finite world length effects

7. IIR **8.** $y(n) = \sum_{k=0}^{\infty} h(k)\, x(n-k)$

9. $y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$

10. $H(z) = \dfrac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \dfrac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$

11. direct form-I, direct form-II, cascade form, parallel form
12. direct form-I **13.** delay elements **14.** FIR

15. $y(n) = \sum_{k=0}^{N-1} h(k)\, x(n-k)$ **16.** $y(n) = \sum_{k=0}^{N-1} b_k\, x(n-k)$

17. $H(z) = \sum_{k=0}^{N-1} b_k z^{-k} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_{N-1} z^{-(N-1)}$

18. direct form, cascade form, linear phase realization
19. impulse response
20. multipliers

Answers to Objective Type Questions

1. (a) **2.** (b) **3.** (b) **4.** (d)

CHAPTER 5

Answers to Fill in the Blanks

1. $X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$

2. $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$

3. signal spectrum

4. 2π

5. integration, summation

6. $-\infty$ to ∞, $-\pi$ to π

7. $x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$

8. unit circle

9. $X(\omega) = X(z)\big|_{z=e^{j\omega}}$

10. continuous and periodic

11. impulse response

12. frequency response

13. frequency response, transfer function

14. continuous

15. magnitude function, phase function

16. real

17. imaginary

18. $x(n) * h(n)$

Answers to Objective Type Questions

1. (b)	**2.** (b)	**3.** (a)	**4.** (a)
5. (c)	**6.** (a)	**7.** (d)	**8.** (c)
9. (b)	**10.** (a)	**11.** (b)	**12.** (b)
13. (a)	**14.** (d)	**15.** (a)	

Answers to Problems

1. (a) $X(\omega) = 2 - e^{-j\omega} + 3e^{-j2\omega} + 2e^{-j3\omega}$

(b) $X(\omega) = 16\,\frac{e^{j2\omega}}{1-(1/4)\,e^{-j\omega}}$

(c) $X(\omega) = \frac{1}{1-0.2e^{-j\omega}} + \frac{1}{1-2e^{-j\omega}}$

(d) $X(\omega) = \frac{1 - a\cos\omega_0 e^{-j\omega}}{1 - 2a\cos\omega_0 e^{-j\omega} + a^2 e^{-j2\omega}}$

2. (a) $X(\omega) = e^{-j3\omega}\,\frac{3/4}{(5/4)-\cos\omega}$

(b) $X(\omega) = e^{-j4\omega}\,\frac{1}{1-(1/2)\,e^{-j\omega}}$

(c) $X(\omega) = -\frac{e^{j\omega}}{(1-e^{j\omega})^2}$

(d) $X(\omega) = \frac{1}{1-e^{-j(\omega-2)}}$

(e) $X(\omega) = -\frac{2e^{j\omega}}{(1-2e^{j\omega})^2}$

3. $y(n) = \{1, 4, 4, 10, 7, -5, -1\}$

4. $x(n) = \{1, 2, \underset{\uparrow}{3}, 2, 1\}$

5. $X(\omega) = (1 + 4\cos\omega)\, e^{-j\omega}$

6. $y(n) = \dfrac{1}{3}[a^{n+1}\, u(n+1) + a^n u(n) + a^{n-1} u(n-1)]$

7. $H(\omega) = \dfrac{1 - \cos\omega}{2}$

8. $H(\omega) = \dfrac{1 + 0.81e^{-j\omega} + 0.81e^{-j2\omega}}{1 + 0.45e^{-j2\omega}}$, $|H(\omega)| = \left[\dfrac{2.31 + 2.94\cos\omega + 1.62\cos 2\omega}{1.2 + 0.9\cos 2\omega}\right]^{1/2}$,

$\angle H(\omega) = \tan^{-1}\left[\dfrac{-0.45\sin\omega - 0.36\sin 2\omega}{1.36 + 1.17\cos\omega + 1.26\cos 2\omega}\right]$

CHAPTER 6

Answers to Fill in the Blanks

1. continuous, 2π
2. DFT
3. DFT
4. $X(k) = X(\omega)\big|_{\omega = \frac{2\pi k}{N}}$
5. unit circle
6. finite number of, unit circle
7. $X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi nk}{N}}, \quad k = 0, 1, 2, \ldots, N-1$
8. $x(n) = \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi nk}{N}}, \quad n = 0, 1, 2, \ldots, N-1$
9. $X(k) = X(z)\big|_{z = e^{j\frac{2\pi k}{N}}}$
10. W_N^{nk}
11. W_N
12. $\dfrac{1}{N}[\text{DFT}\,\{X^*(k)\}]^*$
13. $X(k + N) = X(k)$
14. $\text{DFT}\,\{ax_1(n) + bx_2(n)\} = aX_1(k) + bX_2(k)$
15. $X(k) = \sum_{n=0}^{N-1} x(n)\cos\left(\dfrac{2\pi nk}{N}\right)$
16. $X(k) = \sum_{n=0}^{N-1} x(n)\sin\left(\dfrac{2\pi nk}{N}\right)$
17. $X[(-k), \bmod N]$
18. $X[(k - l), \bmod N]$
19. $X(k)\, e^{-j\frac{2\pi n_0 k}{N}}$
20. $\dfrac{1}{N}[X_1(k) \oplus X_2(k)]$
21. $X_1(k)X_2(k)$
22. $\dfrac{1}{N}\sum_{k=0}^{N-1} X_1(k)\, X_2^*(k)$

23. $X(k)\ Y^*(k)$ **24.** $\sum_{n=0}^{N-1} x(n)$ **25.** $(-1)^n x(n)$ **26.** circular

27. fast convolution **28.** slow convolution **29.** sectioned

30. overlap-add, overlap-save **31.** $N-1$, $N-1$

32. overlapped, non-overlapped **33.** circular convolution

34. $N-1$

Answers to Objective Type Questions

1. (c)	**2.** (b)	**3.** (a)	**4.** (c)
5. (a)	**6.** (d)	**7.** (c)	**8.** (b)
9. (c)	**10.** (a)	**11.** (c)	**12.** (b)
13. (c)	**14.** (a)	**15.** (a)	**16.** (a)
17. (b)	**18.** (a)	**19.** (a)	**20.** (b)

Answers to Problems

1. (a) $X(k) = \{2,\ 1-j,\ 0,\ 1+j\}$, (b) $X(k) = (1/5)[1+\cos(2\pi k/3)]$, $k = 0, 1, 2, \ldots, N-1$

2. $X(k) = \{12, -1.5+j2.6, -1.5+j0.866, 0, -1.5-j0.866, -1.5-j2.60\}$

$|X(k)| = \{12, 3, 1.73, 0, 1.73, 3\}$ $\quad \angle X(k) = \left\{0, -\frac{\pi}{3}, -\frac{\pi}{6}, 0, \frac{\pi}{6}, \frac{\pi}{3}\right\}$

3. $X(k) = \{1, 1-j1.414, 1, 1+j1.414\}$

4. $x(n) = \{2.5, -0.5-j0.5, -0.5, -0.5+j0.5\}$

5. $X(k) = \{1, 0, 1, 0\}$ **6.** $x(n) = \{2, 0, 0, 1\}$ **7.** $X(k) = \{3, 2+j, 1, 2-j\}$

8. $x(n) = \{1, 1, 0, 0\}$ **9.** $X(k) = \{4, 2, 0, 4\}$

10. $x(n) = \{1, -1, 2, -2\}$ **11.** $x(n) = \{1, -2, 3, 2\}$

12. $x(n) = \{2, 1, 2\}$ **13.** $-4+j3, 3-j6, -1+j5, 3+j4, 1-j3, -2-j2$

14. $A = 3+j5, B = 2-j1, C = 1+j3$ **15.** $A = 5$, $B = 1$

16. 7.75 **17.** (a) 5, (b) 11, (c) 8, (d) 228

18. (a) $\{2, j3, 0, -j3\}$, (b) $\{2, j3, 0, -j3\}$, (c) $\{2, -j3, 0, j3\}$, (d) $\{5.5, -j3, -4.5, j3\}$ (e) $\{4, -9, 0, -9\}$, (f) 5.5

19. (a) $\{2, j, -2, 0\}$, (b) $\{16, 4, 16,. 0\}$, (c) $\{8, 4, 9, 4\}$, (d) 9

20. (a) $y(n) = \{2, -3, 8, 1, 6, 4\}$, (b) $y(n) = \{6, 23, 33, 38, 15, 20\}$, (c) $y(n) = \{2, 4, 3, 2, 1\}$

21. (a) $y(n) = \{-2, 2, -2, 2\}$, (b) $y(n) = \{6, 7, 6, 5\}$, (c) $y(n) = \{14, 16, 14, 16\}$

22. (a) $y(n) = \{-4, 0, 1, 0, 1, 0, 1, 0, 1\}$, (b) $y(n) = \{1, 2, 0, 5, 2, 2, 5, 0, 2, 1\}$
(c) $y(n) = \{4, -8, 5, -9, 10, -7, 10, -6, 1\}$, (d) $y(n) = \{5, 0, 9, 0, 7, 0, 5, 0, 2\}$

CHAPTER 7

Answers to Fill in the Blanks

1. $\{A\}$ **2.** $\{A + B, A - B\}$ **3.** $N \times N$, $N \times (N-1)$, **4.** $4N^2$, $4N(N - 1)$

5. $\frac{N}{2}\log_2 N, N\log_2 N$ **6.** algorithm or method **7.** divide and conquer

8. DIT FFT, DIF FFT **9.** symmetry, periodicity **10.** $N = 2^m$

11. radix, stages **12.** bit reversed, normal **13.** normal, bit reversed

14. $\frac{1}{N}\left[\sum_{k=0}^{N-1} X^*(k)\, W_N^{nk}\right]$ **15.** five **16.** six **17.** butterfly

18. $N/2$ **19.** slow **20.** fast **21.** four, eight

22. $x(n)$, $X(k)$

Answers to Objective Type Questions

1. (b) **2.** (d) **3.** (a) **4.** (d)
5. (b) **6.** (b) **7.** (d) **8.** (a)
9. (c) **10.** (a)

Answers to Problems

1. (a) $X(k) = \{1, 1 - j2, -1, 1 + j2\}$ (b) $X(k) = \{1, 0, 1, 0\}$
(c) $X(k) = \{3, 2 + j, 1, 2 - j\}$ (d) $X(k) = \{0, 2, 0, 2\}$

2. (a) $x(n) = \{2, 1, 4, 3\}$ (b) $x(n) = \{1, 1, -1, -1\}$
(c) $x(n) = \{2, 2, 2, 0\}$ (d) $x(n) = \{-2, -2, 2, 0\}$

3. (a) $X(k) = \{0, -\sqrt{2} + j3.414, 2 - j2, \sqrt{2} - j0.585, 4, \sqrt{2} + j0.585, 2 + j2, -\sqrt{2} - j3.414\}$
(b) $X(k) = \{6, -\sqrt{2} - j4.828, -2 + j2, \sqrt{2} - j0.828, -2, \sqrt{2} + j0.828, -2 - j2, -\sqrt{2} + j4.828\}$
(c) $X(k) = \{36, -4 + j9.656, -4 + j4, -4 + j1.656, -4, -4 - j1.656, -4 - j4, -4 - j9.656\}$
(d) $X(k) = \{4, 1 - j2.414, 0, 1 - j0.414, 0, 1 + j0.414, 0, 1 + j2.414\}$

4. (a) $x(n) = \{1, 0, 0, 0, 0, 0, 0, 0\}$ (b) $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$
(c) $x(n) = \{0, 1, 2, 3, 4, 5, 6, 7\}$ (d) $x(n) = \{1, 2, 2, 1, 1, 2, 2, 1\}$

5. (a) {1, 1.25} (b) {14, 16, 14, 16} (c) {–2, 2, –2, 2} (d) {6, 7, 6, 5}

6. (a) {1, 1, 2, 2} (b) {1, 1, 1, –3}
(c) {1, 1, 0, 2, –2, 0, –1, –1, 0} (d) {1, –2, 2, –3, 2, –1, 1, 0}

7. $X(k) = \{7, -1, 1 + j3.464, -1, 1 - j3.464, -1\}$

8. $X(k) = \{0, 1.501 + j0.866, 4.499 - j2.598, 0, 4.499 + j2.598, 1.501 - j0.866\}$

CHAPTER 8

Answers to Fill in the Blanks

1. all the infinite **2.** linear **3.** magnitude
4. only a finite number **5.** ideal frequency response
6. sampling frequency **7.** transforming, equivalent
8. half
9. approximation of derivatives, impulse invariant transformation, bilinear transformation
10. causality, stability **11.** ripples **12.** impulse invariant **13.** left half
14. exterior **15.** unit circle **16.** $2\pi/T$
17. aliasing **18.** aliasing **19.** impulse invariant
20. many-to-one, one-to-one
21. distortion in frequency axis **22.** magnitude response, prewarping
23. bilinear **24.** Butterworth, Chebyshev **25.** Butterworth
26. $1/\sqrt{2}$ **27.** type-1 Chebyshev
28. type-2 Chebyshev **29.** inverse Chebyshev **30.** $1/\sqrt{1+\varepsilon^2}$

Answers to Objective Type Questions

1. (a)	**2.** (a)	**3.** (a)	**4.** (b)
5. (a)	**6.** (b)	**7.** (b)	**8.** (c)
9. (a)	**10.** (b)	**11.** (a)	**12.** (a)
13. (a)	**14.** (a)	**15.** (b)	**16.** (b)
17. (a)			

Answers to Problems

1. (a) $H(z) = \dfrac{1}{5 - z^{-1}}$, (b) $H(s) = \dfrac{1}{26 - 2z^{-1} + z^{-2}}$

(c) $H(z) = \dfrac{1}{26.44 - 2.4z^{-1} + z^{-2}}$

2. (a) $$H(z) = \frac{0.031z^{-1}}{(1-0.049z^{-1})(1-0.018z^{-1})}$$ (b) $$H(z) = \frac{1+0.810z^{-1}}{1+1.621z^{-1}+0.670z^{-2}}$$

(c) $$H(z) = \frac{0.303z^{-1}+0.219z^{-2}}{1-0.883z^{-1}+0.774z^{-2}-0.367z^{-3}}$$

3. $$H(z) = \frac{4.3+0.6z^{-1}-3.7z^{-2}}{34.49+0.18z^{-1}+29.69z^{-2}}$$

4. (a) $$H(z) = \frac{4(1+z^{-1})^2}{15-2z^{-1}-z^{-2}}$$ (b) $$H(z) = \frac{2(1-z^{-2})}{7+2z^{-2}}$$

5. $$H(z) = \frac{1.453(1+z^{-1})}{6.359+2.359z^{-1}}$$

6. $$H(z) = \frac{0.005(1+z^{-1})^4}{[1-1.321z^{-1}+0.633z^{-2}][1-1.048z^{-1}+0.296z^{-2}]}$$

7. $$H(z) = \frac{0.511}{1-0.6z^{-1}} + \frac{-0.511+0.309z^{-1}}{1-0.873z^{-1}+0.6z^{-2}}$$

8. $$H(z) = \frac{-1.450-0.232z^{-1}}{1-0.131z^{-1}+0.300z^{-2}} + \frac{1.450+0.184z^{-1}}{1-0.386z^{-1}+0.055z^{-2}}$$

9. (a) $$H(z) = \frac{0.109(1+z^{-1})}{2.109-1.891z^{-1}}$$ (b) $$H(z) = \frac{0.030(1+z^{-1})}{1-0.938z^{-1}}$$

10. $$H(z) = \frac{0.000996(1+z^{-1})^6}{(1-1.404z^{-1}+0.847z^{-2})(1-1.378z^{-1}+0.796z^{-2})(1-1.171z^{-1}+0.527z^{-2})}$$

11. $$H(s) = \frac{0.16s^2}{\Omega_c^2 s^4 + 5\Omega_c s^3 + (32\Omega_c^2+4)s^2 + 80\Omega_c s + 256\Omega_c^2}$$

12. $$H(s) = \frac{s}{3s+\Omega_c^*}$$

CHAPTER 9

Answers to Fill in the Blanks

1. frequency response $H(\omega)$
2. weighting function, spectral shaping
3. Constant
4. nonlinear
5. Phase
6. synonymous
7. 0 to 2π
8. frequency, impulse
9. FIR, IIR
10. low-pass, high-pass, band pass, band stop
11. symmetric
12. antisymmetric
13. non-causal
14. transition region
15. group delay, phase delay
16. frequency
17. antisymmetric
18. symmetric
19. symmetric
20. antisymmetric
21. Oscillations
22. Gibbs phenomenon
23. $z^{-(N-1)/2}$
24. window sequence
25. main lobe
26. Gibbs oscillation
27. $4\pi/N$
28. Hamming
29. Blackman
30. Kaiser
31. Blackman
32. Hamming
33. Blackman

Answers to Objective Type Questions

1. (b) **2.** (a) **3.** (c) **4.** (a)
5. (b) **6.** (c) **7.** (c) **8.** (d)
9. (d) **10.** (a) **11.** (b) **12.** (c)
13. (d)

Answers to Problems

1. $H(z) = 0.1378z^{-1} + 0.2756z^{-2} + 0.3333z^{-3} + 0.2756z^{-4} + 0.1378z^{-5}$

$H(\omega) = e^{-j3\omega}\,[0.3333 + 0.5512\cos\omega + 0.2756\cos 2\omega]$

2. $H(z) = -\dfrac{1}{3\pi} + \dfrac{1}{\pi}z^{-2} + \dfrac{1}{2}z^{-3} + \dfrac{1}{\pi}z^{-4} - \dfrac{1}{3\pi}z^{-6}$

$H(\omega) = e^{-j3\omega}\left[\dfrac{1}{2} + \dfrac{2}{\pi}\cos\omega - \dfrac{2}{3\pi}\cos 2\omega\right]$

3. $h(0) = -0.159,\ h(1) = 0.225,\ h(2) = 0.75,\ h(3) = 0.225,\ h(4) = -0.159$

$H(\omega) = e^{-j2\omega}\,[0.75 + 0.45\cos\omega + 0.318\cos 2\omega]$

4. $H(z) = 0.0149 + 0.1447z^{-1} + 0.2678z^{-2} + 0.318z^{-3} + 0.2678z^{-4} + 0.1447z^{-5} + 0.0149z^{-6}$

$H(\omega) = e^{-j3\omega}\,[0.318 + 0.5356\cos\omega + 0.288\cos 2\omega + 0.0298\cos 3\omega]$

5. $H(z) = -0.0149 - 0.1447z^{-1} - 0.2678z^{-2} + 0.6816z^{-3} - 0.2678z^{-4} - 0.1447z^{-5} - 0.0149z^{-6}$

$H(\omega) = e^{-j3\omega}\,[0.6816 - 0.5356\cos\omega - 0.2894\cos 2\omega - 0.0298\cos 3\omega]$

6. $H(z) = -0.0344z^{-1} + 0.0135z^{-2} + 0.3183z^{-3} + 0.0135z^{-4} - 0.0344z^{-5}$

$H(\omega) = e^{-j3\omega}\,[0.3183 + 0.0270\cos\omega - 0.0689\cos 2\omega]$

7. $H(z) = 0.0035 + 0.0822z^{-1} - 0.0166z^{-2} + 0.6817z^{-3} - 0.0166z^{-4} + 0.0822z^{-5} + 0.0035z^{-6}$

$H(\omega) = e^{-j3\omega}\,[0.6817 - 0.0332\cos\omega + 0.1644\cos 2\omega + 0.0070\cos 3\omega]$

CHAPTER 10

Answers to Fill in the Blanks

1. single rate **2.** multirate **3.** decimation, interpolation
4. decimation, interpolation **5.** Dth **6.** down, up
7. filtering, down sampling **8.** anti-aliasing **9.** increased
10. variant **11.** $X(e^{j\omega I})$ **12.** imaging **13.** image spectra
14. up-sampling, filtering **15.** anti-imaging **16.** sampling rate **17.** one sample period
18. periods **19.** cascading **20.** co-prime **21.** transpose
22. multistage **23.** analysis, synthesis **24.** dual **25.** 32, 44.1, 48
26. aliasing, pseudo images **27.** sub, down, under
28. Nyquist rate **29.** Nyquist period **30.** Interpolation, decimation

Answers to Objective Type Questions

1. (a) **2.** (b) **3.** (c) **4.** (a)
5. (d) **6.** (a) **7.** (b) **8.** (b)
9. (c) **10.** (b) **11.** (b) **12.** (a)
13. (a) **14.** (b) **15.** (a) **16.** (c)

Answers to Problems

1. $x(2n) = \{3, 8, -2\}$, $x(3n) = \{3, 9\}$, $x(4n) = \{3, -2\}$
$x(n/2) = \{3, 0, 6, 0, 8, 0, 9, 0, -2, 0, -1\}$
$x(n/3) = \{3, 0, 0, 6, 0, 0, 8, 0, 0, 9, 0, 0, -2, 0, 0, -1\}$
$x(n/4) = \{3, 0, 0, 0, 6, 0, 0, 0, 8, 0, 0, 0, 9, 0, 0, 0, -2, 0, 0, 0, -1\}$

2.

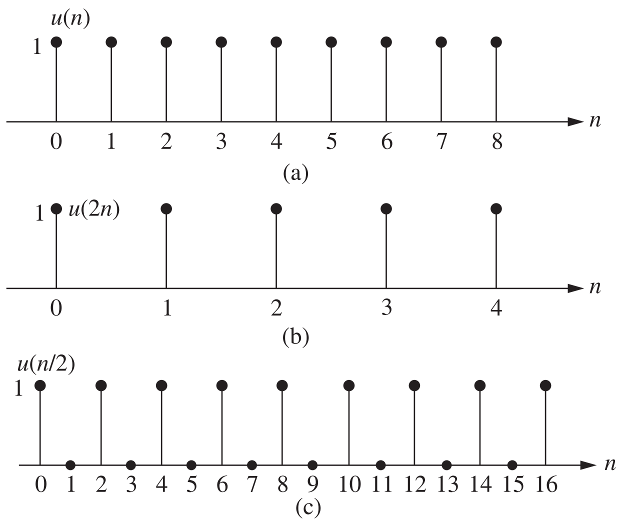


Figure A.2 Answer to problem 2.

3.

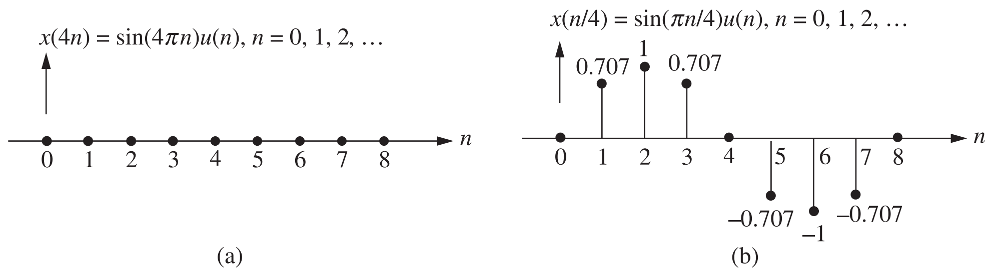


Figure A.3 Answer to problem 3.

4.

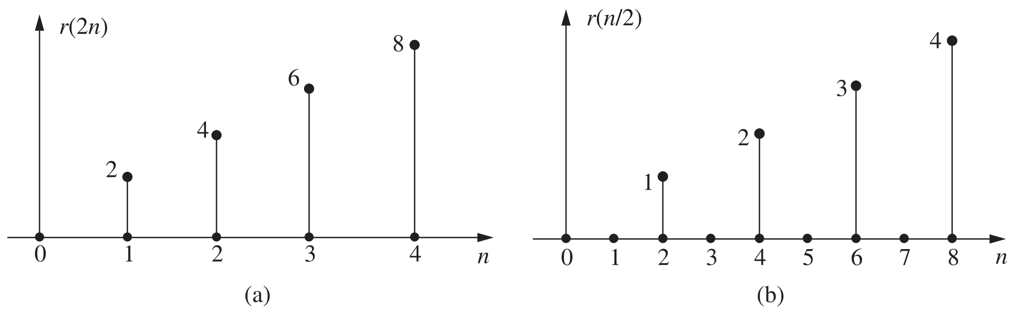


Figure A.4 Answer to problem 4.

5. (a) $P_0(z^2) = \dfrac{1 + 6z^{-2}}{1 - 9z^{-2}}, \quad P_1(z^2) = \dfrac{-5}{1 - 9z^{-2}}$

(b) $P_0(z^2) = \dfrac{1 + 1.8z^{-2} + 1.2z^{-4}}{1 + 0.56z^{-2} + 0.36z^{-4}}, \quad P_1(z^2) = \dfrac{0.2 - z^{-2}}{1 + 0.56z^{-2} + 0.36z^{-4}}$

6. $y(n) = 1, \quad \text{for } n = 0, \pm 3, \pm 6, \ldots$

$\quad = 0, \quad \text{elsewhere}$

7. (a) $X(\omega) = \dfrac{e^{j\omega}}{(e^{j\omega}-1)^2}, \quad |X(\omega)| = \dfrac{1}{\sqrt{6-8\cos\omega + 2\cos 2\omega}}$

(b) $Y(\omega) = \dfrac{4e^{j\omega}}{(e^{j\omega}-1)^2}, \quad |Y(\omega)| = \dfrac{4}{\sqrt{6-8\cos\omega + 2\cos 2\omega}}$

8. (a) $X(\omega) = \dfrac{e^{j\omega}}{e^{j\omega}-a}, \quad |X(\omega)| = \dfrac{1}{\sqrt{1-2a\cos\omega + a^2}}$

(b) $Y(\omega) = \dfrac{e^{j\omega}}{e^{j\omega}-a^{1/3}}, \quad |Y(\omega)| = \dfrac{1}{\sqrt{1-2a^{1/3}\cos\omega + a^{2/3}}}$

CHAPTER 11

Answers to Fill in the Blanks

1. four	2. pins	3. Increase	4. slow
5. multiple	6. high speed, multiport		
7. modified Harvard	8. costlier		
9. Von Neumann, Harvard, modified Harvard, VLIW			
10. data path	11. Independent	12. 8	13. Instruction
14. RISC, CISC	15. 20%, 30 to 40%	16. smaller	17. RISC
18. CISC	19. one phase	20. depth	
21. cyclic, bit reversed	22. auxiliary	23. 16, fixed	

Answers to Objective Type Questions

1. (d)	2. (b)	3. (c)	4. (c)
5. (b)			

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